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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

SERIES 2.—VOL. 20.—PART 1.

CONTENTS.

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THE LONDON MATHEMATICAL SOCIETY is instituted for the promotion and extension of Mathematical Knowledge.

It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

The Candidate, if elected, is informed of his election by one of the Secretaries, and supplied with a copy of the Memorandum and Articles of Association and By-Laws. He must pay the contribution which is due from him within six months after the day of his election, otherwise his election shall be void.

An entrance fee of two guineas is required to be paid by each newly elected Member.

The Annual Subscription to be paid by each Member is two guineas. The subscription due from a newly elected member for his first year of membership is one guinea if his election takes place between March and June inclusive: any Member may compound for his *annual* subscriptions by the payment of twenty-five guineas in one sum, or in two equal annual instalments. In certain cases the composition fee is reduced.

Every Member is considered liable for his annual subscription until he has signified in writing his desire to resign, and has returned all books and property belonging to the Society.

Annual subscriptions are due at the beginning of each Session (the first Thursday in November of each year), and may be paid to the account of the London Mathematical Society, London County, Westminster and Parr's Bank, 1 St. James's Square, London, S.W.1, or to the Treasurer, Dr. A. E. Western, 35 Essex Street, Strand, London, W.C.2.

The affairs of the Society are directed by the Council and Officers.

The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting in November.

The Officers are a President, Vice-Presidents, a Treasurer, and Secretaries.

The Ordinary Meetings of the Society are now held at Burlington House, W.1, and begin at 5 o'clock in the evening. The Meetings for the Session 1920–21 are on November 11, December 9, January 13, February 10, March 10, April 21, May 12, and June 9. Cards giving the Programmes of the meetings as they occur can be had by Members a few days beforehand on application to C. F. Hodgson & Son, 2 Newton Street, Kingsway, London, W.C.2.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to one or more Members, who report to the Council on its fitness for publication in the *Proceedings*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

Authors of Papers intended for communication to the Society are requested to furnish to the Secretaries short Abstracts of their Papers, indicating the nature of the methods employed and the character of the results obtained. Abstracts of papers submitted for publication in full should be extremely brief; as a rule, such an Abstract should not exceed half a page of the *Proceedings*. When an Abstract refers to a communication which is not to be published in full, more space will be allowed for the Abstract, with a maximum of about two pages.

Communications for the Secretaries may be forwarded to them at the following addresses:—

New College, Oxford—G. H. Hardy, M.A., F.R.S.

The University, Edmund Street, Birmingham—G. N. Watson, Sc.D., F.R.S.

London Mathematical Society, Burlington House, W.1.

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RECORDS OF PROCEEDINGS AT MEETINGS.

SESSION NOVEMBER, 1920–JUNE, 1921.

Thursday, November 11th, 1920.

ANNUAL GENERAL MEETING.

Mr. J. E. CAMPBELL, President, and later Mr. H. W. RICHMOND, President, in the Chair.

Present thirty-five members and two visitors.

N. Sen was elected a member.

Messrs. F. G. W. Brown, R. G. Cooke, S. L. Green, Y. A. J. Limerick, C. N. H. Lock, H. Lowery, T. A. Lumsden, J. B. Maclean, K. B. Madhava, A. R. Richardson, and Miss N. I. Calderwood, were nominated for election.

The Treasurer presented his Report. Lt.-Col. Cunningham was appointed Auditor.

The President announced that Prof. Eddington had consented to deliver a lecture at the February meeting.

The President presented the De Morgan medal to Prof. E. W. Hobson.

The Officers and Council for the Session 1920–21 were elected. The list is as follows :—President, H. W. Richmond ; Vice-Presidents, T. J. I'A. Bromwich, J. E. Campbell ; Treasurer, A. E. Western ; Secretaries, G. H. Hardy, G. N. Watson ; other members of the Council, C. G. Darwin, A. L. Dixon, A. S. Eddington, L. N. G. Filon, H. Hilton, Miss H. P. Hudson, A. E. Jolliffe, J. E. Littlewood, J. W. Nicholson, W. H. Young.

The retiring President then delivered his Presidential Address, "Einstein's Theory of Gravitation as an Hypothesis in Differential Geometry." Prof. Eddington also spoke on the subject of the address.

The following papers were communicated by title from the Chair :—

On the Conformal Transformations of a Space of Four Dimensions : H. Bateman.

*(1). The Differentiation of the Complete Third Elliptic Integral with respect to the Modulus, (2) Note on the Intersection of a Plane Curve and its Hessian at a Multiple Point: F. Bowman.

On Dirichlet's Multiplication of Infinite Series: T. S. Broderick.

*Arithmetic of Quaternions: L. E. Dickson.

*The Classification of Rational Approximations: P. J. Heawood.

Integral Solutions of Ordinary Linear Differential Equations: E. L. Ince.

*On the Series of Polynomials, every Partial Sum of which Approximates n Values according to the Method of Least Squares: Charles Jordan.

*On some Solutions of the Wave Equation: H. J. Priestley.

*An Example of a thoroughly Divergent Orthogonal Development: H. Steinhaus.

*The Group of the Linear Continuum: N. Wiener.

*On the Partial Derivates of a Function of many Variables: Mrs. G. C. Young.

ABSTRACTS.

On the Conformal Transformations of a Space of Four Dimensions and Lines of Electric Force

Prof. H. BATEMAN.

The system of eighteen differential equations

$$\frac{\partial(x', y')}{\partial(y, z)} = \pm \frac{\partial(z', t')}{\partial(x, t)}, \quad c^2 \frac{\partial(x', t')}{\partial(y, z)} = \pm \frac{\partial(y', z')}{\partial(x, t)},$$
$$\ldots \quad \ldots \quad \ldots, \quad \ldots \quad \ldots \quad \ldots,$$

may be solved directly, giving the relations

$$\begin{aligned} a(la'-u\beta'-p) &= -na'+w\beta'+r+\beta(-ma'+v\beta'+q), \\ a(ua'+lb'-e) &= -wa'-nb'+g-\beta(va'+mb'-f), \\ a(-ma'+v\beta'+q) &= \quad ha'+j\beta'+k-b(la'-u\beta'-p), \\ a(va'+mb'-f) &= \quad ja'-hb'+s+b(ua'+lb'-e), \end{aligned}$$

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* Printed in this volume.

where $\alpha = z' \pm ct'$, $\beta = x' + iy'$, $a = z' \mp ct'$, $b = x' - iy'$,

$$\alpha' = z - ct, \quad \beta' = x + iy, \quad a' = z + ct, \quad b' = x - iy,$$

and $l, m, n, u, v, w, p, q, r, e, f, g, h, j, k, s$ are arbitrary constants. These equations are equivalent to a conformal transformation from (x, y, z, ict) to (x', y', z', ict').

If $\alpha' = \phi + \theta\beta'$, $\theta a' = \psi - b'$, $a = \phi' + \theta'\beta$, $\theta' a = \psi - b$,

the two sets of parameters (θ, ϕ, ψ), (θ', ϕ', ψ') are connected by a projective transformation

$$\chi\theta = w\theta' - v\phi' + u\psi' + j, \quad \chi\psi = g\theta' - f\phi' + e\psi' - s,$$
$$\chi\phi = r\theta' - q\phi' + p\psi' + k, \quad \chi = n\theta' - m\phi' + l\psi' - h,$$

in accordance with a well known theorem.

A set of parameters θ, ϕ, ψ which are functions of a variable parameter τ may sometimes define a line of electric force in an electromagnetic field. The Riccatian equations, which must be satisfied by θ, ϕ, and ψ in order that they may give a line of electric force of a moving electric pole, are written down, and some interesting transformations of these equations are considered.

The Classification of Rational Approximations

Prof. P. J. HEAWOOD.

The object of this paper is to settle certain questions raised by Mr. J. H. Grace, in a paper published in Vol. 17 of the *Proceedings*, with respect to the rational approximations x/y, to a given number θ, which satisfy the condition

$$\left| \frac{x}{y} - \theta \right| < \frac{1}{ky^2},$$

where k is a given number. The special points relate to the cases where k is equal to, or in the neighbourhood of, the critical value 3, and the questions that arise are as to the special forms of θ for which there will be only a finite number of such approximations. It is first shown that, however slightly k exceeds 3, there are not only algebraic but transcendental numbers θ for which there are only a finite number of approximations x/y which satisfy the above condition, a result suggested but left undecided in the paper referred to. The main investigation, however, is of the possible forms of θ for which this is true when $k = 3$ and when

$k < 3$; and the final conclusion is that the result, based by Mr. Grace on certain investigations of Markoff, that in these cases θ must be a quadratic surd, holds in the latter case but not the former.

On some Solutions of the Wave Equation

Prof. H. J. PRIESTLEY.

The wave equation, expressed in spheroidal coordinates, is satisfied by

$$\psi = M(\mu)\, Z(\zeta)\, e^{\iota(m\theta+pt)},$$

provided that

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dM}{d\mu}\right]+\left[n(n+1)-\frac{m^2}{1-\mu^2}\right]M = k^2a^2(1-\mu^2)\,M, \qquad (1)$$

$$\frac{d}{d\zeta}\left[(1+\zeta^2)\frac{dZ}{d\zeta}\right]-\left[n(n+1)-\frac{m^2}{1+\zeta^2}\right]Z = -\,k^2a^2\,(1+\zeta^2)\,Z, \qquad (2)$$

where $k = p/c$ and n is any constant.

As a preliminary to the solution of (1) and (2) the writer discusses the equation

$$\frac{d}{dx}\left[P\frac{dy}{dx}\right]+Qy = \lambda Ry,$$

and exhibits the solution $w(x)$ as the solution of the integral equation

$$w(x) = \chi(x)-\frac{\lambda}{C}\int_a^x R(t)\begin{vmatrix} y_1(x), & y_2(x) \\ y_1(t), & y_2(t)\end{vmatrix}\, w(t)\,dt,$$

where $\chi(x)$, $y_1(x)$, $y_2(x)$ are solutions of

$$\frac{d}{dx}\left[P\frac{dy}{dx}\right]+Qy = 0,$$

and C, a are constants.

The results obtained are first applied to equation (1) and a solution $W_n^{-m}(\mu)$ is found such that $W_n^{-m}(\mu)/(1-\mu^2)^{\frac{1}{2}m}$ is finite throughout the range $-1 < \mu \leqslant 1$.

It is shown that, if $\dfrac{d}{d\mu}W_n^{-m}(\mu) = 0$ at $\mu = 0$, the following theorems hold :—

(I) $W_n^{-m}\mu)$ is even.

(II) $W_n^{-m}(\mu)$ is the solution of a homogeneous Fredholm equation.

(III) If m is real, the values of n are real and separate.

(IV) The values of n are infinite in number.

(V) Any function of μ, which with its first two derived functions is continuous over the range $0 \leqslant \mu \leqslant 1$ and of which the first derivative vanishes at μ, can be expanded in a series of functions $W_n^{-m}(\mu)$.

Analogous theorems hold when $W_n^{-m}(0) = 0$.

The results of the preliminary discussion are then used to find a Volterra equation for a solution of (2) which behave like $e^{-ika\zeta}/\zeta$ when ζ tends to infinity.

On the Partial Derivates of a Function of many Variables

Mrs. G. C. YOUNG.

The results obtained in this paper correspond to those given in an earlier communication for a single variable, and include a somewhat extended form and a revised proof of one of the earlier theorems. They are as follows, the primitive function $f(x, y) \equiv f(x, y_1, y_2, \ldots, y_{n-1}, \ldots)$ being supposed finite and measurable for each fixed ensemble y.

(1) *The points at which the upper partial derivate on one side with respect to x is less than the lower partial derivate on the other side, form a set of plane content zero, whose section by every line $y =$ constant is a countable set.*

(2) *The points at which the upper partial derivate with respect to x on one side has the value $+\infty$, while the lower partial derivate on the other side has a value other than $-\infty$, form a set of plane content zero, whose section by every line $y =$ constant has zero linear content.*

(3) *The points at which there is a forward or a backward partial differential coefficient, or a partial differential coefficient, $\partial f/\partial x$ which is infinite with determinate sign, form a set of plane content zero, whose section by $y =$ constant is of zero linear content.*

(4) *The points at which one of the upper (lower) partial derivates with respect to x, being finite, is not equal to the lower (upper) derivate on the other side, form a set of plane content zero, whose section by $y =$ constant is a set of linear content zero.*

(4*b*) *The points, if any, at which one of the upper partial derivates with respect to x, and one of the lower partial derivates are finite and different from one another, form a set of plane content zero, whose section by y = constant is a set of linear content zero.*

Corresponding results are given when the primitive function

$$f(x, y) \equiv f(x, y_1, y_2, \ldots, y_{n-1})$$

assumes infinite values. In particular (2) now takes the following form :—

(2 *bis*) *The points at which $f(x, y)$ has an infinite partial forward or backward differential coefficient with determinate sign, consist of the infinities of $f(x, y)$ and possibly an additional set of plane content zero, whose section by y = constant is a set of linear content zero.*

For a partial differential coefficient $\partial f(x, y)/\partial x$, however, (2) remains true, even when $f(x, y)$ assumes infinite values.

Thursday, December 9th, 1920.

Mr. H. W. RICHMOND, President, in the Chair.

Present thirteen members.

The Auditor's report was received, and a vote of thanks to the Auditor was carried unanimously.

Messrs. F. G. W. Brown, R. G. Cooke, S. L. Green, Y. A. J. Limericke, C. N. H. Lock, H. Lowery, T. A. Lumsden, J. B. Maclean, K. B. Madhava, A. R. Richardson, and Miss N. I. Calderwood were elected members of the Society.

Messrs. C. W. Bartram and T. W. J. Powell were nominated for election.

Messrs. G. F. S. Hills and C. G. Darwin were admitted into the Society.

The Secretaries reported that 41 new members were elected during the Session 1919–20, 9 had died, and 2 resigned. The number of members is now 342.

Dr. Watson read a paper "The Product of Two Hypergeometric Functions."*

Lt.-Col. Cunningham and Prof. Hardy made informal communications.

* Printed in this volume.

The following papers were communicated by title from the Chair:—

*The Algebraic Theory of Algebraic Functions of One Variable: S. Beatty.

The Construction of Magic Squares: F. Debono.

Developable Surfaces through a Couple of Guiding Curves in Different Planes: A. R. Forsyth.

*The Distribution of Energy in the Neighbourhood of a Vibrating Sphere: J. E. Jones.

*(1) On the Reciprocity Formula for the Gauss's Sums in a Quadratic Field, †(2) A New Class of Definite Integrals: L. J. Mordell.

*Approximate Solutions of Linear Differential Equations: R. H. Fowler and C. N. H. Lock.

(1) Integration over the Area of a Surface and Transformation of the Variables in a Multiple Integral, (2) A New Set of Conditions for a Formula for an Area: W. H. Young.

ABSTRACTS.

The Product of Two Hypergeometric Functions

Dr. G. N. WATSON.

In this paper I investigate a relation which connects the product of two hypergeometric functions (which have the same constant elements) with the fourth type of Appell's hypergeometric function of two variables. In the case of terminating series the relation assumes the simple form

$$F(-n, \beta+n;\ \gamma;\ z) \times F(-n, \beta+n;\ \gamma;\ Z)$$

$$= (-)^n \frac{(\beta-\gamma+1)_n}{(\gamma)_n} F_4[-n, \beta+n;\ \gamma, \beta-\gamma+1;\ zZ, (1-z)(1-Z)],$$

where $$(\gamma)_n \equiv \gamma(\gamma+1)(\gamma+2) \dots (\gamma+n-1).$$

The Algebraic Theory of Algebraic Functions of One Variable

Mr. S. BEATTY.

The general aim kept in view in preparing the paper has been to attain the simplicity and flexibility of treatment implied in deriving pro-

* Printed in this volume. † (2) does not appear in this volume.

perties relative to a given basis from properties relative to certain appropriate bases, the study of which presents less difficulty. Use is made of order numbers of a certain type of adjointness relative to a given value of the variable. A lower limit is obtained for the number of linearly independent reduced forms of rational functions which are built on certain bases relative to a given value of the variable and contain none but negative powers of the element. Upper and lower limits are obtained for the number of linearly independent reduced forms of rational functions built on a basis—in the latter case a basis of a certain type. The proof of the complementary-theorem is effected by noting that, were it to fail in any given case, certain of the numbers obtained as lower limits would not be such. A well known formula of Dr. Fields is obtained for the number of conditions applicable to the reduced form of a rational function of a certain general type to build it on a given basis relative to a given value of the variable.

Approximate Solutions of Linear Differential Equations

Mr. R. H. FOWLER and Mr. C. N. H. LOCK.

This paper deals with the problem of the determination of the asymptotic expansions of solutions of a system of linear differential equations for large values of a parameter. The solutions are considered over a definite fixed range of values of the independent variable. In the case of *homogeneous* linear equations the asymptotic expansions of solutions have been obtained by Schlesinger (*Math. Ann.*, Vol. 63, p. 277) and Birkhoff (*Trans. Amer. Math. Soc.*, Vol. 9, p. 219) for real values of the independent variable. Non-homogeneous linear equations have hardly been considered—in other words, the expansions of the complementary function are known, but those of the particular integral have not been obtained. The need for expansions of both types arises in connection with the authors' investigations of the motion of a spinning shell, in which problem the leading terms of such asymptotic expansions provide valuable approximate solutions of the equations of motion.

In this paper therefore asymptotic expansions of the particular integrals of a system of non-homogeneous linear differential equations are obtained for large values of a parameter, thus completing the theory for real values of the independent variable. At the same time we adhere to a simplified method of attack which enables us to extend the results for both complementary function and particular integral to a region of complex values of the independent variable, and to analyse the whole problem

of the determination of these asymptotic expansions into its essential component parts.

Integration over the Area of a Curve and Transformation of the Variables in a Multiple Integral

Prof. W. H. YOUNG.

The present paper forms a pendant to the previous one on "A Formula for an Area," and contains the elaboration of the theory of integration over the area of a curve, and the transformation of the variables in such an integral, already foreshadowed in that paper. First the integral of a continuous function is defined completely, beginning with a polygon as area of integration, and proceeding thence to a curve, by means of a limiting process applied to polygons inscribed in the curve in its prescribed sense, the lengths of the sides tending simultaneously to zero. The polygons and curves employed will in general cut themselves and the latter may even do so any finite or infinite number of times. From a continuous function the author passes to any bounded function, using the method of monotone sequences and thence further, in the usual way, to unbounded functions, possessing integrals over our curve which may be called *absolutely convergent*, and we obtain the restrictions imposed on such functions by this integrability.

The simplicity of the theory in the case of bounded functions would seem to be due largely to the fact that a set of zero content in the usual sense possesses *zero content with respect to our curve.* Here content with respect to the curve is defined as the integral with respect to the curve of the function which is unity at the points of the set and zero elsewhere. Two functions which have the same integral in the usual sense have thus the same integral over the area of the curve.

The curves with which we are concerned include those termed by the author, viz. the coordinates $x = x(u)$ and $y = y(u)$, $(u_0 \leqslant u \leqslant U)$ are such that both $x(u)$ and $y(u)$ are continuous, and one at least, say $y(u)$, has bounded variation. The contour integral expression for our integral is then

$$\iint_C f(x, y)\, dx\, dy = \int_{u_0}^{U} F\{x(u), y(u)\}\, dy(u),$$

where

$$F(x, y) = \int f(x, y)\, dx.$$

In the case where C is a semi-rectifiable curve, which does not cut itself, the integral is shown to be the usual one.

The conditions obtained for the validity of the formula

$$\iint_C f(x, y)\, dx\, dy = \iint_R f\{x(u, v),\ y(u, v)\}\ \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv$$

for transformation of the variables in an integral over a curve C which is the image of a fundamental rectangle R, are the following:—

(1) *That the formula for an area* [*i.e. the above, with* $f(x, y) = 1$] *should hold, not only for the fundamental rectangle, but for every homothetic rectangle, that is one whose sides are parallel to those of the fundamental rectangle.*

(2) *When the fundamental rectangle is divided up into sub-rectangles* S, *by means of parallels to the axes of* u *and* v, *and these sub-rectangles are halved by means of their diagonals, sloping down from left to right, the triangles* Δ' *in the* (x, y)*-plane, whose vertices are the* 3*-point images of the semi-rectangles* Δ, *are such that* $\Sigma\ \Delta' \mid$ *is less than a fixed quantity, however the semi-rectangles be constructed.*

As the second of these conditions is fulfilled in point of fact by those obtained in the author's first paper on the subject entitled "On a Formula for an Area," it follows as a special case of the fundamental theorem proved in the present paper that a transformation of the variables in a multiple integral is always allowable, whenever the conditions for validity of the formula for an area given in that earlier paper are fulfilled. This result, though virtually contained in a footnote in the paper last mentioned, is now stated explicitly for the first time. The result may also be extended to the case of any number of variables.

Thursday, January 13*th*, 1921.

Mr. H. W. RICHMOND, President, in the Chair.

Present twenty members.

Messrs. C. W. Bartram and T. W. J. Powell were elected members of the Society.

Messrs. W. H. Glaser, R. F. Whitehead, and Prof. Olive C. Hazlett were nominated for election.

Messrs. S. L. Green and A. J. Thompson were admitted into the Society.

Prof. A. S. Eddington read a paper "On Dr. Sheppard's Method of Reduction of Error by Linear Compounding."*

Dr. W. F. Sheppard spoke on Prof. Eddington's paper, and also made a communication "Conjugate Sets of Quantities."*

Dr. Watson communicated a paper by Dr. M. Kössler "On the Zeros of Analytic Functions."*

The following papers were communicated by title from the Chair:—

*On a Problem concerning the Maxima of certain Types of Sums and Integrals: E. A. Milne and S. Pollard.

On the Linear Differential Equation of the Second Order: H. J. Priestley.

The Theory of a Thin Elastic Plate, Bounded by Two Circular Arcs, and Clamped: A. C. Dixon.

Determination of all the Characteristic Sub-Groups of an Abelian Group: G. A. Miller.

SPECIAL GENERAL MEETING.

The following Extraordinary Resolutions were carried unanimously:—

1. That Article No. 19 be altered by the substitution of the words "two guineas" for the words "one guinea," and by the addition at the end of the Article of the following provision:—"The subscription due from a newly elected member for his first year of membership shall be one guinea if his election takes place after February." And that these alterations shall take effect on and after 11th November, 1920.

2. That Article No. 20 be altered by the substitution of the words "two guineas" for the words "one guinea."

3. That Article No. 13 be altered by the omission of the words "in the case of candidates not residing in the United Kingdom" and of the words "provided that seven members shall be present thereat."

4. That Article No. 27 be altered by the addition at the end thereof of the words, "The accidental omission to give notice to any of the members, or the non-receipt by any of the members of any notice, shall not invalidate any resolution passed, or any proceedings which may take place at any General Meeting. When it is proposed to pass a Special Resolution, the two Meetings may be convened by the same notice."

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* Printed in this volume.

5. That Article No. 29 be cancelled, and that the following Article be substituted for it:—

"29. (1) Every question submitted to a General Meeting (except the election of Council and Officers and candidates for membership) shall be decided in the first instance by a show of hands.

"(2) Any Resolution proposed as an Extraordinary Resolution or a Special Resolution shall require to be carried in accordance with the provisions of the Companies' (Consolidation) Act, 1908, Sec. 69, or any Statutory modification thereof for the time being in force.

"(3) Any Resolution to alter the By-laws shall require a majority of two-thirds of the votes given."

6. That Article No. 32 be altered by the substitution of the words "three members or by the Chairman" for the words "fifteen members," and of the word "conclusive" for the word "sufficient."

7. That Article No. 35 be altered by the addition of the words "or Special" after the word "Annual," and that Article No. 36 be cancelled, and that Article No. 37 be renumbered No. 36.

8. That Article No. 38 be renumbered No. 37 and that the following new Article be adopted:—

"38. Votes may be given either personally or by proxy. A proxy shall be a member, and shall be appointed in writing signed by the member appointing the proxy. And the document appointing a proxy shall be delivered to one of the Secretaries, or deposited at the registered office, not less than 24 hours before the time for holding the Meeting or Adjourned Meeting as the case may be, at which the proxy proposes to vote. Every document appointing a proxy shall be in the form or to the effect following:—

"I, being a member of the London Mathematical Society, hereby appoint a member of the Society, or failing him, another member of the Society, to be my proxy to vote for me and on my behalf at the (Annual or Special or Ordinary) General Meeting of the Society to be held on the day of and at any adjournment thereof As Witness my hand this day of ."

The following Resolutions were also carried unanimously:—

9. That By-law II, Clauses 1, 2, and 3, be cancelled, and that the following By-law be substituted for it:—

"II. Of the Life Composition.

"(1) Any member may compound for future Annual Subscriptions by the payment of 25 guineas.

"(2) The Life Composition Fee shall be reduced in the case of members who shall have already paid Annual Subscriptions as follows:—

"10 Annual Subscriptions	...	21 guineas;
"20 do.	...	17 guineas;
"30 do.	...	12 guineas.

"(3) All Life Compositions may be paid in two equal annual instalments."

10. That By-law IX (4) be altered by the substitution of the words "the volume of the *Proceedings* current at the date of his election and of each Part of the *Proceedings* subsequently published while he remains a member," for the words "the *Proceedings* which shall be published after the date of his election."

It was agreed that Resolutions 1–8 be submitted for confirmation to a Special General Meeting to be held on Thursday, February 10th, 1921.

ABSTRACTS.

On the Zeros of Analytic Functions

Dr. Miloš Kössler.

I start with the equation

$$(1)\qquad \phi(x)-uf(x) = 0,$$

where $\phi(x)$ and $f(x)$ are analytic functions.

If $a_1, a_2, a_3, \ldots$, the roots of $\phi(x) = 0$ are supposed known, I form the power series

$$(2)\qquad x_k = \sum_{n=0}^{\infty} a_n^{(k)} u^n,$$

where

$$(3)\qquad a_0^{(k)} = a_k,\quad a_m^{(k)} = \frac{1}{m!}\frac{d^{m-1}}{dx^{m-1}}\left[\left(\frac{x-a_k}{\phi(x)}\right)^m f^m(x)\right]_{x=a_k}\qquad (k = 1, 2, 3, \ldots).$$

These power series, which represent the roots of (1), are convergent inside

a definite circle $|u| = R$. I transform them into the polynomial developments of Mittag-Leffler,

$$x_k = \sum_{m=0}^{\infty} P_m^{(k)}(u), \tag{4}$$

which are convergent in the whole star, and it is now possible to calculate the roots of (1) for every value of u.

In the case of multiple roots of $\phi(x) = 0$, it is necessary to make a slight modification of the series (2).

This method is very general and powerful; the three following results are obtained as special cases:—

(I) The roots of the general algebraic equation

$$x^n - (a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n) = 0,$$

are expressible in the form

$$x_k = \sum_{m=1}^{\infty} \frac{e^{2km\pi i/n}}{m!} \frac{d^{m-1}}{dx^{m-1}} \left[(a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n)^{m/n}\right]_{x=0}$$
$$(k = 0, 1, 2, \dots, n-1),$$

if the coefficients $a_1, a_2, \dots, a_n$ satisfy certain definite conditions; and the roots are expressible in the form

$$x_k = \sum_{m=1}^{\infty} P_m^{(k)}(e^{2k\pi i/n}),$$

when the coefficients have arbitrary values.

(II) *All* the zeros of such functions as

$$R(x, e^x), \quad R(x, \sin x), \quad R(x, e^{h(x)}), \quad R[\wp(x), e^x],$$

where $R(u, v)$ denotes a rational function of u and v, $h(x)$ is a polynomial in x and $\wp(x)$ is the Weierstrassian elliptic function, can be developed in expansions of the type (4).

(III) All the zeros of a given integral function $F(x)$ can be developed in this manner by using the equation

$$\sin x - u[F(x) + \sin x] = 0,$$

and calculating the zeros when $u = 1$.

As an example consider the zeros of

$$F(x) \equiv \sin x - ie^x.$$

For small values of $|u|$ we solve the equation

$$\sin x - ue^x = 0,$$

by an ascending series

$$x_k = \sum_{m=0}^{\infty} a_m^{(k)} u^m \quad (k = 0, 1, 2, 3, \ldots),$$

where $\quad a_0^{(k)} = \pm k\pi, \quad a_m^k = \dfrac{1}{m!} \dfrac{d^{m-1}}{dx^{m-1}} \left[\left(\dfrac{x \mp k\pi}{\sin x} \right)^m e^{mx} \right]_{x=\pm k\pi}.$

The zeros of $F(x)$ are then given by Borel's formula

$$x_k = \int_0^{\infty} e^{-t} F_k(it)\, dt,$$

by putting $\quad F_k(u) = \displaystyle\sum_{m=0}^{\infty} \frac{a_m^{(k)} u^m}{m!}.$

On Dr. Sheppard's Method of Reduction of Error by Linear Compounding

Prof. A. S. EDDINGTON.

Dr. W. F. Sheppard's theory (*Phil. Trans.*, Vol. 221, A, pp. 199–237) is here treated according to the methods and notation of the tensor calculus. In this way great compactness is attained, and the symmetry of the formulæ becomes apparent. A geometrical interpretation is given of the significance of the processes employed. This method of treating the problem is likely to appeal chiefly to those who already have some familiarity with the theory of tensors; but since it provides an illustration of the elementary notions of tensors, it may also be of use as a first introduction to that subject.

On the Linear Differential Equation of the Second Order

Prof. H. J. PRIESTLEY.

The following results, arrived at in a paper to be communicated to the forthcoming meeting of the Australasian Association for the Advancement of Science, may be of interest to the members of the London Mathematical Society.

1. If the equation

$$\frac{d^2y}{dx^2}+(x-c)^{-1}P(x)\frac{dy}{dx}+(x-c)^{-2}Q(x)y=0, \tag{1}$$

where $P(x)$ and $Q(x)$ are regular in the neighbourhood of $x=c$, be transformed by the substitutions

$$\operatorname{Exp}\left[\int(x-c)^{-1}P(x)\,dx\right]=\phi(x),$$

$$\int[\phi(x)]^{-1}\,dx=z,$$

it becomes
$$\frac{d^2y}{dz^2}=-[(x-c)^{-1}\phi(x)]^2\,Q(x)\,y. \tag{2}$$

The solutions of this equation can be expressed as solutions of a Volterra integral equation. A discussion of this equation shows that solutions of (2) which are regular at $x=c$ can be obtained under the following conditions:—

(*a*) $P(c)\geqslant 1,\quad Q(c)<0;$

(*b*) $P(c)\geqslant 1,\quad Q(c)=0;$

(*c*) $P(c)<1,\quad Q(c)<0;$

(*d*) $P(c)<1,\quad Q(c)=0;$

(*e*) $P(c)<1,\quad 0<Q(c)\leqslant\frac{1}{4}[1-P(c)]^2.$

The behaviour of y and $\phi(x)\dfrac{dy}{dx}$ at $x=c$, in these five cases, is given below

(*a*) $y=0,\quad \phi(x)\dfrac{dy}{dx}=0;$

(*b*) $y=1,\quad \phi(x)\dfrac{dy}{dx}=0;$

(*c*) $y=0,\quad \phi(x)\dfrac{dy}{dx}=0;$

(*d*) $y=0,\quad \phi(x)\dfrac{dy}{dx}=1;$

(*e*) $y=0,\quad \phi(x)\dfrac{dy}{dx}\to\infty.$

2. The equations

$$\frac{d}{dx}\left[\phi(x)\frac{dy_n}{dx}\right]+\psi(x)\,y_n=\frac{An^2+Bn+C}{\alpha n^2+\beta n+\gamma}\,y_n, \tag{3}$$

and

$$\frac{d}{dx}\left[\phi(x)\frac{dy}{dx}\right]+\psi(x)\,y\ =0,$$

are of the above type if $\phi(x)$ contains the factor $(x-c)$. In that case $Q(c)=0$ for both equations, and therefore solutions of both exist satisfying conditions (b) or (d) at $x=c$. These solutions will be referred to as solutions of type A.

By Hilbert's well known method, a solution of (3) of type (A) which also satisfies the condition

$$py_n+q\,\frac{dy_n}{dx}=0 \quad \text{at} \quad x=a, \tag{B}$$

can be expressed as the solution of

$$y_n(x)=\frac{An^2+Bn+C}{\alpha n^2+\beta n+\gamma}\int_a^c K(x,t)\,y_n(t)\,dt,$$

where $K(x, t)$ is symmetrical.

It follows, as in my paper in *Proc. London Math. Soc.*, Vol. 18, pp. 266, 267, that, when A, B, C, α, β, γ are real, the appropriate values of n are real and separate.

It also follows from Hilbert's work* that a function which, with its first and second derivatives, is continuous in the range $a<x<c$, which is of type A at $x=c$ and satisfies condition (B), can be expanded in a series of $y_n(x)$; the coefficients being calculated in Fourier's manner.

The Singularities of the Algebraic Trochoids.

Prof. D. M. Y. SOMMERVILLE.

I am indebted to Prof. H. Hilton for referring me to an article by Elling Holst: "Ueber algebraische cykloidische Kurven," *Arch. Math. Naturvid., Kristiania*, Vol. 6 (1881), pp. 125–152, which anticipates my paper with the above title, *Proc. London Math. Soc.*, Ser. 2, Vol. 18 (1919), pp. 385–392. In this article, using rather different methods and

* Hilbert, "Grundüzge einer allgemeinen Theorie der linearen Integralgleichungen," Chap. VII.

with a different notation, he arrives at the same results which I found regarding the numbers of the various singularities, both in the finite region and at infinity. It is of interest to note that he determines the singularities at infinity separately, and then finds the number of finite singularities by subtraction from the total Plückerian numbers, while I adopted the reverse order. He bases his results on the known facts that the curve $y^q = \mu x^p$ $(p > q)$ has a singularity at the origin consisting of $\frac{1}{2}(p-3)(q-1)$ double points, $\frac{1}{2}(p-3)(p-q-1)$ double tangents, $q-1$ cusps and $p-q-1$ inflexions.

Thursday, February 10*th*, 1921.

Mr. H. W. RICHMOND, President, and later Mr. J. E. CAMPBELL, Vice-President, in the Chair.

Present thirty-seven members and twelve visitors.

Messrs. W. H. Glaser and R. F. Whitehead, and Prof. Olive C. Hazlett, were elected members of the Society.

Dr. H. Levy was nominated for membership.

Prof. H. S. Carslaw was admitted into the Society.

Prof. A. S. Eddington delivered a lecture "World Geometry (with particular reference to Weyl's electromagnetic theory)."

The following papers were communicated by title from the chair:—

*Note on the Electromagnetic Equations: J. Brill.

Researches in the Theory of the Riemann Zeta-Function: J. E. Littlewood.

A New Condition for Cauchy's Theorem: S. Pollard.

*(1) On the Torsion of a Prism, one of the Cross Sections of which remains Plane; *(2) The Analogy with Membranes in the case of the Bending of a Prism: S. Timoschenko.

SPECIAL GENERAL MEETING.

The Extraordinary Resolutions carried at the Special General Meeting of January 13th, 1921 (see *Records of Proceedings at Meetings* for that date), were submitted for confirmation and confirmed unanimously.

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* Printed in this volume.

ABSTRACT.

Researches in the Theory of the Riemann ζ-Function

Mr. J. E. LITTLEWOOD.

It would occupy too much space to give any detailed description of the methods used in these researches, or any full account of previous work in the same subjects, and I have confined myself in the main to a bare statement of results.

1. *Theorems on mean values.*

We have

(1.1)

$$\int_T^{T+H} |\zeta(\sigma+it)|^2\,dt = L_\sigma(T+H)-L_\sigma(T)+O(T^{1-\sigma+\epsilon})+O(T^\epsilon)+O(HT^{-\frac{1}{2}\sigma+\epsilon})$$

uniformly in $\quad 0 \leqslant H \leqslant T, \quad \frac{1}{2} \leqslant \sigma \leqslant 2,$

where $\quad L_\sigma(t) = \zeta(2\sigma)\,t+(2\pi)^{2\sigma-1}\zeta(2-2\sigma)\dfrac{t^{2-2\sigma}-1}{2-2\sigma},$

and limiting values are to be taken when $\sigma = \frac{1}{2}$ *or* $\sigma = 1$.

In particular we have, uniformly for $0 \leqslant H \leqslant T$,

(1.11) $$\int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|^2\,dt = 2\pi[P(T+H)-P(T)]+O(T^{\frac{1}{2}+\epsilon})+O(HT^{-\frac{1}{2}+\epsilon}),$$

where $\quad 2\pi P(t) = t\log t-(1+\log 2\pi)\,t.$

An easy deduction from the special case $H = T$ is

(1.12) $$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt = 2\pi P(T)+O(T^{\frac{1}{2}+\epsilon}).$$

To the same order of ideas belongs the following theorem, which is important in certain applications:—

Given any positive δ, *there is a* $K = K(\delta)$ *and a* $T_0 = T_0(\delta)$, *such that*

(1.2) $$\begin{cases} |\zeta(\sigma+it)| < K(\log T)^{\frac{1}{2}} & (\sigma \geqslant \frac{1}{2}), \\ |\zeta'(\sigma+it)| < K(\log T)^{\frac{3}{2}} & (\sigma \geqslant \frac{1}{2}), \\ \displaystyle\int_{\frac{1}{2}}^{\infty} |\zeta(\sigma+it)|\,d\sigma < K, & \end{cases}$$

for $T > T_0$, *and some* t *satisfying* $T \leqslant t \leqslant T+T^{\frac{1}{2}+\delta}$.

2. *Results concerning $S(T)$, $N(\sigma, T)$, independent of the Riemann hypothesis.*

We suppose $T > 0$, and, for simplicity, that $t = T$ contains no zero of $\zeta(s)$. Let $N(T)$ denote, as usual, the number of zeros of $\zeta(s)$ whose imaginary parts lie between O and T. Let $N(\sigma, T)$ denote the number of these for which, in addition, the real parts are greater than σ. The Riemann hypothesis is equivalent to $N(\frac{1}{2}, T) = 0$. It is known that*

$$N(T) = P(T)+c+S(T),$$

where c is a constant,

$$S(T) = \frac{1}{\pi} If(\tfrac{1}{2}+iT),$$

$f(s)$ is the value of $\log\zeta(\sigma+it)$ obtained by continuous variation from $\log\zeta(2+it)$ as σ varies from 2 to σ, and $\log\zeta(2+it)$ is the branch defined by the ordinary Dirichlet's series.

I prove that

$$(2.11) \qquad \mathfrak{R}\int_0^T f(\sigma+it)\,dt = 2\pi\int_\sigma^1 N(\sigma, T)\,d\sigma - I\int_\sigma^\infty f(\sigma+iT)\,d\sigma,$$

$$(2.12) \qquad I\int_0^T f(\sigma+it)\,dt = \mathfrak{R}\int_\sigma^\infty f(\sigma+iT)\,d\sigma + c(\sigma),$$

where $c(\sigma)$ is independent of T, results which have analogues for more general functions $f(s) = \log\phi(s)$.

Taking $\sigma = \frac{1}{2}$ in (2.12), we have

$$(2.2) \qquad \int_0^T S(t)\,dt = \int_{\frac{1}{2}}^\infty \log|\zeta(\sigma+iT)|\,d\sigma + c_1.$$

Let us write

$$(2.21) \qquad \int_0^T S(t)\,dt = S_1(T)+c_1.$$

Starting from (2.2) I prove

$$(2.3) \qquad S_1(T) = O(\log T).\dagger$$

* See Backlund, *Acta Mathematica*, Bd. 41 (1918).

† H. Cramér, *Mathematische Zeitschrift*, Bd. 4, pp. 122–130, proves, by an entirely different method, that

$$S_1(T) = O(T^\epsilon).$$

Equation (2.11) may be written

(2.31)

$$2\pi \int_{\sigma}^{1} N(\sigma, T)\, d\sigma = \int_{0}^{T} \log |\zeta(\sigma+it)|\, dt + I\int_{\sigma}^{2} f(\sigma+iT)\, d\sigma + I\int_{2}^{\infty} f(\sigma+iT)\, d\sigma.$$

It is known that $If(\sigma+iT) = O(\log T)$, $\sigma \geqslant \frac{1}{2}$.

The second integral on the right of (2.31) is $O(1)$; hence

(2.32) $$2\pi \int_{\sigma}^{1} N(\sigma, T)\, d\sigma = \int_{0}^{T} \log |\zeta(\sigma+it)|\, dt + O(\log T).$$

A remarkable theorem due to F. Carlson states that for *fixed* $\sigma > \frac{1}{2}$,

$$N(\sigma, T) = O(T^{1-4(\sigma-\frac{1}{2})^2+\epsilon}).$$

Equation (2.32) can be used to effect minor improvements in the proof of this, but does not lead to any appreciable refinement of the result. It does, however, lead to new results of some interest when σ is not fixed, and $\sigma-\frac{1}{2}$ is a small function of T. Thus (2.32) leads easily to

(2.33) $$\int_{\frac{1}{2}}^{1} N(\sigma, T)\, d\sigma = O(T \log\log T),$$

whence, if $\phi(t) \to \infty$, however slowly, as $t \to \infty$,

(2.34) $$N(\sigma, T) = o(T \log T), \quad \left(\sigma \geqslant \tfrac{1}{2} + \phi(T)\frac{\log\log T}{\log T}\right).$$

Thus all but an infinitesimal proportion of the complex zeros of $\zeta(s)$ lie in the region

$$|\sigma - \tfrac{1}{2}| < \phi(t)\frac{\log\log t}{\log t}.$$

3. Before proceeding to results which depend, in the main, on the Riemann hypothesis, I mention next one or two of a different character.

There is a $K = K(\delta)$ *and a* $T_0 = T_0(\delta)$ *such that, when* $T > T_0$, $\zeta(s)$ *has a zero in every rectangle*

$$\tfrac{1}{2}-\delta \leqslant \sigma \leqslant 1, \quad T - \frac{K}{\log\log\log T} \leqslant t \leqslant T + \frac{K}{\log\log\log T}.$$

4. In a paper written in collaboration with Prof. G. H. Hardy, which we hope will be published shortly, it is shown that $\zeta(\frac{1}{2}+it) = O(t^{\frac{1}{6}+\epsilon})$, that intermediate upper bounds exist for σ's between $\frac{1}{2}$ and 1, and that (with special reference to the neighbourhood of $\sigma = 1$) *there is a constant A such*

that

$$\zeta(\sigma+it) = O\left(\frac{\log t}{\log\log t}\exp\left[A\,(1-\sigma)\log t \Big/ \log\frac{1}{1-\sigma}\right]\right),$$

uniformly in $\frac{1}{2} \leqslant \sigma \leqslant 1$. Starting from the last of these results I prove:

There is a positive c *and a* t_0 *such that* $\zeta(s)$ *has no zeros in the region*

$$\sigma \geqslant 1 - \frac{c\log\log t}{\log t} \quad (t \geqslant t_0).$$

Further, if $c' < c$, *we have, in*

$$\sigma \geqslant 1 - \frac{c'\log\log t}{\log t},$$

and in particular for $\sigma = 1$,

$$\zeta(s) = O\left(\frac{\log t}{\log\log t}\right), \tag{4.1}$$

$$\frac{\zeta'(s)}{\zeta(s)} = O\left(\frac{\log t}{\log\log t}\right), \tag{4.2}$$

$$\frac{1}{\zeta(s)} = O\left(\frac{\log t}{\log\log t}\right). \tag{4.3}$$

5. *The functions* $S(T)$, $S_n(T)$ *on the Riemann hypothesis.*

If we assume the Riemann hypothesis, so that $N(\sigma, T) = 0$ for $\sigma \geqslant \frac{1}{2}$, and define $S_n(T)$ by the equations

$$\begin{cases} S_0(T) = S(T), \\ S_{2n}(T) = (-)^n\, I \displaystyle\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\cdots\int_{\sigma}^{\infty} f(\sigma+iT)(d\sigma)^{2n} \quad (n \geqslant 1), \\ S_{2n-1}(T) = (-)^{n-1}\, \mathfrak{R} \displaystyle\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\cdots\int_{\sigma}^{\infty} f(\sigma+iT)(d\sigma)^{2n-1} \quad (n \geqslant 1), \end{cases} \tag{5.1}$$

we obtain, by successive integrations of (2.11) and (2.12),

$$S_n(T) = \int_0^T S_{n-1}(t)\,dt + c_n. \tag{5.2}$$

Thus each S is substantially the integral of the preceding one. I prove further

$$S(T) = O\left(\frac{\log T}{\log\log T}\right), \tag{5.3}$$

$$S_n(T) = O\left(\frac{\log T}{(\log\log T)^{n+1}}\right), \tag{5.4}$$

(5.5) $$|\zeta(\tfrac{1}{2}+iT)| < \exp\left(\frac{A\log T}{\log\log T}\right).$$

The proofs are difficult, and there seems reason to suppose that any improvement of the result for $S(T)$, if indeed possible, must depend on exceedingly deep considerations.

It follows from the results of the next section that

(5.6) $$|\zeta(\tfrac{1}{2}+iT)| > \exp\{(\log T)^{\frac{1}{2}-\epsilon}\}$$

for arbitrarily large values of T, and that, for fixed σ satisfying $\frac{1}{2}<\sigma<1$,

(5.7) $$|\zeta(\sigma+iT)| > \exp\{(\log T)^{1-\sigma-\epsilon}\}.$$

The relations (5.5) and (5.6) express the present extent of our knowledge of the order of $\zeta(s)$ on the line $\sigma=\frac{1}{2}$, the Riemann hypothesis being assumed. It may be observed that it is by no means impossible for *both* (5.5) and (5.7) to be "best possible" results.

6. *Further results concerning S and S_n.*

It is known that a positive a exists such that, for every positive ϵ,

$$S(T) \neq O[(\log T)^{a-\epsilon}].$$

Let a be the greatest such a, and let a_n be the corresponding index for S_n. Further, for σ fixed and greater than $\frac{1}{2}$, let $\tau(\sigma)$ be the least index τ such that, for every positive ϵ,

$$\frac{\zeta'(s)}{\zeta(s)} \neq O[(\log t)^{\tau-\epsilon}],$$

and let $$a' = \lim_{\sigma\to\frac{1}{2}+0} \tau(\sigma)/(1-\sigma).$$

The following theorem is fundamental in the proof of much that remains to be stated.

THEOREM A.—*If δ, δ' are any positive constants,*

(6.1)
$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{x} \Lambda(n)n^{-s} + O[(\log t)^{-\delta}x^{1-\sigma}\log x] + O[x^{\frac{1}{2}-\sigma}\log x\,(\log t)^{a+2\delta}]$$

and

(6.2)
$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{x} \Lambda(n)n^{-s} + O[(\log t)^{-\delta}x^{1-\sigma}(\log x)^{n+1}] + O[x^{\frac{1}{2}-\sigma}(\log x)^{n+1}(\log t)^{a_n+2\delta}]$$

uniformly for $2\leqslant x\leqslant t$, $\sigma\geqslant\frac{1}{2}+\delta'$.

I prove the following relations between the a's,

$$1 \geqslant a \geqslant a_n \geqslant a_{n+1} \geqslant a' \geqslant \tfrac{1}{2} \tag{6.3}$$

($1 \geqslant a \geqslant a' > 0$ is known already). The most interesting of these results is $a' \geqslant \frac{1}{2}$: it is a particular case of

$$\tau(\sigma) \geqslant \tfrac{1}{2}(1-\sigma) \quad (\tfrac{1}{2} < \sigma \leqslant 1). \tag{6.4}$$

It is further true that the numbers $1, a, a_1, \ldots, a_n \ldots$ have the property of "convexity."

Again, starting from Theorem A, I prove

$$\frac{1}{T}\int_0^T |S(t)|\,dt = O(\log\log T), \tag{6.5}$$

$$\frac{1}{T}\int_0^T |S_n(t)|^2\,dt = O(1) \quad (n \geqslant 1). \tag{6.6}$$

More generally, δ being any positive constant less than 1,

$$\frac{1}{H}\int_T^{T+H} |S(t)|\,dt = O(\log\log T), \tag{6.51}$$

$$\frac{1}{H}\int_T^{T+H} |S_n(t)|^2\,dt = O(1), \tag{6.61}$$

uniformly for $T^\delta \leqslant H \leqslant T$. Thus, while the "order" a of $S(T)$ as a function of $\log T$ is at least $\frac{1}{2}$, its average order is zero.

7. *Upper and lower bounds for $\zeta(s)$, etc., on the line $\sigma = 1$.*

In this subject I have obtained results of considerable precision. *It is true, without any hypothesis, that*

$$\overline{\lim_{t\to\infty}} \frac{|\zeta(1+it)|}{\log\log t} \geqslant e^\gamma, \tag{7.1}$$

where γ is Euler's constant. On the other hand, we have, on the Riemann hypothesis,

$$\overline{\lim_{t\to\infty}} \frac{|\zeta(1+it)|}{\log\log t} \leqslant 2a'e^\gamma \leqslant 2e^\gamma. \tag{7.2}$$

This last result remains true if we replace $\zeta(1+it)$ by $1/\zeta(1+it)$. It appears from (7.1) and (7.2) that we obtain the exact value of the left-hand side if it is true that $a' = \frac{1}{2}$. Similar results hold for $\dfrac{\zeta'(s)}{\zeta(s)}$.

There are interesting analogues concerning the number $h(k)$ of classes

of ideals of the corpus $P(\sqrt{-k})$, where $-k$ is a negative fundamental discriminant. It is well known that

$$h(k) = \frac{\sqrt{k}}{\pi} L(1),$$

where $$L(s) = \Sigma\chi(n)\, n^{-s} \quad \text{and} \quad \chi(n) = \left(\frac{-k}{n}\right).$$

Here $\left(\frac{-k}{n}\right)$ is the Kronecker symbol of quadratic reciprocity: it is a real primitive character mod k. I prove that, *assuming the hypothesis that all the $L(s)$ have no zeros in $\sigma > \frac{1}{2}$, we have, on the one hand,*

$$\overline{\lim_{k\to\infty}} \frac{L(1)}{\log\log k} \geqslant \tfrac{1}{2}e^{\gamma}. \tag{7.3}$$

and on the other hand

$$\overline{\lim_{k\to\infty}} \frac{|L(1)|}{\log\log k} \leqslant 2e^{\gamma}. \tag{7.4}$$

There is a factor $\frac{1}{2}$ on the right-hand side of (7.3) which is absent from (7.1). There exist *some* moduli k', and corresponding real primitive characters χ, such that

$$L(1, \chi) > (1-\epsilon)\, e^{\gamma} \log\log k',$$

but I have not succeeded in proving this inequality for the special set of characters in which we are interested.

Another analogue is: *There is an $A = A(\delta)$ such that, for all sufficiently large k, $L(s, \chi)$ has a zero in $\sigma \geqslant \frac{1}{2}-\delta$, $|t| \leqslant \dfrac{A}{\log\log\log k}$.*

8. I conclude by mentioning a result in a different field. *Assuming the Riemann hypothesis, we have,* in the usual notation of the prime number theory,

$$\psi(x) - x = \sum_{|\rho| \leqslant X} \frac{x^{\rho}}{\rho} + O(x^{\frac{1}{2}} \log x) \tag{8.1}$$

uniformly for $$X \geqslant x^{\frac{1}{2}}.$$

Thursday, March 10th, 1921.

Mr. H. W. RICHMOND, President, in the Chair.

Present ten members.

Dr. H. Levy was elected a member of the Society.

Messrs. P. J. Daniell, H. G. Forder, A. H. Pope, and Miss C. W. M. Sherriff were nominated for election.

The President announced the death of Lord Moulton.

Mr. J. Brill read a paper "Note on the Electrodynamic Equations." *

Mr. J. E. Littlewood communicated two papers by himself and Prof. Hardy: (1) "The Approximate Functional Equation in the Theory of Riemann's Zeta-Function," (2) "Summation of a certain Multiple Series."

The following papers were communicated by title from the chair:—

A Method for the Solution of certain Linear Partial Differential Equations: T. W. Chaundy.

*An Extension of Two Theorems on Jacobians: C. W. Gilham.

*On certain Classes of Mathieu Functions: E. G. C. Poole.

ABSTRACTS.

The Approximate Functional Equation in the Theory of Riemann's Zeta-Function, with Applications to the Divisor-Problems of Dirichlet and Piltz

Prof. G. H. HARDY *and* Mr. J. E. LITTLEWOOD.

The approximate functional equation may be stated as follows. *Suppose that*

$$s = \sigma + it, \quad -H \leqslant \sigma \leqslant H, \quad x > K, \quad y > K, \quad 2\pi xy = |t|,$$

where H and K are positive constants. Then

$$\zeta(s) = \sum_{n<x} n^{-s} + 2(2\pi)^{s-1} \sin \tfrac{1}{2}s\pi \Gamma(1-s) \sum_{n<y} n^{s-1} + O(x^{-\sigma}) + O(|t|^{\frac{1}{2}-\sigma} y^{\sigma-1}),$$

uniformly in σ, x, and y.

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* Printed in this volume.

By means of this theorem it is shown that

$$\int_{-T}^{T} |\zeta(\tfrac{1}{2}+it)|^4 \, dt = O\{T(\log T)^4\},$$

and that

$$\Delta_k(x) = O(x^{(k-2)/k+\epsilon}),$$

for $k \geqslant 4$ and for every positive ϵ, $\Delta_k(x)$ being the "error term" in Piltz's generalisation of Dirichlet's divisor problem.

Summation of a certain Multiple Series

Prof. G. H. HARDY *and* Mr. J. E. LITTLEWOOD.

The series in question is

$$S_m = \sum_{p_1, q_1;\, p_2, q_2;\, \dots;\, p_m, q_m} \chi(q_1)\chi(q_2)\dots\chi(q_m)\chi(Q)\, e\left(\frac{a_1 p_1}{q_1}+\frac{a_2 p_2}{q_2}+\dots+\frac{a_m p_m}{q_m}\right).$$

Here q_r runs through all positive integral values, and p_r through all such values less than and prime to q_r, and Q is the denominator of

$$\frac{p_1}{q_1}+\frac{p_2}{q_2}+\dots+\frac{p_m}{q_m} = \frac{P}{Q},$$

expressed in its lowest terms. The arithmetical function $\chi(q)$ is defined by

$$\chi(q) = \frac{\mu(q)}{\phi(q)},$$

where $\mu(q)$ and $\phi(q)$ are the well known functions of Möbius and Euler. Finally, the a's are unequal positive integers, and

$$e(x) = e^{2\pi i x}.$$

The sum of the series is

$$S_m = \prod_{\varpi} \left\{ \left(\frac{\varpi}{\varpi-1}\right)^m \left(\frac{\varpi-\nu}{\varpi-1}\right) \right\},$$

where ϖ assumes all prime values, and ν is the number of distinct residues of the group of numbers $0, a_1, a_2, \dots, a_m$ to modulus ϖ. It is plain that $\nu = m+1$ from a certain point onwards.

The series is of very great interest, for it is the series on which the asymptotic distribution of groups of primes

$$p,\ p+a_1,\ p+a_2,\ \dots,\ p+a_m$$

appears to depend. The details of the summation, and some indication of the concordance of the results suggested with the evidence of computation, are included in a memoir to appear in the *Acta Mathematica*.

Thursday, April 21st, 1921.

Mr. H. W. RICHMOND, President, in the Chair.

Present ten members.

Messrs. P. J. Daniell, H. G. Forder, A. H. Pope, and Miss C. W. M. Sherriff were elected members of the Society.

Dr. J. F. Tinto and Dr. N. Wiener were nominated for election.

Prof. Hardy communicated a paper by Mr. L. J. Mordell, "Note on papers by Mr. Darling and Prof. Rogers."*

Prof. Hilton and Major MacMahon made informal communications.

The following papers were communicated by title from the chair :—

*(1) Cyclotomic Quinquesection, †(2) On a Generalisation of a Theorem of Booth: Pandit Oudh Upadhyaya.

Properties of Eulerian and Prepared Bernoullian Numbers: C. Krishnamachary and M. Bhimasena Rao.

ABSTRACT.

Note on Papers by Mr. Darling and Prof. Rogers

Mr. L. J. MORDELL.

These papers are concerned with certain theorems enunciated by Ramanujan, some of which may be stated as follows. Let

$$G = \frac{1}{(1-r)(1-r^4)\dots}, \quad H = \frac{1}{(1-r^2)(1-r^3)\dots},$$

where the factor $1-r^n$ occurs in G if $n \equiv 1, 4 \pmod 5$ and in H if $n \equiv 2, 3 \pmod 5$, and let

$$f = f(r) = r^{\frac{1}{5}} H/G, \quad f_1 = f(r^2).$$

Then (1) $$f^2 - f_1 + f f_1^2 (f^2 + f_1) = 0,$$

(2) $$f^{-5} - f^5 - 11 = \frac{1}{r}\left\{\frac{(1-r)(1-r^2)(1-r^3)\dots}{(1-r^5)(1-r^{10})(1-r^{15})\dots}\right\}^6,$$

* Printed in this volume.

† (2) does not appear in this volume.

or
$$HG^{11}-r^2GH^{11} = 1+11rG^6H^6,$$

$$(3)\quad f^{-1}-f-1 = \frac{1}{r^{\frac{1}{5}}}\,\frac{(1-r^{\frac{1}{5}})(1-r^{\frac{2}{5}})(1-r^{\frac{3}{5}})\ldots}{(1-r^5)(1-r^{10})(1-r^{15})\ldots},$$

$$(4)\quad \sum_0^\infty p(5n+4)\,r^n = 5\,\frac{\{(1-r^5)(1-r^{10})(1-r^{15})\ldots\}^5}{\{(1-r)(1-r^2)(1-r^3)\ldots\}^6},$$

and so forth. In this paper all of these formulæ are deduced in a comparatively simple manner from the general theory of the elliptic modular functions.

Thursday, May 12*th*, 1921.

Mr. H. W. RICHMOND, President, in the Chair.

Present ten members.

Dr. J. F. Tinto and Dr. N. Wiener were elected members of the Society.

Miss F. M. Wood was nominated for election.

Lt.-Col. Cunningham read a paper on "Multifactor Quadrinomials."

Prof. Hardy communicated a paper, written in collaboration with Mr. Littlewood, "Some Problems of Diophantine Approximation; The Lattice-Points of a Right-Angled Triangle" (second paper).*

A paper by Mr. H. W. Turnbull, "Invariants of Three Quadrics,"* was communicated by title from the chair.

ABSTRACTS.

Invariants of Three Quadrics

Mr. H. W. TURNBULL.

The accompanying paper is an attempt to find the irreducible concomitants of three quadrics. In the *Math. Annalen*, Vol. 56, Gordan discussed the system of two quadrics, which I recently showed† to be

* Printed in this volume.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 18 (1920), pp. 69–94.

capable of reduction to 125 forms. Little seems to be known of the invariants of three quadrics. In the new edition of Salmon's *Analytical Geometry of Three Dimensions* (§ 235), the editor, Rogers, discusses three important invariants by starting from geometrical considerations.

The following pages employ the symbolic method and, starting from the fundamental bracket factors $(abcd)$, $(abcu)$, (abp), a_x, proceed to an expression of the symbols in the *prepared* form, analogous to the form used by Gordan for ternary or quaternary quadratics. This *prepared system* of factors (§ 14) illustrates very clearly the importance of reciprocation, and the central place that line coordinates, rather than point or plane coordinates, hold. In § 23 a list of 44 irreducible invariants is given, a list which may be capable of further reduction, although, as in other cases where the symbolic method is used, it necessarily includes all possible reducible invariants. The highest degree which occurs is 6: thus any invariant of degree greater than 6 in the coefficients of either of the three quadrics must be reducible.

Multifactor Quadrinomials

Lt.-Col. ALLAN CUNNINGHAM, R.E.

1. *Introduction.*—The object of this paper is to present a number of quadrinomials (N) which have a large number of (algebraic) factors.

2. THEOREM I.—Let

$$N_1 = (ab\xi^4)^{\alpha\beta} - (a\xi\eta)^{2\alpha\beta} - (b\xi\eta)^{2\alpha\beta} + (ab\eta^4)^{\alpha\beta},$$

$$N_2 = \quad ,, \quad - \quad ,, \quad + \quad ,, \quad - \quad ,, \quad ,$$

$$N_3 = \quad ,, \quad + \quad ,, \quad - \quad ,, \quad - \quad ,, \quad ,$$

$$N_4 = \quad ,, \quad + \quad ,, \quad + \quad ,, \quad + \quad ,, \quad ,$$

where the two members of the pairs (a, b), (α, β), (ξ, η) have no common factor, and α, β are *odd*.

Then, if (a, b) have the values $(\alpha, 1)$, $(1, \beta)$, (α, β), the four functions N_1, N_2, N_3, N_4 have the numbers of (algebraic) factors shown in the table below, depending on the form of $\alpha, \beta = 4i \pm 1$.

α	β	$\alpha\beta$	a	b	Factors in N_1	N_2	N_3	N_4	α	β	$\alpha\beta$	a	b	Factors in N_1	N_2	N_3	N_4
$4i+1$	$4j+1$	$4i+1$	α,	1	12,	10,	10,	8	$4i+1$	$4j-1$	$4i-1$	α,	1	12,	10,	10,	8
			1,	β	12,	10,	10,	8				1,	β	8,	10,	10,	12
			α,	β	10,	9,	9,	8				α,	β	8,	9,	9,	10
$4i-1$	$4j-1$	$4i+1$	α,	1	8,	10,	10,	12	$4i-1$	$4j+1$	$4i-1$	α,	1	8,	10,	10,	12
			1,	β	8,	10,	10,	12				1,	β	12,	10,	10,	8
			α,	β	10,	9,	9,	8				α,	β	8,	9,	9,	10

Demonstration.—Write

$$x = a\xi^2, \quad y = b\eta^2; \quad u = b\xi^2, \quad v = a\eta^2.$$

$$X = (x^{\alpha\beta} - y^{\alpha\beta}), \quad X' = (x^{\alpha\beta} + y^{\alpha\beta}); \qquad U = u^{\alpha\beta} - v^{\alpha\beta}, \quad U' = u^{\alpha\beta} + v^{\alpha\beta},$$

whence $\quad N_1 = XU, \quad N_2 = X'U, \quad N_3 = XU', \quad N_4 = X'U'.$

Since α, β are both *odd*, and have no common factor, therefore each of X, X', U, U' is a product of *four* (algebraic) factors, so that each of N_1, N_2, N_3, N_4 is always a product of *eight* (algebraic) factors (the normal number).

Write $\quad Z_1 = x - y, \quad Z_\alpha = (x^\alpha - y^\alpha)/Z_1, \quad Z_\beta = (x^\beta - y^\beta)/Z_1,$

$$Z'_1 = x + y, \quad Z'_\alpha = (x^\alpha + y^\alpha)/Z'_1, \quad Z'_\beta = (x^\beta + y^\beta)/Z'_1,$$

$$Z_{\alpha\beta} = XZ_1/(x^\alpha - y^\alpha)(x^\beta - y^\beta), \quad Z'_{\alpha\beta} = X'Z'_1/(x^\alpha + y^\alpha)(x^\beta + y^\beta),$$

and take W_1, W_α, W_β, $W_{\alpha\beta}$; W'_1, W'_α, W'_β, $W'_{\alpha\beta}$ the *same functions* of u, v that Z_1, Z_α, Z_β, $Z_{\alpha\beta}$; Z'_1, Z'_α, Z'_β, $Z'_{\alpha\beta}$ are of x, y. Then

$$X = Z_1 Z_\alpha Z_\beta Z_{\alpha\beta}, \qquad X' = Z'_1 Z'_\alpha Z'_\beta Z'_{\alpha\beta};$$

$$U = W_1 W_\alpha W_\beta W_{\alpha\beta}, \qquad U' = W'_1 W'_\alpha W'_\beta W'_{\alpha\beta}.$$

Now use the symbols A_ρ, A'_ρ to denote the *Aurifeuillian* functions of order ρ, *i.e.*

$$A_\rho = (h^{2\rho} - \rho^\rho k^{2\rho})/(h^2 - k\rho^2) \quad [\text{when } \rho = 4i+1],$$

$$A'_\rho = (h^{2\rho} + \rho^\rho k^{2\rho})/(h^2 + k\rho^2) \quad [\text{when } \rho = 4j-1].$$

It is known that A_ρ, A'_ρ are always (algebraically) resolvable into two factors, say $A_\rho = L.M$, $A'_\rho = L'.M'$.

It will be seen now that several of the functions Z, Z', W, W' are of one or other of the forms A_ρ, A'_ρ. See the detail in the table below.

The factors Z, Z', W, W' which are of either of the forms A_ρ, A'_ρ increase the number of algebraic factors in N_1, N_2, N_3, N_4 beyond the normal number (8) up to 9, 10, or 12. The results will be found detailed in the table below.

α, β, $\alpha\beta$	a b	Z & Z'; W & W' A & A'	Factors in N_1 N_2 N_3 N_4
$4i+1$, $4j+1$, $4m+1$	α, 1	Z_α, $Z_{\alpha\beta}$; W_α, $W_{\alpha\beta}$ = A_α	12, 10, 10, 8
	1, β	Z_β, $Z_{\alpha\beta}$; W_β, $W_{\alpha\beta}$ = A_β	12, 10, 10, 8
	α, β	$Z_{\alpha\beta}$; $W_{\alpha\beta}$ = $A_{\alpha\beta}$	10, 9, 9, 8
$4i-1$, $4j-1$, $4m+1$	α, 1	Z'_α, $Z'_{\alpha\beta}$; W'_α, $W'_{\alpha\beta}$ = A'_α	8, 10, 10, 12
	1, β	Z'_β, $Z'_{\alpha\beta}$; W'_β, $W'_{\alpha\beta}$ = A'_β	8, 10, 10, 12
	α, β	$Z_{\alpha\beta}$; $W_{\alpha\beta}$ = $A_{\alpha\beta}$	10, 9, 9, 8
$4i+1$, $4j-1$, $4m-1$	α, 1	Z_α, $Z_{\alpha\beta}$; W_α, $W_{\alpha\beta}$ = A_α	12, 10, 10, 8
	1, β	Z'_β, $Z'_{\alpha\beta}$; W'_β, $W'_{\alpha\beta}$ = A'_β	8, 10, 10, 12
	α, β	$Z'_{\alpha\beta}$; $W'_{\alpha\beta}$ = $A'_{\alpha\beta}$	8, 9, 9, 10
$4i-1$, $4j+1$, $4m-1$	α, 1	Z'_α, $Z'_{\alpha\beta}$; W'_α, $W'_{\alpha\beta}$ = A'_α	8, 10, 10, 12
	1, β	Z_β, $Z_{\alpha\beta}$; W_β, $W_{\alpha\beta}$ = A_β	12, 10, 10, 8
	α, β	$Z'_{\alpha\beta}$; $W'_{\alpha\beta}$ = $A'_{\alpha\beta}$	8, 9, 9, 10

3. THEOREM II.—Let

$$N = (a\xi^4)^{2n} - (a\xi\eta)^{4n} - (\xi\eta)^{4n} + (a\eta^4)^{2n},$$

where a is an *odd* prime, and $n = a^r$.

Then N has always $(6r+4)$ *algebraic* factors.

Demonstration.—Write

$$x = \xi^2, \quad y = a\eta^2; \qquad u = a\xi^2, \quad v = \eta^2.$$

Then
$$N = (x^{2n} - y^{2n})(u^{2n} - v^{2n})$$
$$= (x^n - y^n)(x^n + y^n)(u^n - v^n)(u^n + v^n).$$

Put $X = x^n - y^n$, $X' = x^n + y^n$; $U = u^n - v^n$, $U' = u^n + v^n$.

Then
$$N = XX'.\,UU'.$$

Write $Z_1 = x - y$, $Z_a = (x^a - y^a)/Z_1$, $Z_{2a} = (x^{a^2} - y^{a^2})/Z_a$, ..., &c. ...

..., $Z_{ra} = (x^{a^r} - y^{a^r})/Z_{(r-1)a}$.

Write $Z_1 = x+y$, $Z_a = (x^a+y^a)/Z_1'$, $Z_{2a}' = (x^{a^2}+y^{a^2})/Z_a'$, ..., &c. ...

$$\ldots,\ Z_{ra}' = (x^{a^r}+y^{a^r})/Z_{(r-1)a}'.$$

And let W_1, W_a, W_{2a}, &c.; W_1', W_a', W_{2a}', &c., be the same functions of u, v that Z_1, Z_a, Z_{2a}, &c.; Z_1', Z_a', Z_{2a}', &c., are of x, y.

Then $X = \Pi(Z)$, $X' = \Pi(Z')$; $U = \Pi(W)$, $U' = \Pi(W')$.

Thus each of X, X', U, U' is a product of $(r+1)$ *algebraic* factors.

Further, when $a = 4i+1$, all the Z (except Z_1), and all the W (except W_1), are *Aurifeuillians* of the same order a, and are thus each of them a product of *two* (algebraic) factors (say $= L.M$).

Also, when $a = 4i-1$, all the Z' (except Z_1'), and all the W' (except W_1'), are *Aurifeuillians* of the same order a, and are thus each of them a product of *two* (algebraic) factors (say $= L'.M'$).

Hence, one of the products XW, $X'W'$ has always $(4r+2)$ *algebraic* factors, and the other product $X'W'$ or XW has $(2r+2)$ *algebraic* factors.

Then, finally, $N = XX'WW'$ has always $(6r+4)$ *algebraic* factors.

3*a*. Theorem 2*a*.—It is easy now to see that if (with a, n as above)

$$\begin{aligned} N_2 &= (a\xi^4)^{2n} - (a\xi\eta)^{4n} + (\xi\eta)^{4n} - (a\eta^4)^{2n}, \\ N_3 &= \quad ,, \quad + \quad ,, \quad - \quad ,, \quad - \quad ,, \quad , \\ N_4 &= \quad ,, \quad + \quad ,, \quad + \quad ,, \quad + \quad ,, \quad , \end{aligned}$$

then N_2, N_3 have only $(5r+4)$ *algebraic* factors, and N_4 has only $(2r+2)$ such; because in N_2 and N_3 only one of the products XW, $X'W'$ contains Aurifeuillians, and N_4 has no Aurifeuillians.

4. Theorem III.—Let

$$N_1 = (2a\xi^4)^{2n} + (2a\xi\eta)^{4n} + (\xi\eta)^{4n} + (2a\eta^4)^{2n},$$

let
$$N_2 = (2a\xi^4)^{2n} + (2\xi\eta)^{4n} + (a\xi\eta)^{4n} + (2a\eta^4)^{2n},$$

where a is an *odd* prime, and $n = a^r$.

Then N_1 and N_2 have always $(4r+2)$ *algebraic* factors.

Demonstration.—Write, in N_1,

$$x = \xi^2, \quad y = 2a\eta^2: \quad u = 2a\xi^2, \quad v = \eta^2,$$

and in N_2
$$x = 2\xi^2, \ y = a\eta^2; \quad u = 2\xi^2, \quad v = a\eta^2.$$

Then N_1 and N_2 are each $(x^{2n}+y^{2n})(u^{2n}+v^{2n})$.

Write $Z_2 = (x^2+y^2)$, $Z_{2a} = (x^{2a}+y^{2a})/Z_2$, $Z_{2a^2} = (x^{2a^2}+y^{2a^2})/Z_{2a}$, ...;

$$\ldots,\ Z_{2a^r} = (x^{2a^r}+y^{2a^r})/Z_{2a^{r-1}}.$$

And let W_2, W_{2a}, W_{2a^2}, &c., be the same functions of u, v, that Z_2, Z_{2a}, Z_{2a^2}, &c., are of x, y.

Then $\qquad (x^{2n}+y^{2n}) = \Pi(Z), \quad u^{2n}+v^{2n}) = \Pi(W).$

Thus $\Pi(Z)$ and $\Pi(W)$ contain always $(r+1)$ *algebraic* factors.

Further, all the Z (except Z_2), and all the W (except W_2) are *Aurifeuillians* of same order $(2a)$, and are thus each of them a product of two *algebraic* factors (say $= LM$).

Hence, each of $\Pi(Z)$, $\Pi(W)$ contains $(2r+1)$ *algebraic* factors; and, finally, since N_1 and N_2 are of the forms $\Pi(Z)$, $\Pi(W)$, each contains $(4r+2)$ *algebraic* factors.

4*a*. THEOREM III*a*.—It is easy to see that if—with the same a, n as above—either the 2nd and 4th, or the 3rd and 4th signs in the above N_1, N_2 be *minus*, then N_1 and N_2 will have $(4r+3)$ *algebraic* factors, because only one of the products $\Pi(Z)$, $\Pi(W)$ will contain Aurifeuillians; and that if the 2nd and 3rd signs be *minus*, then N_1 and N_2 will have $(4r+4)$ *algebraic* factors, because there will be no Aurifeuillians in either.

Thursday, June 9th, 1921.

Mr. H. W. RICHMOND, President, in the Chair.

Present fifteen members.

Miss F. M. Wood was elected a member of the Society.

Mr. J. Prescott was nominated for election.

Prof. J. L. S. Hatton read a paper "The Inscribed, Circumscribed, and Self-Conjugate Polygons of Two Conics."

Prof. M. J. M. Hill read a paper "The Differential Equations of the First Order derivable from an Irreducible Algebraic Primitive."*

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* Printed in this volume.

The following informal communications were made :—

The Congruence $2^{p-1}-1 \equiv 0 \pmod{p^2}$: Lieut.-Col. A. Cunningham.
Diophantine Equations : Dr. T. Stuart.
A Chapter from Ramanujan's Note-Book : Prof. G. H. Hardy.

The following papers were communicated by title from the chair :—

Curvature and Torsion in Elliptic Space : Prof. M. J. Conran.
Note on the Resultant of a Number of Polynomials of the same Degree : Dr. F. S. Macaulay.
An Analytic Treatment of the Three-Bar Curve : Mr. F. V. Morley.
Bemerkung zu unserer Abhandlung "On the Diophantine Equation $ay^2+by+c = dx^n$" : E. Landau and A. Ostrowski (communicated by Prof. G. H. Hardy).

ABSTRACTS.

On the Differential Equations of the First Order derivable from an Irreducible Algebraic Primitive

Prof. M. J. M. HILL.

If $\phi(x, y, c)$ be an irreducible polynomial in the variables x, y and the arbitrary constant c, then it is proved in this paper that the differential equation satisfied by the curves

$$\phi(x, y, c) = 0 \tag{I}$$

is of the form

$$[f(x, y, p)]^m = 0, \tag{II}$$

where $p = dy/dx$, where m is a positive integer, and where $f(x, y, p)$ is an irreducible polynomial in x, y, and p.

If the integer m is greater than unity, it is proved that m must be a factor of n, and if in this case $m = n/s$, then the degree of $f(x, y, p)$ in p is s.

Further, in this case it is possible to replace the primitive (I) by another

$$\psi(x, y, C) = 0, \tag{III}$$

which is of degree s in C, where m values of c correspond to each value of C. So far as the relation between x and y is concerned, the two primitives (I) and (III) are equivalent.

Next it is proved that the differential equation

$$f(x, y, p) = 0 \tag{IV}$$

can have no primitive containing an arbitrary constant independent of (III).

Any other primitive, involving an arbitrary constant, which it may possess, is obtainable from (III) by replacing C by some function of c.

If the degrees of two primitives of (IV) in their respective parameters are the same, it is shown that there must be a lineo-linear relation between these parameters, which relation does not involve the variables.

Lastly, it is shown that if a primitive exist, which does not involve an arbitrary constant, it must be obtainable by eliminating c between

$$\phi(x, y, c) = 0 \tag{I}$$

and

$$\frac{\partial \phi(x, y, c)}{\partial c} = 0. \tag{V}$$

Bemerkung zu unserer Abhandlung "On the Diophantine Equation $ay^2+by+c = dx^n$"

E. LANDAU and A. OSTROWSKI (communicated by G. H. HARDY).

Durch eine freundliche Mitteilung von Herrn STÖRMER wurden wir auf die Abhandlung von Herrn THUE aufmerksam gemacht: "Über die Unlösbarkeit der Gleichung $ax^2+bx+c = dy^n$ in grossen ganzen Zahlen x und y [*Archiv for Mathematik og Naturvidenskab*, Bd. XXXIV (1917), No. 16, S. 1–6]. Hierin beweist er im Wesentlichen unser Hauptresultat. Sein Beweis ist elementarer, aber komplizierter als der unsere. Wir bedauern, dass uns die THUEsche Arbeit erst jetzt bekannt werden konnte; der Archivband traf erst 1921 in der Göttinger Universitätsbibliothek ein, und in der *Revue semestrielle des publications mathématiques*, die uns bis Bd. XXVIII$_2$ (Oktober 1919–April 1920) vorliegt, ist der Band bisher nicht besprochen.

LIBRARY

Presents.

BETWEEN December 31st, 1920, and December 31st, 1921, the following presents were made to the Library by their respective authors and publishers:—

Bhattacharyya, D.—Thesis on Vector Calculus.

Böttcher, L.—Copies of eight papers by the author.

Byerley, W. E.—Fourier's Series and Spherical Harmonics.

Duarte, F. J.—Détermination des positions géographiques par les méthodes des hauteurs égales.

Edwards, Joseph.—A Treatise on the Integral Calculus.

Goedhart, J. G. A.—The Spiral Orbit of Celestial Mechanics, Parts I, II.

Halkyard, Edward.—The Fossil Foraminifera of the Blue Marl of the Côte des Basques, Biarritz.

Hurwitz, Frau.—Copies of thirty-seven papers by her husband and copies of six papers by other authors.

Klein, Felix.—Gesammelte Mathematische Abhandlungen, Band 1.

Newton, Isaac.—Principia Philosophiae, Editio tertia (presented by the family of the late Walter Bailey, M.A.).

Prasad, Ganesh.—On Mathematical Research in the last twenty years.

Willis, Edward J.—The Mathematics of Navigation.

Åbo Academy: Acta Humaniora, no. 2.

Amsterdam: Royal Academy of Sciences, Proceedings, vol. 20, parts 1-10; Verhandelingen, deel 12, no. 5.

Brussels: Académie Royale de Belgique, Bulletin de la Classe des Sciences, tome 6, nos. 9-12; tome 7, nos. 1-10. Tables générales des Bulletins, 1899-1910, 1911-1914; Annexe aux Bulletins, 1915. Mémoires, 2me série, tome 6, fasc. 8.

Brussels: Académie Royale des Sciences, Annuaire, 87me année, 1921.

Calcutta University: Post-graduate Teaching in the University of Calcutta, 1919-20.

Journal für die reine und angewandte Mathematik, band 151, hefte 1-4.

Kansas University: Science Bulletin, vol. 11; vol. 12, nos. 1, 2.

Kyoto: Imperial University, College of Science, Memoirs, vol. 3, no. 11; vol. 4, nos. 1-6.

La Haye: Société Hollandaise des Sciences, Œuvres complètes de Christiaan Huyghens, tomes 13, 14.

London: Conjoint Board of Scientific Societies, Confirmed Minutes, 18th, 19th, 20th, and 21st Meetings; Fourth Annual Report.

London: Institution of Civil Engineers, List of Members, 1921; Record of origin and progress.

Madrid: Junta para Ampliación de Estudios e Investigaciones Científicas, Publicaciones del Laboratorio y Seminario Matemático, tome 3, memoria 5; tome 4, memoria 1.

Masaryk University: Faculté des Sciences, Publications, nos. 1-4.

Mathematical Gazette, vol. 10, nos. 150-155.

Paris: L'Enseignement Mathématique, 21me année, nos. 3, 4.
Nation and Athenæum, vol. 28, nos. 21–26 ; vol. 29 ; vol. 30, nos. 1–14.
Nautical Almanac, 1923.
Sendai: Tôhoku Imperial University, Science Reports, vol. 9, no. 6 ; vol. 10, nos. 1–4.
Sendai: Tôhoku Mathematical Journal, vol. 18, nos. 3, 4 ; vol. 19, nos. 1–4.
South Kensington : Science Museum, List of Short Titles of current Periodicals in the Science Library.
Technology, vol. 10.
Tokyo : Physico-Mathematical Society of Japan, Proceedings, vol. 2, no. 11 ; vol. 3, nos. 1-10.

Exchanges.

American Journal of Mathematics, vol. 42, no. 1 ; vol. 43, nos. 1–3.
Athens : Société Mathématique de Grèce, Bulletin, vol. 2, nos. 1, 2.
Benares: Mathematical Society, Proceedings, vol. 2, pt. 2.
Berlin: Mathematische Zeitschrift, band 8, 9, 10 ; band 11, hefte 1, 2.
Boston (Mass.) : American Academy of Arts and Sciences, Proceedings, vol. 55, no. 10 vol. 56, nos. 1–11.
Bulletin des Sciences Mathématiques, vol. 44, nos. 10, 12 ; vol. 45, nos. 1–12.
Calcutta : Indian Association for the Cultivation of Science, Proceedings, vol. 5, pt. 2 ; vol. 6, pts. 1–4. Convention for the year 1918.
Calcutta : Mathematical Society, Bulletin, vol. 9, nos. 1, 2 ; vol. 10, no. 1 ; vol. 11, no. 4 vol. 12, nos. 1, 2.
Cambridge Philosophical Society, Proceedings, vol. 20, pts. 2, 3.
Dublin: Royal Irish Academy, Proceedings, vol. 35, Section A, nos. 1–4.
Edinburgh : Royal Society, Proceedings, vol. 38, pt. 3 ; vol. 39, pt. 1.
Florence : Biblioteca Nazionale Centrale, Bollettino, nos. 234–245.
Hamburg University : Abhandlungen aus dem Mathematischen Seminar, band 1, heft 1.
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OBITUARY NOTICES

LORD RAYLEIGH.

John William Strutt, afterwards Baron Rayleigh, was born on November 12th, 1842. His early education was mainly at a private school. He entered as a Fellow Commoner at Cambridge in 1861, read mathematics with Routh, and, after a brilliant undergraduate career, graduated as Senior Wrangler in 1865. He was shortly afterwards elected to a Fellowship of his College (Trinity). He succeeded to the peerage in 1873.

Within a few years of his degree he began that career of original scientific investigation which continued without intermission almost to the day of his death, and was ultimately to establish his fame, after the departure of his great compeers Stokes, Kelvin, and Maxwell, as the supreme authority in physical science. It is unnecessary here to attempt a record of the manifold distinctions which were conferred on him or the important offices to which he was called. One or two matters may however be mentioned. In response to a pressing invitation he accepted the Cavendish Professorship of Physics in 1879, in succession to Maxwell; this he held till 1884. He was Secretary of the Royal Society from 1885 to 1896, and President from 1905 to 1908. His long and intimate connection with our own Society dates from 1871. He was President in the years 1876–7, and received the De Morgan medal in 1890. It is a matter of some pride to recall that much of his earlier and most characteristic work on Sound and Vibrations made its first appearance in our *Proceedings*. Nor should we forget his solicitude for the welfare of the Society, and the generous contribution which he made to its funds, at a time of financial stress.

Rayleigh's closest affinities were to the great dynamical school of which the three great physicists already named were exponents. In respect of the massive solidity of his work, and serene breadth of judgment, he stands nearest perhaps to Stokes, for whom indeed he had an intense admiration. This found eloquent expression in the obituary notice which he wrote for the Royal Society. One sentence, among others, may be picked out from this memoir as equally applicable to himself: "Instinct

amounting to genius and accuracy of workmanship are everywhere manifest; and in scarcely a single instance can it be said that he has failed to lead in the right direction."

A survey of his achievements from the physical point of view must be sought elsewhere.* In these pages some account may be looked for of his characteristics as a mathematician. It must be recognised that his main interest was in the unravelling of physical phenomena, and that mathematics was to him chiefly valuable as an auxiliary. Moreover, just as in his experimental work he had recourse by preference to the simplest devices, so the mathematical apparatus, whenever possible, was of the most elementary character. There was always, however, enormous mathematical power in reserve, and whenever the occasion called for it the utmost degree of skill of this kind was brought to bear. One striking instance among others was his application of Hill's highly original methods in the Lunar Theory to the optics of stratified media. But perhaps the most original feature in his own mathematical work was the development of approximate methods, by which problems utterly refractory (in their rigorous form) to analysis receive a solution fully adequate for practical purposes. An early instance is the treatment of the Helmholtz resonator as a system of one degree of freedom. The treatise on *Sound* contains many other examples.

His earliest papers relate chiefly to Sound and Optics. The book on *Sound* just referred to is remarkable for the great development given to the theory of Vibrations. This theory, originated by Bernoulli and Lagrange, and further elucidated by Thomson and Tait, was greatly extended by him, and runs as a leading thread through the whole book. The work as a whole ranks as a classical achievement, and has entirely transformed the subject. Many of the theorems which it contains have applications not only in other branches of mechanics but in such subjects as Electricity and Heat.

In Optics he proceeded at first on the basis of the old elastic theory of the æther, until he became convinced that it was untenable, or rather as he expressed it, "too wide for the facts." His later work was in terms of the electromagnetic theory, although many investigations are independent of the particular hypothesis adopted. One of the earliest problems which he took up was the scattering of light by small particles. To this he returned more than once, with the final conclusion that the scattering by the molecules of the air, apart from the influence of grosser particles,

* See the memoir by Sir Arthur Schuster, *Proc. Roy. Soc.*, (A), Vol. 98.

would account for the blue of the sky. Other investigations were on the theory of gratings, which he simplified, and on the resolving powers of spectroscopes, and of optical instruments in general. This theory is in fact largely of his creation. As an instance of his power in putting old matters in a new light and dispelling obscurities we may cite his elucidation of "Huyghens' principle," which had long been a perplexity to serious students of the subject. By great good fortune he was induced to write a connected account of the theory of Light, as he regarded it, in the form of two articles contributed to the *Encyclopædia Britannica*. These are included of course in his collected papers, but might well be published separately. They constitute by far the best textbook on the subject which has ever appeared.

From the theory of Sound and Vibrations to Hydrodynamics, especially in relation to problems of small oscillation about a state of equilibrium or of steady motion, was a natural transition. His first paper on the subject deals with water waves, and reproduces the fundamental results of Airy, Stokes, and others, by an elegant and simplified analysis. The "solitary wave" of Scott Russell was also elucidated, and it was explained in particular why such a wave must necessarily be one of elevation only. Rayleigh was scrupulous here, as in all similar cases, to point out where he had been anticipated. Boussinesq in this instance shares the credit of clearing up a matter which had long been obscure. The theory of deep-water waves of permanent type, which had been the subject of a classical research by Stokes, had a lasting fascination for Rayleigh, who returned to it again and again, continually improving the approximations. The influence of capillarity on water-waves had been considered by Kelvin in 1871. The subject was taken up and completed by Rayleigh, who investigated the train of waves and ripples set up by a travelling disturbance. The paper referred to, which appeared in Vol. IX of our own *Proceedings*, is remarkable for a characteristic analytical artifice which subsequent writers have found very useful. A mathematical indeterminateness which presents itself in various problems of steady motion (owing to the implicit inclusion of free waves of a certain period), when dissipation is neglected, is evaded by the temporary introduction of frictional forces varying as the velocity, whose coefficient is ultimately made to vanish. The result must be the same as if the true law of viscosity had been employed, but the analysis is much simpler. In this connection we may recall the beautiful investigation of the oscillations of a liquid globule, and the vibrations of a jet, and also the lucid set of papers in which Laplace's theory of Capillarity is explained, criticised, and amended. Reference may also be made to the theory of "group-velocity." The discrimination between this and wave-velocity had been

made by Scott Russell, and the group-velocity had been identified by Reynolds with the rate of transmission of energy, for the case of water-waves. A general proof applicable to any type of wave-motion was given by Rayleigh, who also pointed out the importance of the conception in various fields.

The mathematically elegant theory of discontinuous motions in frictionless liquids had been started by Helmholtz and Kirchhoff in two classical papers. The work of the latter suggested to Rayleigh a theory of the resistance experienced by a plane lamina moving through a stream, and he completed Kirchhoff's solution from this point of view. The results, though necessarily imperfect as a picture of what really takes place, were a great improvement on previous explanations, and have stimulated much subsequent investigation. A cognate subject to which Rayleigh devoted much attention, partly no doubt owing to its acoustical bearings, and later for its own sake, was the question of stability of fluid motions. It had been remarked by Helmholtz, and further insisted on by Kelvin, that a surface of discontinuity would in a frictionless liquid necessarily be unstable. Rayleigh's first enquiry was: to what degree is the instability affected if the discontinuity is eased off, as it actually is by viscosity? He found that the instability remains for disturbances whose wave-length exceeds a certain limit. He further investigated the flow between parallel planes, and later in a pipe, having in view Reynolds's experimental demonstration of a critical velocity. The motion proved to be stable provided the graph of the velocity, as a function of the distance from the axis, is free from inflexions. Rayleigh was well aware that this conclusion must not be pressed too far. The disturbances contemplated are assumed to be infinitesimal; moreover, although the type of steady motion is such as could be maintained (if there were no disturbance) under the influence of viscosity, the effect of friction on the *disturbed* motion is in fact neglected. In particular, the condition of no slipping at the walls, which appears to be fundamental, is violated. Calculations of the above type were resumed at frequent intervals, and his more recent papers include a masterly review of the subject, in which viscosity is duly considered.

The most casual inspection of the contents of any one of the six volumes of his collected papers will show what large fields of Rayleigh's activity have here been left unnoticed. The electrical researches, for instance, important as they often are from a mathematical as well as a physical point of view, have not even been mentioned. But the main characteristics of the work are the same throughout. If asked to describe in one word the essential character of his genius, we should say that it was in the highest degree *illuminating*. Whatever the subject taken up,

not only is new material contributed, but existing knowledge is reviewed and set in a fresh light, unsuspected analogies and affinities are revealed, and what was often a collection of disconnected fragments becomes an orderly and massive structure. His mind was of a type which we like to think of as peculiarly British, and he maintained to the full the tradition of the great dynamical school of which he was the most conspicuous surviving representative. He died on June 30th, 1919.

H. L.

ADOLF HURWITZ.

JUST a week after the signing of the Peace, there passed away in the person of Adolf Hurwitz one of the most notable representatives of contemporary German mathematical science. Although a Jew by parentage, and for no less than twenty-seven years Professor at the Swiss Technische Hochschule, he retained his German nationality to the end. A product of the German academic system at its best, he can never have felt the temptation, to which so many of his countrymen have yielded, to change, even nominally, his nationality.

Born at Hildesheim in the year in which Riemann became Professor at Göttingen, Hurwitz entered the Andreanum little more than eighteen months after Riemann's untimely death, and was, before he had quite reached his eighteenth year, already at Munich attending the lectures of Klein, the most genial exponent of Riemann's ideas. A year later, he was at Berlin, in the mathematical Seminar, and gaining at first hand, from Weierstrass and Kronecker, a knowledge of the methods in which they were passed masters. But Hurwitz was to be above all a pupil of Klein, and, after three semesters spent at Berlin, we find him once more at Munich, and in October 1880 following Klein thence to Leipzig.

Untrammelled by examinations, Hurwitz was able, even when at Berlin, to collaborate with Klein and to afford him help in one of his most notable papers on elliptic modular functions,* a paper destined, with others of Klein's, almost equally remarkable, to be for many years the pivot on which Hurwitz's mathematical interests were to turn. Hurwitz was peculiarly fitted to carry out Klein's ideas.† He had gained from Schubert,‡ of Abzählende Geometrie fame, his master at the Andreanum, an interest

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* *Math. Ann.*, Vol. 17, pp. 69 and 70. Some idea of the advance due to Klein may be gained by comparing the papers just referred to with H. J. S. Stephen's *B. A. Report*, 1865.

† Klein's strength, it may be remembered, was sometimes regarded as consisting still more in the fertility and the genial character of his ideas than in the power of developing them. Cf. Lie, *Transf. Gruppen.*

‡ We are told that Schubert gave up part of every Sunday to working at Geometry with the schoolboy Hurwitz, and the first of the latter's papers, written when he was still at the Andreanum, was a joint paper. It was also Schubert who persuaded Hurwitz's father to allow him to go to the University and who sent him with warm recommendations to Klein at Munich.

in Geometry and a familiarity with geometrical methods which were bound to serve him in good stead with Klein, and he had already entered on the field of original research. On the other hand, we have Hilbert's authority for the statement that the acquisition of Riemannian ideas, which intercourse with Klein rendered possible, of itself constituted at that time a transfer, so to speak, to a higher class among mathematicians. It is not surprising, then, that we find Hurwitz, a Göttingen *Privatdocent* of barely two years' standing and not yet 25, called in 1884 to Königsberg as *Extraordinarius*, with a record of important published work behind him.

At Königsberg he made the acquaintance of Hilbert, first the student and then the *Privatdocent*, and of Minkowski, whose family lived there, and who, when at home from Bonn for the holidays, joined them in their almost daily walks. During these walks, continued over the whole of the eight-year period of Hurwitz's residence at Königsberg, wellnigh every corner of the then known mathematical world was explored.*

We get some glimpse, incidentically, in studying Hurwitz's career, as to the way in which a professional mathematician may be formed.

From Königsberg, Hurwitz went to Zürich, where he remained until his death.

About a hundred papers were published by Hurwitz. In almost all of them, the influence of Klein,† direct or indirect, is perceptible, and many of them may be characterized as solutions, usually complete, of problems, of a fundamental nature and of no small difficulty, proposed by

* Cf. Hilbert, *G. N.*, 1920. Hilbert adds: "Hurwitz mit seinen ebenso ausgedehnten und vielseitigen wie festbegründeten und wohlgeordneten Kenntnissen war uns dabei immer der Führer.

† How much Hurwitz owed to Dedekind also is evident from his papers and from his own acknowledgments. But it would seem that they had never met, at any rate not before 1895. There is an interesting indication of this in Dedekind's answer to a question as to what he thought of the paper "Über die Theorie der Ideale" (*G. N.*, 1894), the first of Hurwitz's attempts in this direction. Dedekind explains that the mode of treatment of the fundamental theorem in the theory of Ideals there exposed, and based indeed on an algebraical lemma of his own, had been familiar to him for many years, and he gives in detail his reasons for not having adopted it. He had since found and published in Dirichlet's *Zahlentheorie* what he regarded as a much more natural and simple way of building up the subject. He quotes Gauss's "Ausspruch eines grossen Wissenschaftlichen Gedanken, 'die Entscheidung für das Innerliche im Gegensatz zu dem Aeusserlichen'," and then continues: "Hiernach wird man es auch erklärlich finden, dass ich meiner Definition des Ideals durch eine charakteristische innerliche Eigenschaft den Vorzug gebe vor derjenigen durch eine äusserliche Darstellungsform, von welcher Herr Hurwitz in seiner Abhandlung ausgeht. Aus denselben Gründen konnte der . . . . Beweis des Satzes . . . . mich noch nicht völlig befriedigen, weil durch die Einmischung der Functionen von Variabelen die Reinheit der Theorie nach meiner Ansicht getrübt wird." (*Göttinger Nachrichten*, 1895, p. 111.)

Klein. In some cases, the results obtained or the methods employed have an importance far beyond what we know or may presume to have been the occasion for writing them. In particular, the paper "Über algebraische Correspondenzen,"* may be referred to in this connection. Brill had succeeded in proving a theorem of the truth of which Cayley had persuaded himself by inductive processes, without being able however to devise anything of the nature of a demonstration save in a very special case. Hurwitz's work goes far beyond Brill's in generality,† besides being above all remarkable as an application, promised nine years before, of Abelian integrals to Geometry, and as the point of departure of Castelnuovo in his investigations on analogous matters in the theory of surfaces. And it may be said to generalise Abel's Theorem itself.

Other pairs of papers that have become classical are those entitled "Über algebraische Gebilde mit eindeutige Transformationen in sich,"‡ and "Über Riemannsche Flächen mit gegebenen Verzweigungspunkte"§; the second pair are also interesting because they show Hurwitz first failing to obtain the complete solution of the problem, taking up the thread ten years later in the light of a happy suggestion from Lasker, the international Chess Champion, met in the previous summer, and finally completing the solution by the use of a method‖ in the theory of abstract groups discovered in the meanwhile by Frobenius, Hurwitz's predecessor at Zürich.

Of the papers not more or less directly inspired by Klein, among the most original are those on the roots of algebraic and transcendental equations. Hurwitz was a recognised expert in the treatment of problems of this nature. His paper on the zeros of Bessel's functions,¶ which already marked a strikingly new departure, both as regards the methods employed and the character of the results obtained, was followed rapidly by several others on the roots of transcendental equations.** And when, soon after he had gone to Zürich, one of his Swiss colleagues turned to him for help

* *Math. Ann.*, Vol. 28.

† Hurwitz was thus able to repay with interest a debt of Klein to Cayley (*Math. Ann.*, Vol. 17, p. 66). It was in studying the theory of modular correspondences that Hurwitz was led to consider the necessity of investigating correspondences defined by more than one equation on entities of genus p.

‡ *Math. Ann.*, Vols. 32, 41.

§ *Math. Ann.*, Vols. 39, 55.

‖ This method of Frobenius is also interesting to English readers as being closely connected with some of the most important of Burnside's work.

¶ *Math. Ann.*, Vol. 33.

** No fewer than seven papers of Hurwitz's deal with roots of equations

in a technical problem involving the conditions under which an algebraic equation has the real part of its roots all negative, the skill shown by Hurwitz in furnishing the complete solution was noteworthy. Simple as are the conditions arrived at,* namely that certain determinants formed out of the coefficients of the equation have to be positive, the resources Hurwitz disposes of are seen in this, as in so many others of his papers, to be of the most varied description. He avails himself with equal freedom of the ideas and results of Sturm, of Hermite, of Frobenius, of Kronecker, and of Cauchy.

His papers on continued fractions and on the approximate representation of irrational numbers are also very original, as well as curious. And all his algebraical work is marked by a rare insight into underlying principles. One of Hurwitz's greatest triumphs was his complete solution of a question concerned with the reducibility of quadratic forms of any number of variables, a part of which had baffled the united efforts of a Cayley and a Roberts, equipped though they might be with all the resources of an empirical science and of a power of calculation that shunned no labour.†

But perhaps Hurwitz's main interests really lay in the theory of numbers as a whole, including that part of it which attaches itself naturally to the theory of modular functions, such as the relations connecting numbers of classes of quadratic forms. On this latter subject he wrote seven papers, and one of the earliest of his papers written independently was devoted to the proof that a theorem of Stieltjes, giving the number of modes of expressing a prime as the sum of five squares, holds in a generalised form for every integer. The interest of four others lies in their connection with the theory of ideals. And of his last sixteen papers,‡ almost all those whose interest is not chiefly pedagogic, were devoted to the solution of Diophantine equations and analogous problems,

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* *Math. Ann.*, Vol. 46.

† Hurwitz's account of the matter is worth quoting:—"Roberts und Cayley haben sich im 16ten und 17ten Bande des *Quarterly Journal* mit den Nachweis beschäftigt, dass ein Product von Zwei Summen von je 16 Quadraten nicht als Summe von 16 Quadraten darstellbar sei. Ihre äusserst mühsamen auf Probiren beruhenden Betrachtungen besitzen indessen keine Beweiskraft, weil ihnen bezüglich der bilinearen Formen $z_1, z_2, \ldots$ specielle Annahmen zu Grunde liegen, die durch nichts gerechtfertigt sind." (*Gött. Nach.*, 1898, p. 310, Note 1.)

‡ It is noteworthy that in one of these later papers he concerns himself with an equation first employed by Klein in investigations (*Math. Ann.*, Vols. 14, 15) with which Hurwitz's dissertation was connected.

while his only publication in book form is a reprint of one of his papers on "Quaternion Theory of Numbers."*

Brilliant as Hurwitz's researches were known to be, he was honoured at Zürich most as a teacher, and the tradition of his success there is likely to be long preserved.

But he loved to employ the deductive method of exposition, alike in his writings and in his lectures. His papers even weary by their completeness, although this is almost always atoned for by a finished elegance of form. And if but few are merely didactic, a relatively large proportion are concerned with new proofs of known theorems;† while of his pupils, only those in close personal contact with him can have been able to form a just idea of the processes by which he was led to his results. Perhaps had he been less successful as a teacher, he might have been better able to found a great school of mathematics.

Hurwitz always remained a nineteenth century mathematician. One of the first, if not the very first, to utilise Cantor's theorem on the non-countability of the continuum,‡ and possessed, as he several times showed, of an acquaintance with, and the ability to apply, the elements of the theory of sets of points, he had to content himself with appreciating the nature, without fully grasping the magnitude of the revolution brought about in mathematical analysis by the extension of our knowledge of the Real Variable, so characteristic of the century in which we live. The very thoroughness of his early preparation may indeed have rendered him inapt or unwilling to do more than skirmish on ground§ relatively unfamiliar to him, and it is noteworthy that he did not follow Poincaré in the exploitation of the generalisation of the elliptic modular function constituted by the automorphic function. Indeed, though in an improved version‖ of a portion of his dissertation, published in later years, he remarks that the methods he employs are obviously applicable to automorphic functions, his sole

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* *Gött. Nach.*, 1896.

† The interest of these proofs is undeniable, and some have become classical. In one the motive is the desire to give a purely algebraic proof of an algebraic theorem previously established with extreme facility by the use of the Calculus.

‡ *Crelle*, Vol. 95.

§ His interest in new work is shown in more than one of his later papers—for example, in his use of Fejér-Cesàro methods in dealing with Fourier series, in his proof of a theorem of Fatou, and in the application of his old modular elliptic function equipment to a new proof of Landau's extension of Picard's theorem.

‖ *Math. Ann.*, Vol. 58.

contribution to that subject is a paper* in which he shows how the fundamental region may be determined for automorphic functions of any number of variables.

Though he was educated in the Real Gymnasium† section of the Andreanum, and though, curiously enough, the friend of his father to whose benevolence he owed his University career, and to whom he dedicated his dissertation, had the English or at least British name of Edwards, Hurwitz was not sufficiently master of our language to be able to read English mathematical papers, except with very great difficulty. His knowledge of the work of English writers was indirect. But he had great familiarity with French and several of his papers are written in this language. The value of his work was appreciated outside Germany and Switzerland. Some of his papers written in German were translated into other languages, and he was elected honorary or corresponding member of several learned bodies. He became an honorary member of our Society in November 1913.

Hurwitz was very generally liked, not only as a teacher, but as a man, and this in spite of the fact of his life being one long struggle‡ with a wasting disease, which must have rendered him little disposed for social intercourse. In points of honour he was punctilious. It fell to his lot to be anticipated in several of his results, and he was never slow to publish his recognition of the priority of another. And it is related that nothing but his refusal to break his word pledged to the Swiss Schulratspresident, who had secured him for the Technische Hochschule by travelling all the way from Zürich to Königsberg for the purpose, prevented his going to Göttingen as *Ordinarius* in succession to H. A. Schwarz. How great a sacrifice this entailed will be realised if it be borne in mind that a Chair in a Swiss University was, for a German, never, before the War, regarded as more than a stepping stone to a Chair in Germany. That the call

* *Math. Ann.*, Vol. 60.

† It was owing to this circumstance that he did not become *Privatdocent* at the University of Leipzig, as he, a pupil of Klein's, then holding a Chair there, would naturally have done. Evidence of a knowledge of Greek was regarded by the Philosophical faculty of Leipzig as an indispensable requisite for the *venia legendi*. Göttingen was, as would now be said, more advanced in its ideas.

‡ That this struggle was waged with comparative success for so many years appears almost incredible, and can only be accounted for by the constant care and devotion of Hurwitz's wife, a daughter of Professor Samuel, a well known member of the Medical faculty at Königsberg.

was not repeated at a later date may be attributed to the state of Hurwitz's health, supposed to render such a call undesirable.*

W. H. Y.

* Since the above notice was in print, I have received the following statement from Dr. Vermeil, Klein's assistant, written at Klein's request. It confirms in various points what is given above, and adds some details of interest: "Hurwitz hat als Schüler von Schubert schon als Secundaner Resultate im Gebiete der abzählenden Geometrie gefunden. Als er dann im Sommer 1877 nach München kam, stellte ihm Klein sofort die Aufgabe, die Resultate der abzählenden Geometrie auf zuverlässige Grundlagen zu stellen. Leider aber erlitt Hurwitz sehr bald einen Typhusanfall (der Typhus grassierte damals in München), und kehrte darum erst nach mehreren Semestern nach München zurück, wo er die beste Hilfe von Klein im Ausbau der Theorie der elliptischen Modulfunktionen wurde. Seine Leipziger Dissertation, die in den *Math. Ann.* erschienen ist, ist nicht nur durch die selbständige Entwicklung der Eisenstein'schen Methoden bemerkenswert, sondern insbesondere dadurch, dass er $\sqrt[12]{\Delta(\omega_1, \omega_2)}$ als eine Kongruenzform 12ter Stufe erkannte und dadurch die einfachste Grundlage für die neuen Multiplikatorgleichungen schuf. Inzwischen hatte Gierster aus den von Klein gefundenen Modulargleichungen höherer Stufe neue Zahlentheoretische Resultate, zum Teil auf induktivem Wege, abgeleitet, und es bleibt eine der grössten Leistungen von Hurwitz, durch die Theorie der zugehörigen überall endlichen Integrale, die Gierster'schen Resultate endgültig begründet zu haben und überhaupt eine allgemeine Theorie der algebraischen Korrespondenzen auf algebraischen Kurven begründet zu haben. Später machte die räumliche Trennung die Beziehung Zwischen Klein und Hurwitz seltener. Aber Klein wünscht die Förderung an zuerkennen, die Hurwitz der Theorie der endlichen Gruppen linearer Substitutionen von der Theorie der elliptischen Modulfunktionen her erteilt hat. Hurwitz war wesentlich ein zahlentheoretisches Talent und ergänzte dadurch die mehr intuitive Art von Klein auf glückliche Weise."

PAPERS

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EINSTEIN'S THEORY OF GRAVITATION AS AN HYPOTHESIS IN DIFFERENTIAL GEOMETRY

(*Presidential Address.*)

By J. E. CAMPBELL.

[Read November 11th, 1920.]

MR. PRESIDENT, LADIES AND GENTLEMEN,

I am venturing this afternoon to talk to you on Differential Geometry, or rather on a part of that vast and interesting study. The part I wish to bring before you is the geometry of quadratic differential forms, in its relation to Einstein's Theory of Gravitation. I want to show how naturally the law of gravitation arises in this geometry. It might even have been discovered by some pure mathematician, and studied by him as a particular kind of four dimensional geometry; and he would never have dreamt of its wonderful possibilities as an explanation of natural events.

There will be little originality in anything I say; yet it may possibly help some, who, like myself, have a very slight knowledge of physics, to understand the bearing of the new theory on Differential Geometry, and perhaps enable them to contribute to its advancement by familiarising them with the foundations.

As regards the law of gravitation, I owe what knowledge I have of it to Prof. Eddington's report on the relativity theory. I cannot, how-

ever, claim his great authority for anything I say; the errors into which I may fall will be my own contribution to the theory, and he will perhaps warn others of falling into the same in the lecture we are hoping to hear from him in the coming session.

I must ask your indulgence if I have to start off in my talk with some rather formidable symbols. I must also ask you to allow me only to sketch the proofs of my assertions. You would not, I think, find much difficulty in supplying the gaps—agreeing with me if I am right and correcting me if wrong—had you the time and inclination, and, say, Bianchi's *Differential Geometry* to refer to.

Let us begin by considering the expression

$$a_{ik}\,dx_i\,dx_k,$$

which is briefly written for the sum of n^2 such terms, obtained by giving to i, k independently the values 1, 2, ..., n.

The coefficients a_{ik} $(= a_{ki})$ are at present arbitrarily assigned functions of the variables $x_1, x_2, \ldots, x_n$. If, for instance, $n = 2$, the expression is a short way of writing

$$a_{11}\,dx_1^2 + 2a_{12}\,dx_1\,dx_2 + a_{22}\,dx_2^2.$$

Let A_{ik} denote the minor of a_{ik} in the determinant

$$\begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{nn} \end{vmatrix}$$

divided by the determinant itself.

Let $$[ikt] \equiv \tfrac{1}{2}\left(\frac{\partial a_{it}}{\partial x_k} + \frac{\partial a_{kt}}{\partial x_i} - \frac{\partial a_{ik}}{\partial x_t}\right)$$

$$(rkih) \equiv \tfrac{1}{2}\left(\frac{\partial^2 a_{rh}}{\partial x_i \partial x_k} + \frac{\partial^2 a_{ik}}{\partial x_r \partial x_h} - \frac{\partial^2 a_{ir}}{\partial x_h \partial x_k} - \frac{\partial^2 a_{kh}}{\partial x_r \partial x_i}\right)$$
$$+ A_{\lambda\mu}\left([rh\mu][ik\lambda] - [ri\mu][hk\lambda]\right).$$

Just as in the expression

$$a_{ik}\,dx_i\,dx_k,$$

the law of the notation is that wherever a suffix which occurs in one factor of a product is repeated in another factor [as in the expression $(rkih)$ the suffixes λ and μ are repeated] the sum of n^2 such terms are to be taken by giving to the suffixes independently the values 1, 2, ..., n.

We shall only be dealing with the case of $n = 4$; but even in this case, and for fixed values of r, k, i, h, the full expression of the symbol $(rkih)$ would involve 10 expressions of the form

$$A_{\lambda\mu}([rh\mu][ik\lambda]-[ri\mu][hk\lambda]),$$

were it not for the convention introduced; one is therefore soon converted to a belief in its utility.

But there are 256 symbols $(rkih)$, as the integers take independently the values 1, 2, 3, 4, and the need of further contraction is obvious. Happily the 256 symbols can be shown to be simply expressible in terms of 21 two index symbols a_{pq}.

We pass from the four index symbol $(rkih)$ to the two index one a_{pq} by taking 23, in this order, as 1, 31 as 2, 12 as 3, 14 as 4, 24 as 5, 34 as 6, and we notice that $a_{ik} = a_{ki}$. We can easily show that

$$a_{14}+a_{25}+a_{36} \equiv 0,$$

so that in effect our 256 symbols reduce to 20.

Written at length we have

$$(2323) = a_{11},\quad (2331) = a_{12},\quad (2312) = a_{13},\quad (2314) = a_{14},\quad (2324) = a_{15},$$
$$(2334) = a_{16},\quad (3131) = a_{22},\quad (3112) = a_{23},\quad (3114) = a_{24},\quad (3124) = a_{25},$$
$$(3134) = a_{26},\quad (1212) = a_{33},\quad (1214) = a_{34},\quad (1224) = a_{35},\quad (1234) = a_{36},$$
$$(1414) = a_{44},\quad (1424) = a_{45},\quad (1434) = a_{46},\quad (2424) = a_{55},\quad (2434) = a_{56},$$
$$(3434) = a_{66}.$$

From the point x_1, x_2, x_3, x_4 in our four way space we think of two directions going out into this space given by

$$\frac{dx_1}{\xi_1'} = \frac{dx_2}{\xi_2'} = \frac{dx_3}{\xi_3'} = \frac{dx_4}{\xi_4'}$$

and

$$\frac{\partial x_1}{\xi_1''} = \frac{\partial x_2}{\xi_2''} = \frac{\partial x_3}{\xi_3''} = \frac{\partial x_4}{\xi_4''}.$$

Here $\xi_1', \xi_2', \xi_3', \xi_4'$, and $\xi_1'', \xi_2'', \xi_3'', \xi_4''$ are arbitrarily assigned functions of the coordinates x_1, x_2, x_3, x_4, and we may usefully regard them as the coordinates of two points in ordinary Euclidean space.

Let

$$p_1 = dx_2\partial x_3 - dx_3\partial x_2,\quad p_2 = dx_3\partial x_1 - dx_1\partial x_3,\quad p_3 = dx_1\partial x_2 - dx_2\partial x_1,$$
$$p_4 = dx_1\partial x_4 - dx_4\partial x_1,\quad p_5 = dx_2\partial x_4 - dx_4\partial x_2,\quad p_6 = dx_3\partial x_4 - dx_4\partial x_3.$$

We have $$p_1p_4+p_2p_5+p_3p_6 = 0,$$
and may therefore look on $p_1, p_2, p_3, p_4, p_5, p_6$ as the six coordinates of a line in ordinary space.

Consider now the expression
$$a_{ik}p_ip_k.$$
When we regard the point x_1, x_2, x_3, x_4 as fixed and equate this expression to zero, we have the most general quadratic complex of lines in ordinary space. We therefore call the expression the first complex.

The expression formed by expanding
$$(a_{ik}dx_idx_k)(a_{\lambda\mu}\partial x_\lambda\partial x_\mu)-(a_{ik}dx_i\partial x_k)(a_{\lambda\mu}dx_\lambda\partial x_\mu)$$
is written $$\beta_{ik}p_ip_k,$$
and called the second complex.

The ratio of the first complex to the second is what Riemann calls the measure of curvature of the space x_1, x_2, x_3, x_4.

The meaning of this measure of curvature I will now try to explain. The six coordinates $p_1, p_2, p_3, p_4, p_5, p_6$ are proportional to the six coordinates of the line in ordinary space, joining the point whose homogeneous coordinates are $\xi'_1\xi'_2\xi'_3\xi'_4$ and $\xi''_1\xi''_2\xi''_3\xi''_4$. Regarding for the moment x_1, x_2, x_3, x_4 as fixed, and also these two points as fixed, we consider the single infinity of rays in the four way space whose directions are proportional to
$$\xi'_1a+\xi''_1\beta,\quad \xi'_2a+\xi''_2\beta,\quad \xi'_3a+\xi''_3\beta,\quad \xi'_4a+\xi''_4\beta.$$
We then construct in the four way space the geodesic two way surface, made up of the single infinity of geodesic lines issuing from the point $x_1x_2x_3x_4$ in the directions given by
$$\frac{dx_1}{\xi'_1a+\xi''_1\beta}=\frac{dx_2}{\xi'_2a+\xi''_2\beta}=\frac{dx_3}{\xi'_3a+\xi''_3\beta}=\frac{dx_4}{\xi'_4a+\xi''_4\beta}.$$
Here the ratio $a:\beta$ is a mere parameter, and thus the geodesic surface is formed and may be taken to have the element of length on it given by
$$ds^2 = edu^2+2f\,du\,dv+g\,dv^2,$$
where u and v are the Gaussian coordinates which may be taken, say, to be zero at the point from which the geodesics issue forth. The curvature of this geodesic two way surface will express itself in terms of e, f, g and their derivatives: its value at the point x_1, x_2, x_3, x_4 and for the orienta-

tion given by the six coordinates $p_1, p_2, p_3, p_4, p_5, p_6$ is Riemann's measure of the curvature of the space x_1, x_2, x_3, x_4.

Let us now consider the effect of transforming

$$a_{ik}\,dx_i\,dx_k$$

to new variables.

When it is said that this expression is invariant for all transformations it is not meant that the functions a_{ik} preserve their form as functions of x_1, x_2, x_3, x_4. Rather, as in the theory of linear transformation, we regard it as the fundamental quantic.

It can be proved that the two complexes are absolute invariants. This fact is of fundamental importance in the theory of quadratic forms.

It may be verified that the first complex is such an invariant by considering the infinitesimal transformation

$$x_1' = x_1 + t\xi, \quad x_2' = x_2, \quad x_3' = x_3, \quad x_4' = x_4,$$

where t is a small constant, and ξ any arbitrary function of the variables $x_1 x_2 x_3 x_4$. We then have, from the fundamental invariant,

$$a_{ik}' = a_{ik} - t\left(a_{1k}\frac{\partial \xi}{\partial x_i} + a_{1i}\frac{\partial x}{\partial x_k}\right),$$

and by aid of this set of equations it can be proved that the first complex is invariant for this infinitesimal transformation, and, therefore for every transformation.

It is a much simpler matter to prove that the second complex is an invariant.

Riemann's measure of curvature is thus independent of any particular system of coordinates.

Define now a system of functions G_{ik} thus

$$G_{rh} \equiv A_{ki}\,(rkih),$$

where the suffixes ki on the right, being found in two factors of the product, are to indicate summation of the products in accordance with the law explained earlier. It is easily proved that

$$G_{ik}\,dx_i\,dx_k$$

is an absolute invariant.

Einstein's law of gravitation just asserts that this invariant vanishes identically. I only mention this now in passing and will return to it later.

Next let us consider the invariant determinantal equation in λ,

$$a\lambda \equiv \begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & a_{14}-\lambda, & a_{15}, & a_{16} \\ a_{21}, & a_{22}, & a_{23}, & a_{24}, & a_{25}-\lambda, & a_{26} \\ a_{31}, & a_{32}, & a_{33}, & a_{34}, & a_{35}, & a_{36}-\lambda \\ a_{41}-\lambda, & a_{42}, & a_{43}, & a_{44}, & a_{45}, & a_{46} \\ a_{51}, & a_{52}-\lambda, & a_{53}, & a_{54}, & a_{55}, & a_{56} \\ a_{61}, & a_{62}, & a_{63}-\lambda, & a_{64}, & a_{65}, & a_{66} \end{vmatrix} = 0.$$

Take

$$\begin{matrix} m_{11}, & m_{12}, & m_{13}, & m_{14}, & m_{15}, & m_{16}, \\ m_{21}, & m_{22}, & m_{23}, & m_{24}, & m_{25}, & m_{26}, \\ m_{31}, & m_{32}, & m_{33}, & m_{34}, & m_{35}, & m_{36}, \\ m_{41}, & m_{42}, & m_{43}, & m_{44}, & m_{45}, & m_{46}, \\ m_{51}, & m_{52}, & m_{53}, & m_{54}, & m_{55}, & m_{56}, \\ m_{61}, & m_{62}, & m_{63}, & m_{64}, & m_{65}, & m_{66}, \end{matrix}$$

to be six sets of first minors of any row corresponding to the six roots of $a\lambda = 0$.

We can prove that, for any values of i and j,

$$m_{i1}m_{j4}+m_{i4}m_{j1}+m_{i2}m_{j5}+m_{i5}m_{j2}+m_{i3}m_{j6}+m_{i6}m_{j3} = 0.$$

Now the terms in m which we have written down are such that, if we take the m's in any row as the coordinates of a linear complex, we get six linear complexes corresponding to the six roots of $a\lambda = 0$; and these six complexes are naturally in involution.

But if we choose κ so that

$$(m_{i1}+\kappa m_{j1})(m_{i4}+\kappa m_{j4})+(m_{i2}+\kappa m_{j2})(m_{i5}+\kappa m_{j5})+(m_{i3}+\kappa m_{j3})(m_{i6}+\kappa m_{j6}) = 0,$$

we obtain what we may take as the "six coordinates" of two lines in ordinary Euclidean space. In saying this we look on x_1, x_2, x_3, x_4 as fixed, and therefore a_{ik} ... as fixed, and we say that these two lines correspond to the roots λ_i and λ_j of $a\lambda = 0$.

If we divide the roots into three pairs we will have as corresponding lines three pairs of lines; each line of any pair intersecting each line of any other pair, but not the line of its own pair. These lines will then form the six edges of a tetrahedron in ordinary space. We may call this the tetrahedron which corresponds to x_1, x_2, x_3, x_4.

Suppose the coordinates of one of the vertices of this tetrahedron are, say, $(\alpha_1 \beta_1 \gamma_1 \delta_1)$, then—no longer regarding x_1, x_2, x_3, x_4 as fixed, and expressing, as we may, α_1, β_1, γ_1, δ_1 in terms of the functions a_{ik}—the curve

$$\frac{dx_1}{\alpha_1} = \frac{dx_2}{\beta_1} = \frac{dx_3}{\gamma_1} = \frac{dx_4}{\delta_1}$$

is invariant for any transformation which leaves

$$a_{ik} dx_i dx_k$$

invariant.

Corresponding to the tetrahedron of the point x_1, x_2, x_3, x_4, we have therefore four invariant curves going out from the point in our four way space.

So far, although the language of geometry has been used in speaking of curvature, geodesics, complexes, lines and points, this language is merely a convenient artifice to avoid prolixity.

We shall now try to make a little more use of our conceptions of time and space. We look on x_1, x_2, x_3, x_4 as functions of any space coordinates and the time. I mean by space coordinates simply numbers which fix for me the position of a point in space at a given time. I do not think I introduce any new ideas as to what space is. I imagine a triply infinite system of surfaces drawn in this space at any given time, and describe the position of the point by giving the numbers attached to the surfaces which pass through the point. The time coordinate I cannot express otherwise than just as the time coordinate. When I say that two points are "near" one another, I mean that their four coordinates of space and time differ only by small quantities, and I do not attempt any greater accuracy of expression.

We take $$\delta s^2 = a_{ik} dx_i dx_k,$$

and call δs the space-time interval element. Because of the arbitrary nature of the functions a_{ik} of x_1, x_2, x_3, x_4 this space-time interval does not depend merely on the coordinates of the two near points. It depends also on our choice of the functions a_{ik}. That choice may depend on what we want to measure. Thus, to take a very homely illustration which will perhaps explain my meaning, suppose we want what we may call the practical distance between two points in London from the point of view of the man who has to walk from one to the other. Here we need to use only two coordinates, say, x_1 and x_2. The practical distance would be

measured by the minimum value of the integral

$$\delta = \int \sqrt{(a_{11}\,dx_1^2 + 2a_{12}\,dx_1\,dx_2 + a_{22}\,dx_2^2)}$$

taken between the two points. If the two extreme points lie on a horizontal plane, and if there were no obstacles in the intervening path, we could take the integral to be

$$\int \sqrt{(dx_1^2 + dx_2^2)}$$

by proper choice of coordinates and obtain the straight line path. But generally this would not be possible, and the a_{11}, a_{12}, a_{22} would be functions of x_1 and x_2 depending on the traffic at that point. And even if the traffic were to die away and the obstacles in the way of buildings were to disappear, the surface over which the pedestrian would have to travel between the two points might be a curved one, and the a_{11}, a_{12}, a_{22} would still remain as functions of x_1 and x_2. The path he should take would be the geodesic on this surface. The practical distance between the points might remain the same in the two cases, but in the second case we should look on his problem as one merely of geometry.

Now suppose that the inhabitant of some distant planet, if he exists, observed this pedestrian he might give either interpretation to what he observed. He might regard London as a portion of a surface and think the path a geodesic on it, or he might regard the pedestrian as a particle moving under the action of some force. But he would have no such simple law to explain the motion he observed as that which Einstein has formulated to explain gravitation; and Einstein would interpret what is observed as a property of space time, and not of force influencing the path of the particle.

In taking
$$ds^2 = a_{ik}\,dx_i\,dx_k,$$
where a_{ik} is any function of the coordinates as the element of length in some four way space, the geometer is guided, I think, by analogy merely. He knows of two way surfaces in ordinary space where the element of length is given by
$$ds^2 = e\,du^2 + 2f\,du\,dv + g\,dv^2,$$
and e, f, g are functions of u and v.

The geometry on this surface is, in general, quite different from that of plane Euclidean geometry, though he was led to it by considering Euclidean space. He is not concerned as a mathematician with the question whether his abstract Euclidean space is, or is not, the space of ex-

perience, though he knows that at all events it must be very nearly so. So, in taking the space-time interval element as given by

$$\delta s^2 = a_{ik} dx_i dx_k,$$

he is not primarily concerned with the question whether this is built up from the space and time continuum of experience itself, or whether the functions a_{ik} arise in some other way.

He can conceive a four-fold continuum of space and time, but he can only conceive of a four way space by analogy and by his power of imagining the plight of a mathematician to whom experience had only awakened a perception of two way space.

So far no special hypothesis has been made with respect to the functions a_{ik}.

Einstein's law of gravitation is the hypothesis that

$$G_{ik} dx_i dx_k$$

vanishes identically; that is, that G_{ik} is zero. Its naturalness in the study of differential geometry could hardly be exaggerated. The wonderful thing is that it should tell us about facts of the universe.

It gives us a number of differential equations to determine a_{ik}, ..., and when these are determined the path of a particle in empty space will be that which makes

$$\int \sqrt{(a_{ik} dx_i dx_k)}$$

stationary.

But the form given for the gravitation field due to a particle at rest, viz.

$$-\gamma^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta \, d\phi^2 + \gamma dt^2,$$

where $$\gamma = 1 - \frac{2m}{r},$$

is not deduced merely from the gravitation law; other considerations are brought in; and I have a difficulty in seeing the justification of the use made of polar coordinates when Euclidean geometry has been rejected.

I now introduce an hypothesis which will be satisfied by this form, but which, I think, arises more naturally out of the form

$$\delta s^2 = a_{ik} dx_i dx_k,$$

from which our geometry of Einstein is to be built up.

The hypothesis I make is that the four curves which go out from any point, and which we have seen are invariant for any choice of coordinates,

may be taken as the intersections of three of a quadruply infinite system of hyper-surfaces—here three way loci.

This hypothesis distinctly limits the functions a_{ik}, and if it is to be applied in conjunction with Einstein's hypothesis it distinctly limits the law of gravitation.

First we take this hypothesis apart from Einstein's and consider what it leads to just as a geometrical one.

We can take the quadruply infinite system of surfaces to be

$$x_1 = \text{constant}, \quad x_2 = \text{constant}, \quad x_3 = \text{constant}, \quad x_4 = \text{constant},$$

and we call this coordinate system the normal one. Referred to normal coordinates the four invariant curves have the equations

$$dx_2 = dx_3 = dx_4 = 0, \quad dx_1 = dx_3 = dx_4 = 0, \quad dx_1 = dx_2 = dx_4 = 0,$$
$$dx_1 = dx_2 = dx_3 = 0.$$

It follows that the vertices of what we called the tetrahedron of the point $x_1x_2x_3x_4$ will be the point whose coordinates are

$$\begin{matrix} 1, & 0, & 0, & 0, \\ 0, & 1, & 0, & 0, \\ 0, & 0, & 1, & 0, \\ 0, & 0, & 0, & 1, \end{matrix}$$

and the "six coordinates" of its edges will be

$$\begin{matrix} 1, & 0, & 0, & 0, & 0, & 0, \\ 0, & 1, & 0, & 0, & 0, & 0, \\ 0, & 0, & 1, & 0, & 0, & 0, \\ 0, & 0, & 0, & 1, & 0, & 0, \\ 0, & 0, & 0, & 0, & 1, & 0, \\ 0, & 0, & 0, & 0. & 0, & 1. \end{matrix}$$

It may be proved that consequentially all the coefficients in the first complex vanish except

$$a_{11}, \quad a_{22}, \quad a_{33}, \quad a_{44}, \quad a_{55}, \quad a_{66}, \quad a_{14}, \quad a_{25}, \quad a_{36}.$$

In other words, every symbol $(rkih)$ vanishes, in which three and only three distinct indices occur.

It is worth noting in connection with the hypothesis I have made that the necessary and sufficient condition that by a change of variables

$$a_{ik}\,dx_i\,dx_k$$

can be brought to the form

$$dx_1^2+dx_2^2+dx_3^2+dx_4^2,$$

that is, to flat or Euclidean space, is the identical vanishing of *all* the symbols $(rkih)$, taken with respect to *any* coordinate system, whilst this particular hypothesis requires the vanishing of every symbol in which three, and only three, of the indices are distinct when referred to the normal coordinate system.*

We now combine this hypothesis with Einstein's law of gravitation. We have

$$\begin{aligned}
G_{11} &= -a_{33}A_{22}-a_{22}A_{33}-a_{44}A_{44}, & G_{22} &= -a_{33}A_{11}-a_{11}A_{33}-a_{55}A_{44},\\
G_{33} &= -a_{22}A_{11}-a_{11}A_{22}-a_{66}A_{44}, & G_{44} &= -a_{44}A_{11}-a_{55}A_{22}-a_{66}A_{33},\\
G_{23} &= a_{11}A_{23}+(a_{36}-a_{25})A_{14}, & G_{14} &= (a_{36}-a_{25})A_{23}+a_{44}A_{14},\\
G_{31} &= a_{22}A_{31}+(a_{14}-a_{36})A_{24}, & G_{24} &= (a_{14}-a_{36})A_{31}+a_{55}A_{24},\\
G_{12} &= a_{33}A_{12}+(a_{25}-a_{14})A_{34}, & G_{34} &= (a_{25}-a_{14})A_{12}+a_{66}A_{34}.
\end{aligned}$$

Since all the functions G_{ik} vanish for a non-vanishing set of A_{ik}, ..., we see that the product of the determinants

$$\begin{vmatrix} 0, & -a_{33}, & -a_{22}, & -a_{44} \\ -a_{33}, & 0, & -a_{11}, & -a_{55} \\ -a_{22}, & -a_{11}, & 0, & -a_{66} \\ -a_{44}, & -a_{55}, & -a_{66}, & 0 \end{vmatrix} \begin{vmatrix} a_{11}, & a_{36}-a_{25} \\ a_{36}-a_{25}, & a_{44} \end{vmatrix} \begin{vmatrix} a_{22}, & a_{14}-a_{36} \\ a_{14}-a_{36}, & a_{55} \end{vmatrix} \begin{vmatrix} a_{33}, & a_{25}-a_{36} \\ a_{25}-a_{36}, & a_{66} \end{vmatrix}$$

must be zero.

If we go back to the determinantal equation $a\lambda = 0$, we see that the sum of its six roots is zero; and further, now that the first complex takes a simpler form, the roots are given by

$$a_{11}a_{44} = (a_{14}-\lambda)^2, \quad a_{22}a_{55} = (a_{25}-\lambda)^2, \quad a_{33}a_{66} = (a_{36}-\lambda)^2.$$

It can be shown that the determinant product just considered is the

* I owe the correction of a misleading statement here to Prof. Eddington's remark that the symbols in which three of the indices are distinct are not components of a tensor.

square root of the product of the sum of every set of three of the roots of the determinantal equation $a\lambda = 0$.

The sum of some three of these roots is therefore zero.

Excluding the case of equality between any of the roots—a possibility which would repay investigation as it actually occurs in the form given for the gravitation field due to a particle at rest—we can arrange that one of the three roots, whose sum is zero, is found in each of the sets

$$a_{11}a_{44} = (a_{14}-\lambda)^2, \quad a_{22}a_{55} = (a_{25}-\lambda)^2, \quad a_{33}a_{66} = (a_{36}-\lambda)^2.$$

It will be found to follow that none of the determinants

$$\begin{vmatrix} a_{11}, & a_{36}-a_{25} \\ a_{36}-a_{25}, & a_{44} \end{vmatrix} \quad \begin{vmatrix} a_{22}, & a_{14}-a_{36} \\ a_{14}-a_{36}, & a_{55} \end{vmatrix} \quad \begin{vmatrix} a_{33}, & a_{25}-a_{36} \\ a_{25}-a_{36}, & a_{66} \end{vmatrix}$$

can vanish, and therefore we must have

$$A_{23} = A_{14} = A_{13} = A_{24} = A_{12} = A_{34} = 0.$$

The space-time interval must thus take the form

$$a^2\partial x_1^2 + b^2\partial x_2^2 + c^2\partial x_3^2 + d^2\partial x_4^2,$$

and for this form we easily verify that

$$a_{14} = a_{25} = a_{36} = 0.$$

The functions a, b, c, d are no longer arbitrary functions of the variables x_1, x_2, x_3, x_4: the fact that all symbols $(rkih)$, in which at least three of the indices are distinct, must vanish, gives us

$$a_{23} = a_2\frac{b_3}{b} + a_3\frac{c_2}{c}, \quad a_{34} = a_3\frac{c_4}{c} + a_4\frac{d_3}{d}, \quad a_{42} = a_4\frac{d_2}{d} + a_2\frac{b_4}{b},$$

$$b_{31} = b_3\frac{c_1}{c} + b_1\frac{a_3}{a}, \quad b_{14} = b_1\frac{a_4}{a} + b_4\frac{d_1}{d}, \quad b_{43} = b_4\frac{d_3}{d} + b_3\frac{c_4}{c},$$

$$c_{12} = c_1\frac{a_2}{a} + c_2\frac{b_1}{b}, \quad c_{24} = c_2\frac{b_4}{b} + c_4\frac{d_2}{d}, \quad c_{41} = c_4\frac{d_1}{d} + c_1\frac{a_4}{a},$$

$$d_{23} = d_2\frac{b_3}{b} + d_3\frac{c_2}{c}, \quad d_{31} = d_3\frac{c_1}{c} + d_1\frac{a_3}{a}, \quad d_{12} = d_1\frac{a_2}{a} + d_2\frac{b_1}{b},$$

where with respect to a, b, c, d, and only with respect to these functions, suffixes denote differentiation with respect to x_1, x_2, x_3, x_4.

We have not yet used the differential equations arising from the conditions

$$G_{11} = G_{22} = G_{33} = G_{44} = 0,$$

and we now proceed to obtain them.

The first complex has reduced to

$$a_{11}p_1^2+a_{22}p_2^2+a_{33}p_3^2+a_{44}p_4^2+a_{55}p_5^2+a_{66}p_6^2,$$

the second to

$$b^2c^2p_1^2+c^2a^2p_2^2+a^2b^2p_3^2+a^2d^2p_4^2+b^2d^2p_5^2+c^2d^2p_6^2.$$

Recalling the explanation of Riemann's measure of curvature, how it depended, not merely on the point at which it was taken, but also on the six coordinates of its line of orientation, we take the curvatures corresponding to the six edges of the tetrahedron of the point x_1, x_2, x_3, x_4.

Let K_{23} be the measure of curvature corresponding to the edge BC of this tetrahedron $ABCD$. The other curvatures will be K_{31} corresponding to CA, K_{12} to AB, K_{14} to AD, K_{24} to BD, and K_{34} to CD.

The first complex may now be written

$$b^2c^2K_{23}p_1^2+c^2a^2K_{31}p_2^2+a^2b^2K_{12}p_3^2+a^2d^2K_{14}p_4^2+b^2d^2K_{24}p_5^2+c^2d^2K_{34}p_6^2,$$

and we have

$$bcK_{23}+\left(\frac{b_3}{c}\right)_3+\left(\frac{c_2}{b}\right)_2+\frac{b_1c_1}{a^2}+\frac{b_4c_4}{d^2}=0,$$

$$caK_{31}+\left(\frac{c_1}{a}\right)_1+\left(\frac{a_3}{c}\right)_3+\frac{a_2c_2}{b^2}+\frac{a_4c_4}{d^2}=0,$$

$$abK_{12}+\left(\frac{a_2}{b}\right)_2+\left(\frac{b_1}{a}\right)_1+\frac{a_3b_3}{c^2}+\frac{a_4b_4}{d^2}=0,$$

$$adK_{14}+\left(\frac{a_4}{d}\right)_4+\left(\frac{d_1}{a}\right)_1+\frac{a_2d_2}{b^2}+\frac{a_3d_3}{c^2}=0,$$

$$bdK_{24}+\left(\frac{b_4}{d}\right)_4+\left(\frac{d_2}{b}\right)_2+\frac{b_1d_1}{a^2}+\frac{b_3d_3}{c^2}=0,$$

$$cdK_{34}+\left(\frac{c_4}{d}\right)_4+\left(\frac{d_3}{c}\right)_3+\frac{c_1d_1}{a^2}+\frac{c_2d_2}{b^2}=0.$$

The Einstein conditions

$$G_{11}=G_{22}=G_{33}=G_{44},$$

now give

$$K_{14}=K_{23},\quad K_{24}=K_{13},\quad K_{34}=K_{12},\quad K_{23}+K_{31}+K_{12}=0,$$

that is, "The sum of the curvatures which correspond to the edges of the tetrahedron meeting at any vertex is zero."

We thus have four further differential equations which must be satis-

fied by a, b, c, d: that is, sixteen in all of the second order if we are to combine Einstein's law of gravitation with the hypothesis I have made.

I have not worked out the consequences of the consistency of the sixteen equations of the second order to be satisfied by the four functions a, b, c, d. Perhaps the first step ought to be to eliminate these four functions, and thus to obtain any further consequential relations between the derivatives of two of the curvatures, say K_{12} and K_{13}, in terms of which the others can be expressed.

In the expression given for the gravitational field due to a particle at rest $K_{12} = K_{13}$, and thus the more general hypothesis, of a functional relation between them, is suggested as something that might lead to a new geometrical problem.

In a further study of the differential equations one might be led to connect them with other problems in geometry, in a manner which will be familiar to those who have made a study of similar problems in differential geometry of ordinary Euclidean space.

In the gravitational field given for a particle at rest $K_{12} = m/r^3$, so that as we increase our distance from the particle the geometry of space tends to flatness or zero curvature.

SOME PROBLEMS OF DIOPHANTINE APPROXIMATION: THE LATTICE-POINTS OF A RIGHT-ANGLED TRIANGLE

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1. *Introduction.*

1.1. The problem considered in this paper may be stated as follows.

Suppose that ω and ω' are two positive numbers whose ratio $\theta = \omega/\omega'$ is irrational; and denote by Δ the triangle whose sides are the coordinate axes and the line

$$(1.11) \qquad \omega x + \omega' y = \eta > 0,$$

and by $N(\eta)$ the number of lattice-points* which lie inside Δ. *How accurate an approximation can we find for $N(\eta)$ when η is large? And how does the accuracy of the approximation depend upon the arithmetic character of θ?* We call this problem *Problem A.*

Such "lattice-point" problems are, in general, very difficult. It is enough to recall the two most famous of them, the *problem of the circle* (the problem of Gauss and Sierpinski), and the *problem of the rectangular hyperbola* (Dirichlet's divisor problem), both of which have been the subject of numerous researches during the last ten years. The particular problem which we consider here has not, so far as we know, been stated quite in this form before. It is however easily brought into connection with another problem which has attracted a certain amount of attention, and which has been considered, from varying points of view, by Lerch,† by Weyl,‡ and by ourselves.§ This problem, which we shall call

* A lattice-point (*Gitterpunkt*) is a point whose coordinates x and y are both integral.

† M. Lerch, *l'Intermédiaire des Mathématiciens*, Vol. 11 (1904), pp. 145–146 (Question 1547).

‡ H. Weyl, "Über die Gleichverteilung von Zahlen mod. Eins", *Math. Annalen*, Vol. 77 (1916), pp. 313–352.

§ G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation", *Proceedings of the fifth international congress of mathematicians*, Cambridge, 1912, Vol 1, pp. 223–229.

Problem B, is as follows. Suppose that, as usual, $[x]$ denotes the integral part of x, and that

(1.12) $$\{x\} = x-[x]-\tfrac{1}{2}.$$

Then *what is the most that can be said as to the order of magnitude of*

(1.13) $$s(\theta, n) = \sum_{\nu=1}^{n} \{\nu\theta\}$$

when n is large?

1.2. We begin, in § 2, by proving the formula which establishes the connection between Problems A and B, and shows that the first problem is a generalised and more symmetrical form of the second. We prove in fact that

(1.21) $$N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + S(\eta),$$

where $S(\eta)$ is a sum very similar to the sum 1.13.

It is trivial that

(1.211) $$N(\eta) = \frac{\eta^2}{2\omega\omega'} + O(\eta),$$

the area of the triangle, together with an error of the order of the perimeter. The second and third terms of (1.21) occur naturally when we consider, instead of Δ, the similar and similarly situated triangle whose vertex is at (1, 1) instead of the origin; for the area of this triangle is

$$\frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + \frac{1}{2}.$$

But no closer approximation than (1.211) is in any way trivial; and, when θ is rational, $S(\eta)$ is effectively of order η, so that a universal formula, professing to be more precise than (1.211), would necessarily be false.

In § 3 we deduce transformation formulæ for N and S, which are generalisations of a formula given without proof by Lerch, and which enable us to study these sums by means of the expression of θ as a simple continued fraction. In § 4 we prove (a) that

(1.22) $$S(\eta) = o(\eta)$$

for any irrational θ, and (b) that (1.22) is the most that is true for every such irrational. Incidentally we obtain the corresponding results concerning Problem B: the first of them at any rate is in this case familiar.

In § 5 we consider more closely cases in which the rate of increase of the quotients in the continued fraction is comparatively slow, and in particular the case in which they are bounded; and we prove that in this case

$$S(\eta) = O(\log \eta), \tag{1.23}$$

and that this result too is a best possible result of its kind. There are naturally analogous results for Problem B; that corresponding to (1.23) was stated as a new theorem in our communication to the Cambridge congress, but had, as was pointed out to us by Prof. Landau, been given already by Lerch.

Up to this point our argument is entirely elementary, and both methods and results are of a kind to be found in our previous papers on Diophantine approximation or in those of other writers. We have therefore aimed at the maximum of compression and have omitted a good deal of elementary algebraical calculation. The concluding section (§ 6) is more novel. In it we prove that *if θ is algebraic then*

$$S(\eta) = O(\eta^a), \tag{1.24}$$

where $a < 1$. This result is unlike any which we have been able to prove before, and is obtained by entirely different methods, based on the properties of the analytic function

$$\zeta_2(s, a, \omega, \omega') = \sum_{m,\, n=0}^{\infty} \frac{1}{(a+m\omega+n\omega')^s}. \tag{1.25}$$

This function will be recognised as a degenerate form of the "Double Zeta-function" introduced into analysis by Dr. Barnes.*

2. *Reduction of Problem* A.

2.1. We write

$$\frac{\eta}{\omega} = \left[\frac{\eta}{\omega}\right]+f, \quad \frac{\eta}{\omega'} = \left[\frac{\eta}{\omega'}\right]+f', \tag{2.11}$$

where $$0 \leqslant f < 1, \quad 0 \leqslant f' < 1.$$

* E. W. Barnes, "A memoir on the Double-Gamma-function", *Phil. Trans. Roy. Soc.*, (A), Vol. 196 (1901), pp. 265–387; see in particular pp. 314–349. For a study of some of the properties of the degenerate function (for which the ratio ω/ω' is real) see G. H. Hardy, "On double Fourier series, and in particular those which represent the double Zeta-function with real and incommensurable parameters", *Quarterly Journal*, Vol. 37 (1906), pp. 53–79.

Suppose first that there is no lattice-point on the line (1.11), or AB of the figure. Then the number of lattice-points inside OAB is

$$(2.12)\qquad N(\eta) = \sum_{\mu=1}^{\eta/\omega} \left[\frac{\eta-\mu\omega}{\omega'}\right] = \left[\frac{\eta}{\omega}\right]\left[\frac{\eta}{\omega'}\right] + \sum_{\mu=1}^{\eta/\omega} [f'-\mu\theta],$$

Now $[-x] = -[x]-1+\epsilon_x$, where ϵ_x is 1 or 0 according as x is or is not an integer; and $\mu\theta-f'$ cannot be an integer, since then $\eta-\mu\omega$ would be an integral multiple of ω' and there would be a lattice-point on AB. Thus

$$(2.13)\qquad [f'-\mu\theta] = -[\mu\theta-f']-1 = -(\mu\theta-f')+\{\mu\theta-f'\}-\tfrac{1}{2}.$$

Substituting into (2.12), and using (2.11), we obtain, after a little reduction

$$(2.14)\qquad N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + \phi + S(\eta),$$

where

$$(2.141)\qquad \phi = \tfrac{1}{2}f + \tfrac{1}{2}\theta f(1-f)$$

and

$$(2.142)\qquad S(\eta) = \sum_{\mu=1}^{\eta/\omega} \{\mu\theta-f'\}.$$

Since ϕ is bounded, the problem is reduced, substantially, to the discussion of $S(\eta)$.

The preceding argument requires a trifling modification when there is a lattice-point on AB; there cannot be more than one, since θ is irrational. In this case the sum (2.12) gives $N(\eta)+1$ instead of $N(\eta)$. There is one value of μ for which $\mu\theta-f'$ is integral, and for this μ the $-\frac{1}{2}$ in (2.13) is changed into $\frac{1}{2}$. The net result is to leave the final formulæ unchanged.

3. *The Transformation Formulæ.*

3.1. In order to obtain a formula for the transformation of $N(\eta)$ or of $S(\eta)$, we employ the familiar device of adding together the number of lattice-points of the triangles OAB, $O'A'B'$ of the figure.

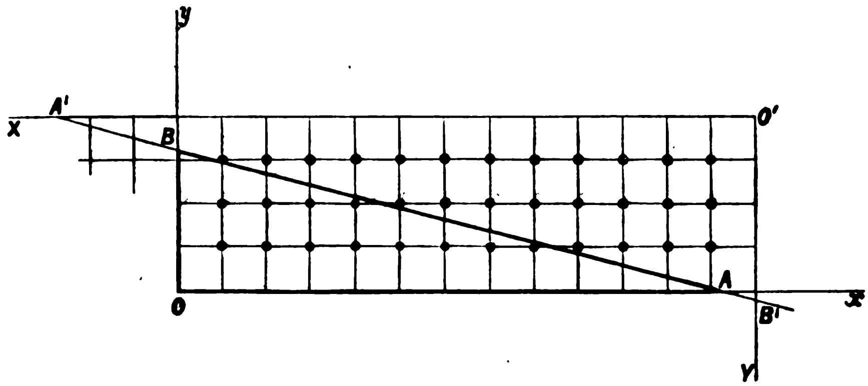
If we take new axes $O'X$, $O'Y$, as shown in the figure, it is plain that

$$x+Y = \left[\frac{\eta}{\omega}\right]+1, \quad X+y = \left[\frac{\eta}{\omega'}\right]+1;$$

and the equation of AB, referred to the new axes, is

(3.11) $$\omega' X + \omega Y = \eta + \omega(1-f) + \omega'(1-f') = H,$$

say. Repeating the arguments of § 2, we find, for the number $N'(H)$ of



lattice-points of $O'A'B'$,

(3.12) $$N'(H) = \frac{H^2}{2\omega\omega'} - \frac{H}{2\omega} - \frac{H}{2\omega'} + \Phi + S'(H),$$

where

(3.121) $$\Phi = \tfrac{1}{2}F' + \frac{F'(1-F')}{2\theta}$$

and

(3.122) $$S'(H) = \sum_{\nu=1}^{H/\omega'} \left\{ \frac{\nu}{\theta} - F \right\},$$

F and F' being defined by

(3.123) $$\frac{H}{\omega} = \left[\frac{H}{\omega}\right] + F, \quad \frac{H}{\omega'} = \left[\frac{H}{\omega'}\right] + F', \quad 0 \leqslant F < 1, \quad 0 \leqslant F' < 1.$$

3.2. We suppose now that $\omega < \omega'$, $\theta < 1$. A glance at the figure shows that

$$\left[\frac{H}{\omega'}\right] = \left[\frac{\eta}{\omega'}\right] + 1.$$

Substituting for H in terms of η, from (3.11), we find at once that

(3.21) $$F' = \theta(1-f).$$

The same argument shows that

(3.22) $$F = \frac{1-f'}{\theta} - p,$$

where p is an integer; it happens that the value of p is not material to the argument.

It is also clear from the figure that

$$N(\eta)+N'(H) = \left[\frac{\eta}{\omega}\right]\left[\frac{\eta}{\omega'}\right]-\epsilon, \tag{3.23}$$

where ϵ is zero unless there is a lattice point on AB, and then unity. Substituting for $N(\eta)$ and $N'(H)$ from (2.14) and (3.12), using (2.11), (3.11), and (3.21), and reducing, we obtain, finally,

$$S+S'+\epsilon = -\tfrac{1}{2}+\tfrac{1}{2}(f+f')-\tfrac{1}{2}\theta f(1-f)+\frac{f'(1-f')}{2\theta}. \tag{3.24}$$

3.3. It is important, in view of Problem B, to show that this formula includes a formula given by Lerch.* Suppose then in particular that $\omega' = 1$, $\omega = \theta < 1$, and write

$$s = \sum_{1}^{n}\{\mu\theta\}, \quad s' = \sum_{1}^{m}\left\{\frac{\nu}{\theta}\right\}, \tag{3.31}$$

where m is the integral part of $n\theta$.

Starting with an arbitrary positive integral n, we write $n\theta = M+\delta$, where M is an integer and $0 < \delta < 1$, and take

$$\eta = M+1 = n\theta+1-\delta.$$

Then
$$f' = 0, \quad F \equiv \frac{1}{\theta} \pmod 1,$$

by (2.11) and (3.22); and there is no lattice point on AB, so that $\epsilon = 0$.

Suppose now that q is a positive integer and

$$q < \frac{1-\delta}{\theta} < q+1.\dagger$$

Then
$$\frac{\eta}{\theta} = n+\frac{1-\delta}{f} = n+q+f, \quad f = \frac{1-\delta}{\theta}-q.$$

Also $H = \eta+1+\theta(1-f)$ lies between $M+2$ and $M+3$. Hence

$$S' = \sum_{\nu=1}^{M+2}\left\{\frac{\nu-1}{\theta}\right\} = -\tfrac{1}{2}+\left\{\frac{M+1}{\theta}\right\}+s'; \tag{3.32}$$

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* M. Lerch, *loc. cit.*

† It is easy to see that $(1-\delta)/\theta$ cannot be integral.

and

(3.321) $$\left\{\frac{M+1}{\theta}\right\} = \left\{\frac{\eta}{\theta}\right\} = \left\{\frac{1-\delta}{\theta}\right\} = \frac{1-\delta}{\theta}-q-\tfrac{1}{2}.$$

Also

(3.33) $$S = \sum_{\mu=1}^{[\eta/\theta]} \{\mu\theta\} = \sum_{1}^{n+q} \{\mu\theta\} = s+\sum_{r=1}^{q} \{(n+r)\theta\} = s+S_0,$$

say. And $(n+1)\theta$, ..., $(n+q)\theta$ have all the integral part M, since $q\theta < 1-\delta < (q+1)\theta$. Hence

(3.34) $$S_0 = \sum_{r=1}^{q} (n\theta+r\theta-M-\tfrac{1}{2}) = \sum_{r=1}^{q} (r\theta+\delta-\tfrac{1}{2}) = \tfrac{1}{2}q(q+1)\theta+q(\delta-\tfrac{1}{2}).$$

Substituting from (3.32), (3.321), (3.33), and (3.34) into (3.24), and reducing, it will be found that

(3.35) $$s+s' = \tfrac{1}{2}\delta-\frac{\delta(1-\delta)}{2\theta},$$

which is the formula of Lerch.

4. *Results concerning an arbitrary irrational θ.*

4.1. THEOREM A1.—*If $\theta = \omega/\omega'$ is irrational, then*

$$N(\eta) = \frac{\eta^2}{2\omega\omega'}-\frac{\eta}{2\omega}-\frac{\eta}{2\omega'}+o(\eta).$$

We may clearly suppose that $\theta < 1$. Suppose that

(4.11) $$\theta = \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots,$$

(4.12) $$\theta = \frac{1}{a_1+\theta_1}, \quad \theta_1 = \frac{1}{a_2+\theta_2}, \quad \ldots.$$

We have, from (3.24),

(4.13) $$S+S' = O(1/\theta),$$

the constant of the O being independent of both η and θ.

We write $\eta = \omega\xi$, so that

$$\frac{H}{\omega'} = \xi\theta+\theta(1-f)+1-f' = \xi\theta+O(1),$$

and we write f_1 and μ_1 in S' instead of F' and ν. Then

$$S' = \sum_{\mu_1=1}^{H/\omega'} \left\{\frac{\mu_1}{\theta}-f_1\right\} = O(1)+\sum_{\mu_1=1}^{\xi\theta} \{\mu_1\theta_1-f_1\} = O(1)+S_1,$$

say; so that

$$(4.14)\qquad S = O(1/\theta) - S_1.$$

Similarly, we have

$$S_1 = O(1/\theta_1) - S_2, \qquad S_2 = O(1/\theta_2) - S_3, \qquad \ldots,$$

where S_2, S_3, ... are sums of the types

$$S_2 = \sum_{\mu_2=1}^{\xi\theta\theta_1} \{\mu_2\theta_2 - f_2\}, \qquad S_3 = \sum_{\mu_3=1}^{\xi\theta\theta_1\theta_2} \{\mu_3\theta_3 - f_3\}, \qquad \ldots,$$

so that
$$S_2 = O(\xi\theta\theta_1), \qquad S_3 = O(\xi\theta\theta_1\theta_2), \qquad \ldots.$$

It follows that

$$(4.151)\qquad S = O\left(\frac{1}{\theta}\right) + O\left(\frac{1}{\theta_1}\right) + \ldots + O\left(\frac{1}{\theta_{\nu-1}}\right) + O(\xi\theta\theta_1 \ldots \theta_{\nu-1})$$

and

$$(4.152)\qquad S = O\left(\frac{1}{\theta}\right) + O\left(\frac{1}{\theta_1}\right) + \ldots + O\left(\frac{1}{\theta_\nu}\right) + O(\xi\theta\theta_1 \ldots \theta_{\nu-1}\theta_\nu).$$

We shall require both of these equations.

4.2. We choose ν so that

$$(4.21)\qquad \xi\theta\theta_1 \ldots \theta_{\nu-1}\theta_\nu < 1 \leqslant \xi\theta\theta_1 \ldots \theta_{\nu-1}.$$

It may be verified at once* that $\theta_s\theta_{s+1} < \frac{1}{2}$ for every s. Hence on the one hand

$$(4.22)\qquad \theta\theta_1 \ldots \theta_{\nu-1} = O(2^{-\frac{1}{2}\nu}),$$

and on the other

$$(4.23)\qquad \frac{1}{\theta} + \frac{1}{\theta_1} + \ldots + \frac{1}{\theta_{\nu-1}} = O\left(\nu \operatorname{Max} \frac{1}{\theta_s}\right) = O\left(\frac{\nu 2^{-\frac{1}{2}\nu}}{\theta\theta_1 \ldots \theta_{\nu-1}}\right) = O(\nu 2^{-\frac{1}{2}\nu}\xi).$$

From (4.151), (4.22), and (4.23), we obtain

$$(4.24)\qquad S = O(\nu 2^{-\frac{1}{2}\nu}\xi) + O(2^{-\frac{1}{2}\nu}\xi) = o(\xi),$$

since ν tends to infinity with ξ; and the theorem follows from (2.14) and (4.24).

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* See our paper "Some problems of Diophantine approximation (II)" [*Acta Mathematica*, Vol. 37 (1914), pp. 193–238 (p. 212)].

4.3. To Theorem A1 corresponds, for Problem B, the well known theorem:

THEOREM B1.—*If θ is irrational, then*

$$s(\theta, n) = \sum_{\mu=1}^{n} \{\mu\theta\} = o(n).$$

The proof of this theorem is included in that of Theorem A1. We have only to take $\eta = k\omega'$, where k is an integer, so that $f' = 0$, and to write $\xi = \eta/\omega = k/\theta$, $n = [\xi]$.

4.4. THEOREM A2.—*If $\psi(\eta)$ is any function of η which tends steadily to infinity with η, then there is an irrational θ such that each of the inequalities*

$$N(\eta) - \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} > \frac{\eta}{\psi(\eta)}, \qquad N(\eta) - \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} < -\frac{\eta}{\psi(\eta)}$$

is satisfied for a sequence of indefinitely increasing values of η.

Thus Theorem A1 is the best possible theorem of its kind.

Making the transformations indicated in 4.3, we see at once that it is enough to prove

THEOREM B2.—*If $\psi(n)$ is any function of n which tends steadily to infinity with n, then there is an irrational θ such that each of the inequalities*

$$s(\theta, n) > \frac{n}{\psi(n)}, \qquad s(\theta, n) < -\frac{n}{\psi(n)}$$

is satisfied for an infinity of values of n.

To prove this we use Lerch's formula (3.35). Writing

(4.41) $$n_1 = [n\theta] = n\theta - \delta,\ n_2 = n_1\theta_1 - \delta_1,\ \ldots,\ n_{r+1} = n_r\theta_r - \delta_r,$$

(4.42) $$\phi_s = \tfrac{1}{2}\delta_s - \frac{\delta_s(1-\delta_s)}{2\theta_s},$$

we have

(4.43) $$s(\theta, n) = \phi_0 - s\left(\frac{1}{\theta}, n_1\right) = \phi_0 - s(\theta_1, n_1) = \phi_0 - \phi_1 + s(\theta_2, n_2)$$

$$= \ldots = \phi_0 - \phi_1 + \ldots + (-1)^r\phi_r + s(\theta_{r+1}, n_{r+1}).$$

We suppose a_{r+1} even, and exceedingly large in comparison with the preceding quotients $a_1, a_2, \ldots, a_r$, and take $n_r = \frac{1}{2}a_{r+1}$. Then $n_{r+1} = 0$, and

δ_r is practically $\frac{1}{2}$, so that $\frac{1}{2}\delta_r(1-\delta_r)$ is certainly greater than $\frac{1}{9}$. Having fixed n_r, we can determine $n_{r-1}, n_{r-2}, \ldots, n_1, n$ from the equations (4.41); and

$$n \leqslant \frac{2n_1}{\theta} \leqslant \frac{2^2 n_2}{\theta\theta_1} \cdots \leqslant \frac{2^r n_r}{\theta\theta_1 \ldots \theta_{r-1}} = \frac{2^{r-1} a_{r+1}}{\theta\theta_1 \ldots \theta_{r-1}}.$$

It is then plain that, if a_{r+1} is sufficiently large in comparison with the preceding partial quotients, $s(\theta, n)$ will have the sign of $(-1)^r$, and

$$(4.41) \qquad |s(\theta, n)| > \tfrac{1}{2}|\phi_r| > \frac{1}{20\theta_r} > \frac{a_{r+1}}{20} > \frac{n}{\psi(n)}.$$

And, by choosing a θ for which sufficiently violent increments in the order of magnitude of the quotients occur at an infinity of stages in the continued fraction, we can secure the truth of (4.41) for an infinity of values of n.

5. *Results concerning special classes of irrationals.*

5.1. Theorem **A3**.—*If the quotients a_n in the continued fraction for $\theta = \omega/\omega'$ are bounded, then*

$$N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + O(\log \eta).$$

Theorem **B3**.—*Under the same condition,*

$$s(\theta, n) = O(\log n).$$

To prove Theorem **A3**, we return to the analysis of 4.1 and 4.2, but use (4.152) instead of (4.151). In this case we have plainly

$$S = O\left(\frac{1}{\theta}\right) + O\left(\frac{1}{\theta_1}\right) + \ldots + O\left(\frac{1}{\theta_\nu}\right) = O(\nu).$$

Since $$2^{\frac{1}{2}\nu} = O\left(\frac{1}{\theta\theta_1 \ldots \theta_{\nu-1}}\right) = O\left(\frac{1}{\theta\theta_1 \ldots \theta_\nu}\right) = O(\xi),$$

we have $\nu = O(\log \xi) = O(\log \eta)$; and the theorem is proved. Theorem **B3** follows *a fortiori*: this is the theorem which, as we explained in 1.2, was claimed as a new theorem in our communication to the Cambridge congress, but is really due to Lerch.

It will easily be verified that, if we assume

$$a_n = O(n^\rho) \quad (\rho > 0),$$

we obtain an error term of the order

$$S = O\{(\log \eta)^{\rho+1}\};$$

if we assume $a_n = O(e^{\rho n})$, where ρ lies below a certain limit, we obtain

$$S = O(\eta^\sigma) \quad (\sigma < 1).^*$$

As so little is known concerning the order of magnitude of the quotients in the continued fractions which express irrationals of particular types, it is hardly worth while to go into further detail.

5.2. THEOREM **A4**. — *There are values of* $\theta = \omega/\omega'$, *with bounded quotients, such that each of the inequalities*

$$N(\eta) - \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} > K \log \eta, \quad N(\eta) - \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} < -K \log \eta,$$

where K is a positive constant, is satisfied for a sequence of indefinitely increasing values of η.

THEOREM **B4**.—*There are values of θ, with bounded quotients, such that each of the inequalities*

$$s(\theta, n) > K \log n, \qquad s(\theta, n) < -K \log n$$

is satisfied for an infinity of values of n.

Thus Theorems **A3** and **B3** are also best possible theorems of their kind. To prove this, it is plainly enough to prove Theorem **B4**; and this we shall do by considering the simplest irrational of all, viz.

$$\theta = \frac{\sqrt{5}-1}{2} = \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\dots.$$

We write
$$\Theta = \frac{1}{\theta} = \frac{\sqrt{5}+1}{2},$$

and take the convergents to θ to be

$$\frac{p_0}{q_0} = \frac{0}{1}, \quad \frac{p_1}{q_1} = \frac{1}{1}, \quad \frac{p_2}{q_2} = \frac{1}{2}, \quad \dots.$$

Then it is easily verified that

$$q_s = \frac{1}{\sqrt{5}}(\Theta^{s+1} + (-1)^s \theta^{s+1}), \qquad p_s = q_{s-1}.$$

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* Compare p. 214 of our memoir in the *Acta Mathematica* referred to above (p. 22).

5.3. We first take $n = q_s$ in the formula (3.31). We find without difficulty that

$$[q_s\theta] = q_{s-1}, \quad \delta = q_s\theta - [q_s\theta] = \theta^{s+1},$$

if s is even, and $\quad [q_s\theta] = q_{s-1}-1, \quad \delta = 1-\theta^{s+1},$

if s is odd; and that in either case

(5.31) $$\sigma_s = \sum_{\nu=1}^{q_s} \{\nu\theta\}$$

satisfies the equation

(5.32) $$\sigma_s + \sigma_{s-1} = \tfrac{1}{2}\left(\theta^{2s+1} + (-1)^{s+1}\theta^{s+2}\right).$$

Using this recurrence equation to express σ_s in terms of

$$\sigma_0 = \{\theta\} = \tfrac{1}{2}\sqrt{5}-1,$$

we find, after reduction, that

(5.33) $$\sigma_s = \frac{\theta^{2s+2}}{2\sqrt{5}} - \tfrac{1}{2}(-1)^{s+1}\theta^{s+1} + (-1)^{s+1}\frac{\theta}{\sqrt{5}}.$$

Suppose now that

(5.34) $$s(\theta, n) = \sum_{1}^{n} \{\nu\theta\} \quad (q_s \leqslant n < q_{s+1}).$$

We can express n in one and only one way in the form

$$n = q_s + q_{s_1} + q_{s_2} + \ldots + q_{s_k} = q_s + Q_1,$$

where $s, s_1, s_2, \ldots$ are descending integers differing by at least 2; and

$$s(\theta, n) = \sigma_s + \sum_{\mu=1}^{Q_1} \{(q_s+\mu)\theta\}.$$

Now $q_s\theta$ differs from an integer by less than does any $\mu\theta$. Hence

$$[(q_s+\mu)\theta] = q_{s-1} + [\mu\theta]$$

and $\quad \{(q_s+\mu)\theta\} = q_s\theta - q_{s-1} + \mu\theta - [\mu\theta] - \tfrac{1}{2} = (-1)^s\,\theta^{s+1} + \{\mu\theta\}.$

$$s(\theta, n) = \sigma_s + (-1)^s\theta^{s+1}Q_1 + s_{Q_1}.$$

We now write

$$Q_1 = q_{s_1} + q_{s_2} + \ldots + q_{s_k} = q_{s_1} + Q_2, \qquad Q_2 = q_{s_2} + Q_3,$$

and so on, and repeat the argument. We thus obtain

(5.35) $$s(\theta, n) = \sigma_s + \sigma_{s_1} + \sigma_{s_2} + \ldots + \sigma_{s_k}$$
$$+ (-1)^s\theta^{s+1}Q_1 + (-1)^{s_1}\theta^{s_1+1}Q_2 + \ldots + (-1)^{s_{k-1}}\theta^{s_{k-1}+1}Q_{s_k}.$$

5.4. If in (5.35) we substitute the values of the σ's given by (5.39), the first two terms of (5.39) will plainly give a contribution bounded for all values of s, so that

$$\sigma_s+\sigma_{s_1}+\ldots+\sigma_{s_k} = -\frac{\theta}{\sqrt{5}}\left((-1)^s+(-1)^{s_1}+\ldots+(-1)^{s_k}\right)+O(1). \tag{5.41}$$

Again

$$Q_1 = \sum_{r=1}^{k} q_{s_r} = \frac{1}{\sqrt{5}}\sum_{r=1}^{k}\left(\Theta^{s_r+1}+(-1)^{s_r}\theta^{s_r+1}\right), \tag{5.42}$$

and the sum of the second terms is numerically less than k, and *a fortiori* than s. The sum of the contributions of all such terms to (5.35) is therefore less in absolute value than

$$s\theta^{s+1}+s_1\theta^{s_1+1}+\ldots = O(1).$$

These terms, then, may be disregarded. Making this simplification, and substituting from (5.41) and (5.42) into (5.35), we obtain, finally,

$$\begin{aligned} s(\theta, n) = O(1) &- \frac{\theta}{\sqrt{5}}\left((-1)^s+(-1)^{s_1}+\ldots+(-1)^{s_k}\right) \\ &+\frac{(-1)^s}{\sqrt{5}}\left(\theta^{s-s_1}+\theta^{s-s_2}+\ldots+\theta^{s-s_k}\right) \\ &+\frac{(-1)^{s_1}}{\sqrt{5}}\left(\theta^{s_1-s_2}+\theta^{s_1-s_3}+\ldots+\theta^{s_1-s_k}\right) \\ &+\ldots+\frac{(-1)^{s_{k-1}}}{\sqrt{5}}\theta^{s_{k-1}-s_k}. \end{aligned} \tag{5.43}$$

5.5. This formula enables us to study the behaviour of $s(\theta, n)$ for different forms of n, and in particular to prove our theorem. Let us take, for example,

$$s = 4k+4, \quad s_1 = 4k, \quad s_2 = 4k-4, \quad \ldots, \quad s_k = 4.$$

Then the right-hand side of (5.43) becomes

$$-\frac{s\theta}{4\sqrt{5}}+\frac{1}{\sqrt{5}}\left(\frac{\theta^4-\theta^s+\theta^4-\theta^{s-4}+\ldots+\theta^4-\theta^8}{1-\theta^4}\right)+O(1) = Cs+O(1),$$

where
$$C = \frac{1}{4\sqrt{5}}\left(\frac{\theta^4}{1-\theta^4}-\theta\right) = -\frac{1}{20} \neq 0;$$

and $s(\theta, n)$ is negative and greater than a constant multiple of s. Similarly, if we were to take

$$s = 4k+3, \; s_1 = 4k-1, \; \ldots, \; s_k = 3,$$

we should find $s(\theta, n)$ to be positive and greater than a constant multiple of s. Since s is greater than a constant multiple of $\log n$, this completes the proof of Theorems **A4** and **B4**.

5.6. We should perhaps, before passing to more transcendental investigations, add a word concerning the case, so far excluded, of a *rational* θ. It is easy to see that, when θ is rational, no such results as we have proved in the irrational case are true: $s(\theta, n)$ is effectively of order n, and the oscillatory part of $N(\eta)$ of order η. Thus, to take a simple case, the series $\Sigma\{\frac{2}{3}\mu\}$ is

$$\tfrac{1}{6}-\tfrac{1}{6}-\tfrac{1}{2}+\tfrac{1}{6}-\tfrac{1}{6}-\tfrac{1}{2}+\tfrac{1}{6}-\tfrac{1}{6}-\tfrac{1}{2}+\ldots,$$

and
$$s(\tfrac{2}{3}, n) \sim -\tfrac{1}{6}n.$$

In general, for a fixed rational $\theta = p/q$, we have $s(\theta, n) \sim A_q n$, where $A_q \to 0$ when $q \to \infty$.

6. *Transcendental methods: results true for all algebraical values of* θ.

6.1. The substance of our concluding section lies somewhat deeper. Our goal is to prove

THEOREM **A5**.—*If* $\theta = \omega/\omega'$ *is an algebraic irrational, then*

$$N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + O(\eta^a),$$

where $a < 1$.

THEOREM **B5**.—*Under the same conditions*

$$s(\theta, n) = O(n^a) \quad (a < 1).$$

We require some preliminary lemmas concerning the function

$$(6.11) \qquad \zeta_2(s, a, \omega, \omega') = \sum_{m,\, n=0}^{\infty} \frac{1}{(a+m\omega+n\omega')^s},$$

where a, ω, and ω' are positive, and $s = \sigma + it$. This function is a degenerate form of the double Zeta-function of Dr. E. W. Barnes. Barnes considers only the case in which (as in the theory of elliptic functions) the ratio $\theta = \omega/\omega'$ is *complex*. The series (6.11) defines the function in the first instance for $\sigma > 2$.

6.21. LEMMA α.—*The function* $\zeta_2(s, a, \omega, \omega')$ *is an analytic function of* s, *regular all over the plane except for simple poles at the points* $s = 2$

and $s=1$, *where it behaves like*

$$\frac{1}{\omega\omega'}\,\frac{1}{s-2},\qquad \frac{\omega+\omega'-2a}{2\omega\omega'}\,\frac{1}{s-1}$$

respectively.

This is proved by Barnes when θ is complex, and his proof, depending on the formula

$$(6.211)\qquad \zeta_2(s,\,a,\,\omega,\,\omega') = \frac{i\Gamma(1-s)}{2\pi}\int \frac{e^{-au}(-u)^{s-1}}{(1-e^{-\omega u})(1-e^{-\omega' u})}\,du,$$

is equally applicable in the case considered here. We should observe that $(-u)^{s-1} = e^{(s-1)\log(-u)}$, where $\log(-u)$ has its principal value, that the contour of integration is the same as in the well-known Riemann-Hankel formulæ for the ordinary Gamma and Zeta functions, and that the formula is valid for all values of s except positive integral values.

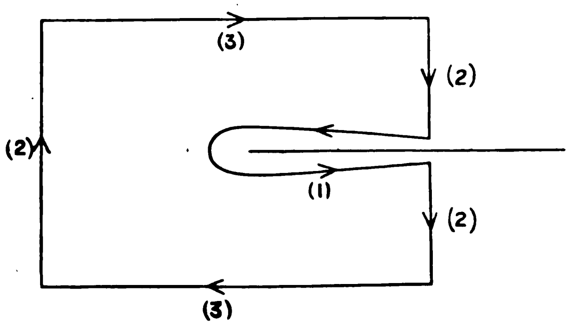
6.22. LEMMA β.—*Suppose that* $0 < a \leqslant \omega+\omega'$, *and that* $\theta = \omega/\omega'$ *is an algebraic irrational. Then there is a* K *such that*

$$(6.221)\qquad \frac{\zeta_2(s,\,a,\,\omega,\,\omega')}{(2\pi)^{s-1}\,\Gamma(1-s)} = \frac{1}{\omega^s}\sum_{m=1}^{\infty}\frac{\sin\left\{\dfrac{2m\pi}{\omega}(\tfrac{1}{2}\omega'-a)+\tfrac{1}{2}(1-s)\pi\right\}}{m^{1-s}\sin\dfrac{m\omega'\pi}{\omega}}$$

$$+\frac{1}{\omega'^s}\sum_{m=1}^{\infty}\frac{\sin\left\{\dfrac{2m\pi}{\omega'}(\tfrac{1}{2}\omega-a)+\tfrac{1}{2}(1-s)\pi\right\}}{m^{1-s}\sin\dfrac{m\omega\pi}{\omega'}}$$

for $\sigma < -K$.

To prove this formula we start from the integral (6.211) and integrate



round the contour shown in the figure. We suppose, as plainly we may,

that the horizontal lines (3) pass at a distance greater than a constant δ from any pole of the subject of integration, and that the loop (1) passes between the origin and the poles $\pm 2\pi i/\omega$, $\pm 2\pi i/\omega'$ nearest the origin. This being so, it is easy to see that the contributions of the rectilinear parts of the contour tend to zero when the sides of the rectangle move away to infinity, and that

$$\zeta_2 = \Gamma(1-s) \lim \Sigma R,$$

where R is a residue of the integrand. A simple calculation shows that the residues yield the two series required. If $\theta = \omega/\omega'$ is algebraic, we have

$$\left| \sin \frac{m\omega'\pi}{\omega} \right| > m^{-c}, \quad \left| \sin \frac{m\omega\pi}{\omega'} \right| > m^{-c},$$

where c is a constant depending on the degree of the algebraic equation which defines θ. It follows that the two series of the lemma are absolutely convergent if σ is negative and sufficiently large.* We shall suppose in what follows that the series are absolutely convergent for $\sigma < -K$. The formula (6.221) may of course hold in a wider region than this.

6.23. LEMMA γ.—*If* $|t| \to \infty$ *then*

$$\zeta_2(s, a, \omega, \omega') = O(e^{\epsilon |t|}),$$

for every positive ϵ, *and uniformly throughout any finite interval of values of* σ.

Suppose that $\sigma_1 \leqslant \sigma \leqslant \sigma_2$. We may suppose the contour of integration in (6.211) deformed in such a manner that

$$|\phi| = |\arg(-u)| \leqslant \tfrac{1}{2}\pi + \tfrac{1}{2}\epsilon$$

at every point of it, and $\quad |\phi| = \tfrac{1}{2}\pi + \tfrac{1}{2}\epsilon$

at all distant points. We have then

$$|(-u)^{s-1}| < A|u|^A e^{|\phi t|} < A|u|^A e^{(\frac{1}{2}\pi + \frac{1}{2}\epsilon)|t|},$$

where A is a number depending on σ_1 and σ_2,

$$|\Gamma(1-s)| = O(e^{-\frac{1}{2}\pi|t|} |t|^{\frac{1}{2}-\sigma}) = O(e^{-(\frac{1}{2}\pi - \frac{1}{2}\epsilon)|t|}),$$

$$\zeta_2 = O\left(e^{\epsilon|t|} \int \frac{|e^{-au}|\,|du|}{|1-e^{-\omega u}|\,|1-e^{-\omega' u}|}\right) = O(e^{\epsilon|t|}).$$

* It is hardly necessary to give fuller details of the proof, as the substance of the lemma is contained in the paper of Hardy referred to in the footnote to p. 17.

6.24. Lemma γ is required only in order to prove a somewhat deeper lemma, viz.:

LEMMA δ.*—*The function* $\zeta_2(s, a, \omega, \omega')$ *is of finite order in any half-plane* $\sigma > \sigma_0$, *and its* μ-*function* $\mu(\sigma)$ *satisfies the relations*

$$\mu(\sigma) = 0 \quad (\sigma > 2), \tag{6.241}$$

$$\mu(\sigma) \leqslant \frac{(\frac{1}{2}+K)(2-\sigma)}{2+K} \quad (-K \leqslant \sigma \leqslant 2), \tag{6.242}$$

$$\mu(\sigma) \leqslant \tfrac{1}{2}-\sigma \qquad (\sigma \leqslant -K). \tag{6.243}$$

Of these relations, (6.241) is obvious, since the series (6.11) is absolutely convergent for $\sigma > 2$; and (6.243) follows from (6.221), since we have

$$(2\pi)^{s-1}\Gamma(1-s)\sin\left\{\frac{2m\pi}{\omega}(\tfrac{1}{2}\omega'-a)+\tfrac{1}{2}(1-s)\pi\right\} = O\{e^{\frac{1}{2}\pi|t|}|\Gamma(1-s)|\}$$
$$= O(|t|^{\frac{1}{2}-\sigma})$$

uniformly in m, and, of course, a similar result in which ω and ω' are interchanged. Finally, (6.242) follows from (6.241), (6.243), and the well-known theorem of Lindelöf.† Lemma γ is used only to show that the conditions of Lindelöf's theorem are satisfied.

6.25. Our last lemma is of a different character. We write

$$a+m\omega+n\omega' = l_p, \tag{6.251}$$

the numbers l_p (no two of which are equal, since θ is irrational) being arranged in order of magnitude. We suppose that ξ is not equal to any l_p, and we put

$$W(\xi) = \sum_{l_p<\xi} 1.$$

LEMMA ϵ. — *Suppose that* $c > 2$, $T > 1$, *and* $\xi = \sqrt{(l_q l_{q+1})}$. *Then there exists a number* H, *independent of* T *and* ξ, *such that*

$$\left| W(\xi) - \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \zeta_2(s)\frac{\xi^s}{s}\,ds \right| < H\frac{\xi^c}{T}.$$

* For explanations concerning the "μ-function" of a function $f(s)$, defined initially by a Dirichlet's series, see G. H. Hardy and M. Riesz, "The general theory of Dirichlet's series," *Cambridge Mathematical Tracts*, no. 18, 1915, pp. 14-18.

† Theorem 14 of the tract referred to above.

We have

(6.252) $$W-\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\zeta_2(s)\frac{\xi^s}{s}\,ds = W-\sum_p \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(\frac{\xi}{l_p}\right)^s\frac{ds}{s}.$$

Since
$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{\xi}{l_p}\right)^s\frac{ds}{s}\begin{matrix} = 1 & (l_p<\xi) \\ = 0 & (l_p>\xi)\end{matrix},$$
the right-hand side of (6.252) may be written in the form

$$\sum_p\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c-iT}+\frac{1}{2\pi i}\int_{c+iT}^{c+i\infty}\right)\left(\frac{\xi}{l_p}\right)^s\frac{ds}{s} = \sum_p U_p,$$

say. Now*
$$|U_p| \leqslant \frac{2}{T}\,\frac{(\xi/l_p)^c}{|\log(\xi/l_p)|}.$$

Hence

(6.253) $$\left|W-\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\zeta_2(s)\frac{\xi^s}{s}\,ds\right| \leqslant \frac{2\xi^c}{T}\sum_p\frac{l_p^{-c}}{|\log(\xi/l_p)|}.$$

If we write $l_p = e^{-\lambda_p}$, $\xi = e^{\rho}$, the series becomes

(6.254) $$\sum\frac{e^{-c\lambda_p}}{|\rho-\lambda_p|},$$

and
$$\rho = \tfrac{1}{2}(\lambda_q+\lambda_{q+1}).$$

Now Bohr,† generalising a result of Landau,‡ has shown that the series (6.254) is bounded, provided only that

(C) *there is a number l, positive or zero, such that*

$$\frac{1}{\lambda_{p+1}-\lambda_p} = O(e^{(l+\delta)\lambda_p})$$

for every positive δ;

and it is easy to verify that the condition (C) is satisfied by our series $\Sigma l^{-s} = \Sigma e^{-s\lambda_p}$. For

$$l_{p+1}-l_p = a+m'\omega+n'\omega'-a-m\omega-n\omega' = h\omega+k\omega' = \omega'(k+h\theta),$$

say, and so, since θ is algebraic and $l_{p+1}<l_p+H$,

$$l_{p+1}-l_p > (|h|+2)^{-H} > H(|m|+|m'|+2)^{-H} > Hl_{p+1}^{-H} > Hl_p^{-H}\ (p>p_0),$$

$$\lambda_{p+1}-\lambda_p = \log\left(1+\frac{l_{p+1}-l_p}{l_p}\right) > Hl_p^{-H};$$

* Landau, *Handbuch*, § 86.

† H. Bohr, "Einige Bemerkungen zum Konvergenzproblem der Dirichletscher Reihen", *Rendiconti del Circolo Matematico di Palermo*, Vol. 37 (1914), pp. 1-16.

‡ *Handbuch*, § 235.

H, wherever it occurs, denoting a positive constant, not of course the same at different occurrences. Thus Bohr's condition is satisfied, and Lemma ϵ follows from (6.253).

6.3. We can now prove our theorems. We take $T = \xi^\gamma$, where $0 < \gamma < 2$. We choose arbitrary positive numbers δ and ϵ, and take $c = 2+\delta$.

We then apply Cauchy's theorem to the integral

$$\frac{1}{2\pi i}\int \zeta_2(s)\,\frac{\xi^s}{s}\,ds,$$

taken round the rectangle

$$(c-iT,\ c+iT,\ -K+iT,\ -K-iT),$$

the sides of which, taken in order, we denote by (1), (2), (3), and (4). Using Lemma a, we obtain

$$(6.31)\quad \frac{1}{2\pi i}\int \zeta_2(s)\,\frac{\xi^s}{s}\,ds = \int_{(1)}+\int_{(2)}+\int_{(3)}+\int_{(4)} = \frac{\xi^2}{2\omega\omega'}+\frac{\omega+\omega'-2a}{2\omega\omega'}\,\xi+\zeta_2(0).$$

Now

$$(6.32)\qquad \int_{(1)} = W(\xi)+O\left(\frac{\xi^c}{T}\right) = W(\xi)+O(\xi^{2+\delta-\gamma}),$$

by Lemma ϵ; and

$$(6.33)\quad \int_{(3)} = O\left(\xi^{-K}\int_{-T}^{T}|t|^{K-\frac12+\epsilon}\,dt\right) = O\left(\frac{T^{K+\frac12+\epsilon}}{\xi^K}\right) = O(\xi^{-K+(K+\frac12)\gamma+\epsilon}),$$

by Lemma δ. It remains to estimate the contributions of the horizontal sides; and it is clear, from Lemma δ, that the contribution of either is of the form

$$O(\operatorname*{Max}_{-K\leqslant\sigma\leqslant c} \qquad {}^{\sigma)-1+\epsilon}) = O(\operatorname{Max}\xi^\eta),$$

where

$$\eta = \sigma+\left\{\frac{(\frac12+K)(2-\sigma)}{2+K}-1\right\}\gamma+\epsilon \quad (-K\leqslant\sigma\leqslant 2),$$

$$\eta = \sigma-\gamma+\epsilon \quad (2\leqslant\sigma\leqslant c).$$

It is clear that η cannot exceed the greater of its values for $\sigma = -K$ and $\sigma = c$, viz.

$$-K+(K-\tfrac12)\gamma+\epsilon, \qquad 2+\delta-\gamma+\epsilon.$$

The possible error-term arising from the first of these values may be absorbed into that already present in (6.33). That corresponding to

the second, as well as that in (6.32), may be absorbed in a single term $O(\xi^{2+\delta-\gamma+\epsilon})$. We have therefore, on collecting our results,

$$(6.34)\qquad W(\xi)=\frac{\xi^2}{2\omega\omega'}+\frac{\omega+\omega'-2a}{2\omega\omega'}\xi+O\{\xi^{-K+(K+\frac{1}{2})\gamma+\epsilon}\}+O(\xi^{2+\delta-\gamma+\epsilon}).$$

We have still γ at our disposal. Taking

$$-K+(K+\tfrac{1}{2})\gamma=2+\delta-\gamma,$$

we obtain

$$\gamma=\frac{2+\delta+K}{\frac{3}{2}+K}$$

(which is, as we supposed, positive and less than 2), and

$$2+\delta-\gamma=\frac{(2+\delta)(\frac{1}{2}+K)-K}{\frac{3}{2}+K}.$$

This is equal to $(1+K)/(\frac{3}{2}+K)<1$ when $\delta=0$, and is therefore less than unity if δ is sufficiently small. We have therefore

$$(6.35)\qquad W(\xi)=\frac{\xi^2}{2\omega\omega'}+\frac{\omega+\omega'-2a}{2\omega\omega'}\xi+O(\xi^{\alpha}),$$

where $\alpha<1$. In order to obtain Theorem **A5**, it is only necessary to attribute to a the particular value $\omega+\omega'$ and to replace ξ by η, since $W(\xi)$ then becomes $N(\eta)$.

Our argument naturally yields a definite value for α. But it becomes clear, when we consider the particular case of a *quadratic* θ, that the value so obtained is, in the light of Theorem **A2**, not the best value possible. We are therefore content to show that α is in any case less than unity.

Additional Note (*March* 13*th*, 1921).

We have developed the transcendental method of § 6 considerably since this paper was first communicated to the Society.

Suppose that $k\geqslant 0$ and

$$W_k(\xi)=\sum_{l_p<\xi}(\xi-l_p)^k.$$

Then

$$W_k(\xi)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\zeta_2(s)\frac{\Gamma(k+1)\Gamma(s)}{\Gamma(k+1+s)}\xi^{s+k}\,ds$$

if $c>2$. We transform this equation by (1) moving back the path of integration to the line $\sigma=-q<0$, with the appropriate corrections for the residues, (2) substituting for $\zeta_2(s)$ from (6.221), and (3) integrating term

by term. This process can be justified if $\theta = \omega/\omega'$ is algebraic and k and q are chosen appropriately, and we obtain an expression for $W_k(\xi)$ in the form of an absolutely convergent series.

We then make use of a lemma which is of some interest in itself, viz.: *if there are constants* $h \geqslant 1$ *and* $H > 0$ *such that*

(1). $$n^h \,|\sin n\theta\pi| > H$$

for all positive integral values of n, *then the series*

$$\sum \frac{1}{n^{h+\epsilon}\,|\sin n\theta\pi|}$$

is convergent for every positive ϵ.

Using this lemma and our series for $W_k(\xi)$, we are able to show that *if* (1) *is true for all positive integral values of* n, *then*

(2) $$N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + O(\eta^{a+\epsilon}),$$

where $a = (h-1)/h$, *for every positive* ϵ. This is included in Theorem **A3** if $h = 1$; but is in all other cases considerably more precise than anything proved in the paper.

In (2) the index $a = (h-1)/h$ of the power of η is the best possible one. For we can also show that *if*

(3) $$n^h \,|\sin n\theta\pi| < H$$

for an infinity of values of n, *then each of the inequalities*

(4) $$N(\eta) - \frac{\eta^2}{2\omega\omega'} + \frac{\eta}{2\omega} + \frac{\eta}{2\omega'} > A\eta^a, \quad N(\eta) - \frac{\eta^2}{2\omega\omega'} + \frac{\eta}{2\omega} + \frac{\eta}{2\omega'} < -A\eta^a,$$

where A *is a positive constant depending on* h *and* H, *is true for a sequence of indefinitely increasing values of* η.

We are further able to obtain an "explicit formula" for $N(\eta)$; viz.

(5) $$N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + \frac{\omega^2+\omega'^2+3\omega\omega'}{12\omega\omega'}$$
$$- \frac{1}{2\pi}\sum\left(\frac{\cos\dfrac{2\mu\pi}{\omega}(\eta-\tfrac12\omega')}{\mu\sin\dfrac{\mu\omega'\pi}{\omega}} + \frac{\cos\dfrac{2\nu\pi}{\omega'}(\eta-\tfrac12\omega)}{\nu\sin\dfrac{\nu\omega\pi}{\omega'}}\right).$$

Here $\theta = \omega/\omega'$ is irrational and algebraic, and the series is to be interpreted as meaning

$$\lim \sum_{\mu < \omega R,\ \nu < \omega' R}$$

when $R \to \infty$ in an appropriate manner.

The most difficult of the remaining problems is that of determining whether there is *any* θ for which the error-term in $N(\eta)$, or the sum $s(\theta, n)$ is *bounded.* The answer is in the negative. We can prove, in fact, that *there exists an $A > 0$ such that, for every irrational θ,*

$$|s(\theta, n)| > A \log n$$

for an infinity of values of n. Further, *given K, there exists a $B = B(K) > 0$ such that, for every θ for which $a_n < K$, the inequalities*

$$s(\theta, n) > B \log n, \qquad s(\theta, n) < -B \log n,$$

are satisfied each for an infinity of values of n.

The corresponding Cesàro means behave rather differently. It is possible to find θ's for which the first Cesàro mean $\sigma(\theta, n)$ of $s(\theta, n)$ is bounded, and others for which $\sigma(\theta, n)/\log n$ tends to a limit other than zero.

We may take this opportunity of correcting a misstatement in our communication to the Cambridge Congress referred to on p. 15. It was stated there that

$$\sum_{\nu=1}^{n} \{\nu\theta\}^2 = \tfrac{1}{12}n + O(1)$$

for *every* irrational θ. This is untrue; but the equation holds for very general classes of values of θ, and in particular for any θ whose partial quotients are bounded.

ON SOME SOLUTIONS OF THE WAVE EQUATION

By H. J. Priestley.

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1. *Preliminary.*

Referred to coordinates μ, ζ, θ defined by the equations

$$x = a\,(1-\mu^2)^{\frac{1}{2}}\,(1+\zeta^2)^{\frac{1}{2}}\cos\theta,$$

$$y = a\,(1-\mu^2)^{\frac{1}{2}}\,(1+\zeta^2)^{\frac{1}{2}}\sin\theta,$$

$$z = a\mu\zeta,$$

the wave equation takes the form

$$\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu}\right]+\frac{\partial}{\partial\zeta}\left[(1+\zeta^2)\frac{\partial\psi}{\partial\zeta}\right]+\left[\frac{\zeta^2+\mu^2}{(1-\mu^2)(1+\zeta^2)}\right]\frac{\partial^2\psi}{\partial\theta^2}$$
$$= \left(\frac{a}{c}\right)^2(\mu^2+\zeta^2)\frac{\partial^2\psi}{\partial t^2}.$$

This equation is satisfied by

$$\psi = M(\mu)\,Z(\zeta)\,e^{i\,(m\theta+pt)},$$

provided that M and Z satisfy the equations

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dM}{d\mu}\right]+\left[n(n+1)-\frac{m^2}{1-\mu^2}\right]M = k^2a^2(1-\mu^2)\,M, \tag{1}$$

$$\frac{d}{d\zeta}\left[(1+\zeta^2)\frac{dZ}{d\zeta}\right]-\left[n(n+1)-\frac{m^2}{1+\zeta^2}\right]Z = -\,k^2a^2(1+\zeta^2)\,Z, \tag{2}$$

where $k = p/c$ and n is any constant.

2. *Solution of an auxiliary differential equation.*

Equations (1) and (2) are of the type

$$\frac{d}{dx}\left[P\frac{dy}{dx}\right]+Qy = \lambda Ry, \tag{3}$$

where P, Q, R are known functions of x. λ is constant, and the solutions of

$$\frac{d}{dx}\left[P\frac{dy}{dx}\right]+Qy=0, \tag{4}$$

are known.

Let $y_1(x)$, $y_2(x)$ be solutions of (4), and assume that

$$y(x)=u(x)\,y_1(x)+v(x)\,y_2(x)$$

is a solution of (3).

Assume further that

$$y_1\frac{du}{dx}+y_2\frac{dv}{dx}=0. \tag{5}$$

Then
$$P\left[\frac{du}{dx}\frac{dy_1}{dx}+\frac{dv}{dx}\frac{dy_2}{dx}\right]=\lambda R\left[uy_1+vy_2\right]. \tag{6}$$

Now, from (5),
$$-\frac{du}{dx}\Big/y_2=\frac{dv}{dx}\Big/y_1. \tag{7}$$

If each of these be put equal to Z, (6) becomes

$$ZP\left[y_1\frac{dy_2}{dx}-y_2\frac{dy_1}{dx}\right]=\lambda R\left[uy_1+vy_2\right]. \tag{8}$$

But, since y_1, y_2 are solutions of (4),

$$P\left[y_1\frac{dy_2}{dx}-y_2\frac{dy_1}{dx}\right]=C, \tag{9}$$

where C is independent of x.

Therefore, if Rw/C be written for Z in (8), that equation becomes

$$w=\lambda\,(uy_1+vy_2), \tag{10}$$

showing that w is a solution of (3).

Now, since $\dfrac{du}{dx}=-y_2Rw/C$, $\quad \dfrac{dv}{dx}=y_1Rw/C$,

$$u(x)=A-\frac{1}{C}\int_a^x R(t)\,w(t)\,y_2(t)\,dt, \tag{11}$$

$$v(x)=B+\frac{1}{C}\int_a^x R(t)\,w(t)\,y_1(t)\,dt, \tag{11'}$$

where A, B and a are arbitrary constants.

On substituting these values in (10) it appears that $w(x)$ satisfies the Volterra equation

$$w(x) = \chi(x) - \frac{\lambda}{C}\int_a^x R(t) \begin{vmatrix} y_1(x), & y_2(x) \\ y_1(t), & y_2(t) \end{vmatrix} w(t)\,dt, \tag{12}$$

where $\chi(x)$ is a solution of (4).

Different solutions of (3) can be obtained from (12) by varying the function $\chi(x)$, or, what amounts to the same thing, changing the limit a.

3. *The odd and even solutions of* (12) *when P and R are even.*

If $P(x)$ is an even function, it follows from (9) that (4) cannot have two independent solutions which are both odd or both even. Hence there must be an odd solution $z_1(x)$ and an even solution $z_2(x)$.

If $y_1(x)$, $y_2(x)$ are not these solutions they are linear functions of them, and therefore

$$\begin{vmatrix} y_1(x), & y_2(x) \\ y_1(t), & y_2(t) \end{vmatrix} \text{ is a multiple of } \begin{vmatrix} z_1(x), & z_2(x) \\ z_1(t), & z_2(t) \end{vmatrix}.$$

From this it follows that

$$\begin{vmatrix} y_1(-x), & y_2(-x) \\ y_1(-t), & y_2(-t) \end{vmatrix} = -\begin{vmatrix} y_1(x), & y_2(x) \\ y_1(t), & y_2(t) \end{vmatrix}.$$

Consequently, if $K(x, t)$ denotes the kernel of the integral equation (12), when $R(t)$ is an even function

$$K(x, t) = -K(-x, -t).$$

Now consider the function $w_1(x)$ defined by the integral equation

$$w_1(x) = z_1(x) - \frac{\lambda}{C}\int_0^x K(x, t)\, w_1(t)\,dt.$$

On changing the sign of x the equation becomes

$$w_1(-x) = z_1(-x) - \frac{\lambda}{C}\int_0^{-x} K(-x, t)\, w_1(t)\,dt$$

$$= -z_1(x) - \frac{\lambda}{C}\int_0^x K(x, t)\, w_1(-t)\,dt.$$

From this it is clear that $w_1(x)$ and $-w_1(-x)$ satisfy the same Volterra equation.

It is known that the solution of such an equation is unique.

Hence $$w_1(x) = -w_1(-x);$$

i.e. $w_1(x)$ is an odd function.

Similarly, it can be shown that $w_2(x)$, defined by

$$w_2(x) = z_2(x) - \frac{\lambda}{C}\int_0^x K(x, t)\, w_2(t)\, dt,$$

is an even function.

It follows from the equations defining $w_1(x)$ and $w_2(x)$ that at the origin

$$w_1 = z_1, \quad w_2 = z_2, \quad \frac{dw_1}{dx} = \frac{dz_1}{dx}, \quad \frac{dw_2}{dx} = \frac{dz_2}{dx}.$$

Hence at the origin

$$w_1\frac{dw_2}{dx} - w_2\frac{dw_1}{dx} = y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx}.$$

But each of these functions is a constant multiple of $[P(x)]^{-1}$.

Hence they are always equal.

4. *Application of the foregoing to equation* (1) *above.*

Solutions of

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dM}{d\mu}\right] + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right]M = 0,$$

are $P_n^m(\mu)$, $P_n^{-m}(\mu)$, where*

$$P_n^m(\mu) = \frac{1}{\Pi(-m)}\left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m} F[n+1, -n, -m+1, \tfrac{1}{2}(1-\mu)].$$

When m is an integer these solutions are not independent, but a second solution is then obtained by considering the limit as m tends to the integral value, of †

$$Q_n^m(\mu) = \tfrac{1}{2}\pi \operatorname{cosec} m\pi\left[\cos m\pi\, P_n^m(\mu) - \frac{\Pi(n+m)}{\Pi(n-m)}P_n^{-m}(\mu)\right].$$

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* Hobson, *Phil. Trans.*, (A) 187, p. 473.

† Macdonald, *Proc. London Math. Soc.*, Ser. 1, Vol. XXI, p. 274.

In the present paper $P_n^{-m}(\mu)$, $Q_n^m(\mu)$ will be used as the independent solutions.

Dougall* has shown that

$$(1-\mu^2)\begin{vmatrix} P_n^m(\mu), & P_n^{-m}(\mu) \\ \dfrac{d}{d\mu}P_n^m(\mu), & \dfrac{d}{d\mu}P_n^{-m}(\mu)\end{vmatrix} = -(2/\pi)\sin m\pi\,;$$

from which it follows that

$$(1-\mu^2)\begin{vmatrix} Q_n^m(\mu), & P_n^{-m}(\mu) \\ \dfrac{d}{d\mu}Q_n^m(\mu), & \dfrac{d}{d\mu}P_n^{-m}(\mu)\end{vmatrix} = -\cos m\pi.$$

[*Note.*—In Dougall's notation the function $P_n^m(\mu)$, which Hobson defines as above, is denoted by $P_n^{-m}(\mu)$.]

Now consider the integral equation

$$(1-\mu^2)^{-\frac{1}{2}m}\,W_n^{-m}(\mu)$$
$$=(1-\mu^2)^{-\frac{1}{2}m}P_n^{-m}(\mu)+k^2a^2\sec m\pi\int_1^\mu\left(\frac{1-t^2}{1-\mu^2}\right)^{\frac{1}{2}m}K_n^m(\mu,\,t)(1-t^2)^{-\frac{1}{2}m}\,W_n^{-m}(t)\,dt,$$

where $$K_n^m(\mu,\,t)=(1-t^2)\begin{vmatrix} Q_n^m(\mu), & P_n^{-m}(\mu) \\ Q_n^m(t), & P_n^{-m}(t)\end{vmatrix}.$$

The kernal $(1-t^2)^{\frac{1}{2}m}(1-\mu^2)^{-\frac{1}{2}m}K_n^m(\mu,t)$ is finite throughout the range $-1<\mu\leqslant t\leqslant 1$.

Also $(1-\mu^2)^{-\frac{1}{2}m}P_n^{-m}(\mu)$ is finite for $-1<\mu\leqslant 1$.

Hence, unless m is half an odd integer, the equation determines a function $W_n^{-m}(\mu)/(1-\mu^2)^{\frac{1}{2}m}$ which is finite throughout the range $-1<\mu\leqslant 1$.

But the equation can be written

$$W_n^{-m}(\mu)=P_n^{-m}(\mu)+k^2a^2\sec m\pi\int_1^\mu K_n^m(\mu,\,t)\,W_n^{-m}(t)\,dt. \tag{13}$$

Therefore it follows from the discussion in § 2 that $W_n^{-m}(\mu)$ is a solution of the equation (3).

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* Dougall, *Proc. Edin. Math. Soc.*, Vol. 18, p. 49.

5. *The odd and even solutions of* (3).

Macdonald* has shown that

$$P_n^m(-\mu) = \cos(m+n)\pi P_n^m(\mu) - (2/\pi)\sin(n+m)\pi Q_n^m(\mu).$$

From this it follows that

$$[\cos\tfrac{1}{2}(n+m)\pi P_n^m(\mu) - (2/\pi)\sin\tfrac{1}{2}(n+m)\pi Q_n^m(\mu)]\cos m\pi \text{ is even}$$

and $[\sin\tfrac{1}{2}(n+m)\pi P_n^m(\mu) + (2/\pi)\cos\tfrac{1}{2}(n+m)\pi Q_n^m(\mu)]\cos m\pi$ is odd.

On substituting for $P_n^m(\mu)$ in these expressions it is easily shown that they reduce to

$$\phi_n^m(\mu) = \cos\tfrac{1}{2}(m+n)\pi\{\Pi(n+m)/\Pi(n-m)\}P_n^{-m}(\mu)$$
$$+(2/\pi)\sin\tfrac{1}{2}(m-n)\pi Q_n^m(\mu)$$

and $\psi_n^m(\mu) = \sin\tfrac{1}{2}(m+n)\pi\{\Pi(n+m)/\Pi(n-m)\}P_n^{-m}(\mu)$

$$+(2/\pi)\cos\tfrac{1}{2}(m-n)\pi Q_n^m(\mu).$$

From these are derived $U_n^m(\mu)$, $V_n^m(\mu)$, even and odd solutions of (3) by means of the Volterra equations

$$U_n^m(\mu) = \phi_n^m(\mu) + k^2a^2 \sec m\pi \int_0^\mu K_n^m(\mu, t)\, U_n^m(t)\,dt, \tag{14}$$

$$V_n^m(\mu) = \psi_n^m(\mu) + k^2a^2 \sec m\pi \int_0^\mu K_n^m(\mu, t)\, V_n^m(t)\,dt. \tag{15}$$

In general $U_n^m(\mu)$, $V_n^m(\mu)$ will be infinite at $\mu = \pm 1$. If, however, the equations are written as integral equations in

$$(1-\mu^2)^{\frac{1}{2}m}\, U_n^m(\mu), \quad (1-\mu^2)^{\frac{1}{2}m}\, V_n^m(\mu),$$

it is easily shown by an argument similar to that used for $W_n^{-m}(\mu)/(1-\mu^2)^{\frac{1}{2}m}$ that $(1-\mu^2)^{\frac{1}{2}m}\,U_n^m(\mu)$ and $(1-\mu^2)^{\frac{1}{2}m}\,V_n^m(\mu)$ are finite.

The argument has to be modified slightly in the case when $m = 0$.

The infinities in $\phi_n(\mu)$, $K_n(\mu, t)$ are logarithmic, and the integral equation for $U_n(\mu)/\log(1-\mu)$ has to be discussed. The logarithm introduces a complication at the origin, but the trouble can be avoided by discussing solutions of (3) obtained by writing in turn α and β in place of 0 as the lower limit in the integral. The solutions so obtained are inde-

* *Loc. cit.*

pendent, and the ratio of each to $\log(1-\mu)$ tends to a finite limit as μ tends to unity. $U_n(\mu)$ and $V_n(\mu)$ are linear functions of these solutions, and consequently $U_n(\mu)/\log(1-\mu)$, $V_n(\mu)/\log(1-\mu)$ are finite at $\mu = \pm 1$.

6. *Expression of $W_n^{-m}(\mu)$ in terms of $U_n^m(\mu)$ and $V_n^m(\mu)$.*

Equation (13) can be written in the form

$$W_n^{-m}(\mu) - k^2a^2 \sec m\pi \int_0^\mu K_n^m(\mu, t)\, W_n^{-m}(t)\,dt$$

$$= P_n^{-m}(\mu)\left[1 + k^2a^2 \sec m\pi \int_0^1 (1-t^2)\, Q_n^m(t)\, W_n^{-m}(t)\,dt\right]$$

$$- Q_n^m(\mu)\, k^2a^2 \sec m\pi \int_0^1 (1-t^2)\, P_n^{-m}(t)\, W_n^{-m}(t)\,dt$$

$$= \alpha_n^m \phi_n^m(\mu) + \beta_n^m \psi_n^m(\mu), \tag{16}$$

where
$$\cos m\pi\, \alpha_n^m = \cos \tfrac{1}{2}(m-n)\pi\, I_1 + \sin \tfrac{1}{2}(m+n)\pi\, I_2,$$

$$\cos m\pi\, \beta_n^m = \sin \tfrac{1}{2}(n-m)\pi\, I_1 - \cos \tfrac{1}{2}(m+n)\pi\, I_2,$$

and
$$I_1 = \frac{\Pi(n-m)}{\Pi(n+m)}\left[1 + k^2a^2 \sec m\pi \int_0^1 (1-t^2)\, Q_n^m(t)\, W_n^{-m}(t)\,dt\right],$$

$$I_2 = \tfrac{1}{2}\pi\, k^2a^2 \sec m\pi \int_0^1 (1-t^2)\, P_n^{-m}(t)\, W_n^{-m}(t)\,dt.$$

But on multiplying (14) by α_n^m, (15) by β_n^m, and adding, it is clear that

$$W_n^{-m}(\mu) = \alpha_n^m U_n^m(\mu) + \beta_n^m V_n^m(\mu)$$

satisfies equation (16).

It is known, moreover, that the solution of the Volterra equation is unique.

Hence the required relation is

$$W_n^{-m}(\mu) = \alpha_n^m U_n^m(\mu) + \beta_n^m V_n^m(\mu). \tag{17}$$

[*Note.*—If m and n are integers and n is less than m, $\phi_n^m(\mu)$, and consequently $U_n^m(\mu)$, vanishes when $m-n$ is even; $\psi_n^m(\mu)$, and consequently $V_n^m(\mu)$, vanishes when $m-n$ is odd.]

In the former case it follows from the relation

$$[\Pi(n-m)]^{-1} = \pi^{-1}\Pi(m-n-1)\sin(m-n)\pi,$$

that $\phi_n^m(\mu)\operatorname{cosec}\frac{1}{2}(m-n)\pi$ does not vanish.

Consequently $U_n^m(\mu)\operatorname{cosec}\frac{1}{2}(m-n)\pi$ does not vanish.

The relation (17) becomes

$$W_n^{-m}(\mu) = [\alpha_n^m \sin\tfrac{1}{2}(m-n)\pi][U_n^m(\mu)\operatorname{cosec}\tfrac{1}{2}(m-n)\pi]+\beta_n^m V_n^m(\mu).$$

In a similar way $\psi_n^m(\mu)\sec\frac{1}{2}(m-n)\pi$ can be used in place of $\psi_n^m(\mu)$ when $m-n$ is odd.

7. *Case when* $\dfrac{d}{d\mu}W_n^{-m}(\mu) = 0$ *at* $\mu = 0$.

If $\dfrac{d}{d\mu}W_m^{-m}(\mu) = 0$ at $\mu = 0$ the following theorems are true:—

(I) $W_n^{-m}(\mu)$ is an even function of μ.

(II) $W_n^{-m}(\mu)$ is the solution of a homogeneous Fredholm integral equation with symmetric kernal.

(III) If m is real the values of n are real and separate.

(IV) The values of n are infinite in number.

(V) Any function of μ, $F(\mu)$ which with its first two derived functions is continuous over the range $0 \leqslant \mu \leqslant 1$, and which satisfies the condition

$$\frac{\partial}{\partial\mu}F(\mu) = 0 \quad \text{when} \quad \mu = 0,$$

can be expanded in an absolutely and uniformly convergent series $\sum\limits_n c_n W_n^{-m}(\mu)$; where

$$c_n = \int_0^1 F(t)\, W_n^{-m}(t)\,dt \Big/ \int_0^1 [W_n^{-m}(t)]^2\,dt.$$

The proofs of these theorems are given below.

(I) $W_n^{-m}(\mu)$ is an even function of μ.

Since $W_n^{-m}(\mu)$ and $U_n^m(\mu)$ are solutions of the equation (1), it follows that

$$(1-\mu^2)\begin{vmatrix} U_n^m(\mu), & W_n^{-m}(\mu) \\ \dfrac{d}{d\mu}U_n^m(\mu), & \dfrac{d}{d\mu}W_n^{-m}(\mu) \end{vmatrix}$$

is constant. Since $\frac{d}{d\mu} U_n^m(\mu)$ and $\frac{d}{d\mu} W_n^{-m}(\mu)$ both vanish at $\mu = 0$, the determinant must vanish for all values of μ. Therefore $W_n^{-m}(\mu)$ is a multiple of $U_n^m(\mu)$, which proves the proposition.

(II) $W_n^{-m}(\mu)$ is the solution of a homogeneous Fredholm equation.

The function $H_r^m(\mu, t)$ defined by

$$H_r^m(\mu, t) = W_r^{-m}(\mu)\, U_r^m(t) \quad (t < \mu),$$

$$H_r^m(\mu, t) = U_r^m(\mu)\, W_r^{-m}(t) \quad (t > \mu),$$

is a continuous solution of

$$\frac{\partial}{\partial t}\left[(1-t^2)\frac{\partial \phi}{\partial t}\right]+\left[r(r+1)-\frac{m^2}{1-t^2}\right]\phi = k^2a^2(1-t^2)\,\phi,$$

which satisfies the three conditions

(*a*) $H_r^m(\mu, t)$ is finite at $t = 1$.

[*Note.*—Unless $m = 0$, $H_r^m(\mu, t) = 0$ at $t = 1$.]

(*b*) $\frac{\partial}{\partial t} H_r^m(\mu, t) = 0$ at $t = 0$.

(*c*) $\frac{\partial}{\partial t} H_r^m(\mu, t)$ is continuous throughout the range $0 < t < 1$, except at $t = \mu$, where

$$(1-\mu^2)\frac{\partial}{\partial t} H_r^m(\mu, t)\Big|_{\mu+}^{\mu-} = -(2/\pi)\,\beta_r^m \frac{\Pi(r+m)}{\Pi(r-m)}\cos^2 m\pi.$$

It follows in the usual way that $W_n^{-m}(\mu)$ satisfies the Fredholm equation*

$$W_n^{-m}(\mu) = (n-r)(n+r+1)/\delta_r^m \int_0^1 H_r^m(\mu, t)\, W_n^{-m}(t)\,dt,$$

where δ_r^m denotes the discontinuity in $(1-\mu^2)\frac{\partial}{\partial t} H_r^m(\mu, t)$ at $t = \mu$.

(III) When m is real the values of n for which $\frac{d}{d\mu} W_n^{-m}(\mu)$ vanishes at $\mu = 0$ are real and distinct.

* Hilbert, *Grundzüge einer Allgemeinen Theorie der Linearen Integralgleichungen*, Kap. 7.

The value of r in Theorem (II) may be taken as real since the only restriction on the choice of r is that β_r^m must not vanish.

Hence the argument used in my note on the values of n which make $\frac{d}{d\mu} P_n(\mu)$ vanish, may be applied to establish the present theorem.*

(IV) The values of n are infinite in number.

Suppose $g(t)$ is a continuous function such that

$$\int_0^1 H_r^m(\mu, t)\, g(t)\, dt = 0.$$

Then
$$W_r^{-m}(\mu)\int_0^\mu U_r^m(t)\, g(t)\, dt + U_r^m(\mu)\int_\mu^1 W_r^m(t)\, g(t)\, dt$$

$$= \int_0^1 H_r^m(\mu, t)\, g(t)\, dt = 0.$$

Differentiation of this equation leads to

$$\frac{d}{d\mu} W_r^{-m}(\mu)\int_0^\mu U_r^m(t)\, g(t) dt + \frac{d}{d\mu} U_r^m(\mu)\int_\mu^1 W_r^m(t)\, g(t) dt = 0.$$

Since $\begin{vmatrix} W_r^{-m}(\mu), & U_r^m(\mu) \\ \frac{d}{d\mu} W_r^{-m}(\mu), & \frac{d}{d\mu} U_r^m(\mu) \end{vmatrix}$ does not vanish, it follows that

$$\int_0^\mu U_r^m(t)\, g(t)\, dt = 0$$

and
$$\int_\mu^1 W_r^{-m}(t)\, g(t)\, dt = 0.$$

Hence
$$g(t) = 0.$$

It is known that under the condition just proved, the integral equation has an infinite number of characteristic constants† λ_s.

The values of n are the roots of the quadratic equations

$$(n-r)(n+r+1) = \lambda_s \delta_r^m \quad (s = 1, 2, 3, \ldots).$$

Hence there are an infinite number of values of n.

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* *Proc. London Math. Soc.*, December 1919.

† Lalescò, *Introduction à la Théorie des Equations Intégrales*, p. 70.

(V) Any function $F(\mu)$ satisfying the conditions stated above can be expanded in a series of $W_n^{-m}(\mu)$.

Hilbert* has shown that if the equation

$$\int_0^1 H_r^m(\mu, t)\, g(t)\, dt = 0,$$

where $g(t)$ is continuous, implies that $g(t) = 0$, then any function of μ which can be expressed in the form

$$\int_0^1 H_r^m(\mu, t)\, f(t)\, dt,$$

where $f(t)$ is continuous, can be expanded in an absolutely and uniformly convergent series $\sum\limits_n c_n W_n W_n^{-m}(\mu)$, the coefficients being found in the Fourier manner.

He has proved also† that if $F(t)$ and its first two derived functions are continuous, and if $F(t)$ satisfies the same boundary conditions as $H_r^m(\mu, t)$, then a function $f(t)$ can be found such that

$$\int_0^1 \left[F(\mu) - \int_0^1 H_r^m(\mu, t)\, f(t)\, dt\right]^2 d\mu < \epsilon,$$

where ϵ is any positive quantity.

Hence he deduces‡ that any continuous function of μ which satisfies the boundary conditions, and which has continuous first and second derived functions, can be expanded as above.

This proves the theorem.

8. *Case when* $W_n^{-m}(0) = 0$.

Analogous theorems can be proved for the function $W_n^{-m}(\mu)$ when $W_n^{-m}(\mu) = 0$ at $\mu = 0$.

9. *Solutions of equation* (2).

Consider first $$\frac{d}{dx}\left[(1+x^2)\frac{dy}{dx}\right] + \frac{1}{1+x^2}\, y = 0.$$

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* *Loc. cit.*, VII, p. 24.

† *Loc. cit.*, XI (Corollary), p. 47.

‡ *Loc. cit.*, XV, p. 51.

On writing $x = \tan z$, this becomes

$$\frac{d^2y}{dz^2}+y = 0,$$

of which solutions are $\sin z$ and $\cos z$.

From these are found, by the methods of § 2, solutions of

$$\frac{d}{dx}\left[(1+x^2)\frac{dy}{dx}\right]+\frac{1}{1+x^2} = -k^2a^2(1+x^2)\,y.$$

The detailed work is as follows :—

The substitution $x = \tan z$ reduces the equation to

$$\frac{d^2y}{dz^2}+y = -k^2a^2\sec^4 z, y,$$

of which a solution $y(z)$ is given by the Volterra equation

$$y(z) = \sin z+k^2a^2\int_0^z \sec^4 t \begin{vmatrix} \cos z, & \sin z \\ \cos t, & \sin t \end{vmatrix} y(t)\,dt.$$

The solution of the integral equation is

$$\sum_0^\infty (ka)^{2r} A_r(z),$$

where $$A_0(z) = \sin z,$$

$$A_{r+1}(z) = \int_0^z \sec^4 t \begin{vmatrix} \cos z, & \sin z \\ \cos t, & \sin t \end{vmatrix} A_r(t)dt.$$

The integrations are easily effected and lead to

$$A_r(z) = (-1)^r \sin z \tan^{2r} z/(2r+1)!\,.$$

Therefore $$y(z) = (ka)^{-1}\cos z \sin(ka\tan z).$$

On dropping the factor $(ka)^{-1}$ and returning to the original variable x, it is clear that

$$u_1(x) = (1+x^2)^{-\frac{1}{2}}\sin kax$$

is a solution of the equation.

From the known solution can be derived a second solution

$$u_2(x) = (1+x^2)^{-\frac{1}{2}}\cos kax.$$

By means of the functions $u_1(\zeta)$, $u_2(\zeta)$, Volterra equations are derived for the solution of (2).

Since $(1+\zeta^2)\begin{vmatrix} u_1(\zeta), & u_2(\zeta) \\ \frac{d}{d\zeta}u_1(\zeta), & \frac{d}{d\zeta}u_2(\zeta) \end{vmatrix} = -ka$

and $$\begin{vmatrix} u_1(\zeta), & u_2(\zeta) \\ u_1(t), & u_2(t) \end{vmatrix} = \frac{\sin ka(\zeta-t)}{(1+\zeta^2)^{\frac{1}{2}}(1+t^2)^{\frac{1}{2}}};$$

the equations are

$$v_1(\zeta) = u_1(\zeta)+(ka)^{-1}\int_a^\zeta G_n^m(\zeta, t)\, v_1(t)\, dt, \tag{18}$$

$$v_2(\zeta) = u_2(\zeta)+(ka)^{-1}\int_a^\zeta G_n^m(\zeta, t)\, v_2(t)\, dt, \tag{19}$$

where $G_n^m(\zeta, t)$ denotes

$$\left[n(n+1)-\frac{m^2-1}{1+t^2}\right]\frac{\sin ka(\zeta-t)}{(1+\zeta^2)^{\frac{1}{2}}(1+t^2)^{\frac{1}{2}}},$$

and a is independent of ζ.

Now, if $$w(\zeta) = (1+\zeta^2)^{\frac{1}{2}}\, v_1(\zeta),$$

$$w(\zeta) = \sin ka\zeta + \frac{1}{ka}\int_a^\zeta \left(\frac{1+\zeta^2}{1+t^2}\right)^{\frac{1}{2}} G_n^m(\zeta, t)\, w(t)\, dt. \tag{20}$$

But $$\int_\zeta^\infty \left| \left(\frac{1+\zeta^2}{1+t^2}\right) G_n^m(\zeta, t) \right| dt = M\int_\zeta^\infty \frac{dt}{1+t^2} = M\cot^{-1}\zeta,$$

where M is finite.

Hence $\int_\zeta^\infty \left(\frac{1+\zeta^2}{1+t^2}\right)^{\frac{1}{2}} G_n^m(\zeta, t)\, dt$ is absolutely convergent.

G. C. Evans* has shown that under this condition the limit a in the integral equation (20) may be made infinite.

Consequently a may be made infinite in (18) and similarly in (19).

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* Evans, "Sopra l'equazione integrale di Volterra," *Atti Lincei*, 1911.

The functions $v_1(\zeta)$, $v_2(\zeta)$ are then solutions of (2) which approximate respectively to $\sin ka\zeta/\zeta$ and $\cos ka\zeta/\zeta$ as ζ tends to infinity.

From these is derived the solution

$$Z_n^{-m}(\zeta) = v_2(\zeta) - \iota v_1(\zeta),$$

which approximates to $e^{-\iota ka\zeta}/\zeta$ as ζ tends to infinity.

I hope, in a subsequent communication, to discuss the application of these solutions to some problems in connection with Sound and Electromagnetic Waves. It may be noted that methods similar to those employed above can be employed to discuss the wave equation referred to elliptic cylindrical coordinates.

ADDRESS BY THE RETIRING PRESIDENT :* SOME PROBLEMS IN WIRELESS TELEGRAPHY

By Prof. H. M. MACDONALD.

[Read November 14th, 1918.]

THE aim of mathematical physics is the application of mathematics to the phenomena of physics to obtain an intelligible representation of these phenomena. Different types of problems can be recognised and of these the first is the selection of the particular problem which promises to admit of successful mathematical treatment, and at the same time to preserve the most essential features of the phenomena to be represented. This involves the careful comparison of the available observations connected with the phenomena under consideration, with the view of ascertaining the most outstanding resemblances between these observations, and thence deducing the most likely underlying physical source of the phenomena. The problem having been strictly defined on the physical side, the next step is to choose the geometrical setting; and if the results obtained by the solution of the problem thus selected are in sufficient agreement with the observations, the primary problem can be regarded as solved. There then remain the other types of problems to be solved, viz. those obtained by varying the geometrical setting and those that result from taking into account other physical causes. In illustrating the foregoing remarks the problems of electrostatics may be cited. The primary problem is that of the perfectly conducting sphere in an indefinitely extended vacuum. When the solution of this problem had been obtained, the next object of investigation was the effect on the result of altering the shape of the conducting body, and this led to the discussion of the case of the ellipsoid, the circular disc, a body with a sharp edge such as a spherical bowl, a ring shaped body such as an anchor ring, and so on. There are further the problems which result from substituting for the perfectly conducting body an imperfectly conducting body or re-

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placing the vacuum outside the body by some dielectric medium either homogeneous or non-homogeneous.

The form in which the solution of a mathematical physical problem is presented is of considerable importance, when it is remembered that comparison has to be made with the results of observation; and the ideal solution is one which gives a simple picture of the phenomena and at the same time admits of rapid reduction to numerical units. The problem of the point charge outside a perfectly conducting sphere affords a perfect example, the effect of the induced distribution in the sphere being equivalent to that of a point charge placed at a definite point inside the sphere.

The particular problems which it is proposed to examine from the above point of view are those of wireless telegraphy, viz. the problems of the emission of electric radiation from a sending station, its reception at another station, and of its transmission from the one to the other.

There are three distinct types of problems connected with the oscillation of a vibrating system; the case of a vibrating system from which there is no loss of energy; the case of a vibrating system from which energy is being radiated freely; and the case of a vibrating system through which the loss of energy by radiation is being made good at exactly the same rate as the energy is being radiated from the system, so that the radiation is steady. Examples of the three different types are the vibration of a gas in the space bounded by a closed surface such as a sphere; the radiation from a nearly closed surface, such as a sphere with an aperture in the surface, of the energy of a disturbance set up in the air inside; and the radiation from such a space when the energy inside is being maintained. The first problem has been completely solved in the case of the sphere; and the result is that any possible system of vibration in the space can be expressed in terms of certain normal types unlimited in number, any one of which can exist separately. The problem of free radiation from the space such as that inside a spherical bowl has not been solved, but it can be predicted that the solution will involve oscillations of different periods and different rates of decay, and further that the oscillations of a particular period cannot necessarily exist separately.

If in the case of the spherical bowl the aperture is small, the periods of the different oscillations will differ but slightly from the periods of oscillations of the air in the closed sphere, and the rates of decay will be small; but it may be expected that, as the aperture is increased in size, the periods will differ more and more from those of the oscillations in the closed space, and at the same time the rates of decay will increase.

The problem of a radiating system, when the loss of energy is being

replaced so that the radiation is steady, admits of being solved at least approximately in several cases, for example the case of the sphere with a circular aperture. The problem to be solved in such a case is that of finding the periods of the oscillation which are most easily maintained, *i.e.* the periods of resonance; and if the aperture is small the periods will differ but slightly from those of the free oscillations in the closed space, but as the aperture is increased the periods will differ appreciably both from those of the free oscillations in the closed spaces and those of the second problem when the energy is being freely radiated without being replaced. The difference between the periods in the three cases can be illustrated from the case of a simple vibrating system with one period $2\pi/n$ when there is no decay of energy; if this system loses energy and the rate of decay is m, the period of the free oscillation is $2\pi/\sqrt{(n^2-\frac{1}{4}m^2)}$, and the period of the oscillation to which the system resonates is $2\pi/\sqrt{(n^2-\frac{1}{2}m^2)}$, from which it appears that when the rate of decay is appreciable the difference between the corresponding periods may be considerable. It should be observed that in the case of maintained oscillations the oscillations of a particular period can be treated separately. It should not however be concluded that any radiating system can act effectively as a resonator; certain conditions have to be satisfied. Taking the case of the sphere with an aperture, it is clear that, if the energy of the oscillation is maintained by a source placed inside the sphere, it would be possible to replace the single aperture by a number of apertures in the surface, the total radiation outwards remaining the same; but, if the source maintaining the oscillations is outside the sphere, the effects of the different apertures will not generally reinforce each other, but will interfere, and resonance will not take place. A necessary condition therefore for effective resonance is that the radiating system which is to be used as a resonator is such that the radiation given out by it is concentrated. Further a radiating system which is to be effective as a source for the emission of radiation must be one for which the rate of decay is appreciable.

These two conditions must be satisfied by any electrical system which is to be efficient for the emission and detection of electrical radiation, and it is therefore necessary to consider the problem of radiation from a conducting body with these conditions in view. The problem of radiation from a perfectly conducting sphere has been completely solved, and taking the simplest case, viz. that in which the initial electrical distribution on the surface is specified by a spherical harmonic of the first order, it appears that in the immediate neighbourhood of the surface the transfer of energy outwards from the sphere takes place from the equator. This result ad-

mits of generalisation for the case of any perfectly conducting body; for the electrical force is everywhere normal to the surface of the body at the surface, and therefore the energy in the immediate neighbourhood of the surface can only flow along the surface at any place where the electric force is finite; hence the energy in the immediate neighbourhood of the surface can only leave the surface at places where the electric force vanishes. Applying this to the case of an ovary ellipsoid of revolution, the initial distribution being one which vanishes only at the equator, the radiation will always take place from the equator; and as the ellipsoid approaches to the form of a straight rod, terminated at both ends, the wave length of the oscillations approximates to double the length of the rod, while at the same time the rate of decay of the oscillations tends to zero. It therefore follows that a straight conducting rod for which the conditions presupposed in this solution, viz. that the surrounding medium can support the electric forces everywhere at the surface, are satisfied, cannot be effective for the emission or detection of oscillations. It should however be observed that as the ellipsoid approaches the form of a straight rod the amplitude of the electric force in the immediate neighbourhood of the ends increases, ultimately being indefinitely great; and when the surrounding medium is air it may be expected that, as in the corresponding case of a charged conductor with a sharp point or edge, electric discharge will take place from the ends. This was first observed by Sarasin and Birkeland, and an examination of their observations shows that radiation is taking place from the end of the wire. An exact solution of the problem of radiation from a freely radiating perfectly conducting straight rod would require a knowledge of the mechanism of the discharge at the ends which is not so far available; but if, as in the experiments above referred to, the energy radiated away is replaced so that the radiation is steady, the flow of energy outwards from the rod cannot differ essentially from the flow from a simple electric oscillator. The problem then admits of solution, and the result is that the wave length of the oscillation of longest period is two and a half times the length of the rod. The determination of the wave length of the radiation from a straight rod has been the subject of experimental investigation by a number of different observers, who have obtained results which range from double the length of the rod to two and a half times its length; but an examination of the conditions, where these have been sufficiently detailed, would seem to show that these differences are to be accounted for by the fact that the arrangements are different and that different phenomena are being observed.

In some of the experiments the arrangements are clearly such that the energy associated with the distribution which has been set up on the rod

by the external oscillation is practically being freely radiated away, and therefore the observed wave length is neither that which corresponds to steady radiation, when the energy is maintained, nor that which corresponds to the case of a rod where the surrounding medium is such that no radiation takes place from the ends. In other cases the arrangements are such that radiation from the ends is prevented, as for example when the rod is immersed in some non-conducting oil.

The wave length corresponding to the other possible periods are readily obtained; and it is important to observe that in these cases the distance between successive nodes along the rod is the wave length of the oscillations in question, a result which has been verified for wires by various observers. The solution of the problem for the case of an imperfectly conducting rod can be obtained; and the result is that (if the specific resistance can be assumed to be approximately the same in the case of oscillation as that for metallic conductors in which there are steady currents) the relation between the wave lengths of the oscillation and the length of the rod only differs by very small quantities from the relation between the wave lengths and the length of the rod when the rod is perfectly conducting. This assumes that the magnetic permeability of the material of the rod is the same as that of the surrounding medium or does not differ greatly from it. If the magnetic permeability of the material of the rod is of the same order as that of iron, the difference in the relation would be appreciable. These results are in agreement with observation; in particular it has been observed that for copper wires the difference between successive nodes along the wire is equal to the wave length of the oscillation, while the distance between successive nodes along an iron wire differs from the wave length by an amount which, though small, is appreciable.

It follows from the above that the difference between the observed wave lengths for the straight rod and double its length cannot be referred to imperfect conductivity for two reasons, viz. that the effect of imperfect conductivity is too small if it is of the same order as in the case of a steady current, and that the distance between successive nodes along the rod does not differ appreciably from the wave length of the oscillations. An idea of the magnitude of the rate of radiation from the rod, when it is radiating steadily, can be obtained by comparing it with a simple vibrating system; in such a system the amplitude of the oscillations would diminish by approximately one-fourteenth for each oscillation when radiating freely, if the relation between the wave length when there is no radiation and when the radiation is steady were as 4 to 5. Hence it may be concluded that the radiation from the rod is not small; and as, further, this radia-

tion takes place from the ends, the rod can act effectively both for the emission of radiation and for resonating to radiation from other sources. The simplest arrangement of a sending or receiving station consists essentially of a vertical antenna in which the radiation is emitted or collected by the free end; and the presence of points or angles from which radiation can take place is an essential feature of the arrangements of all wireless telegraph stations. It has been observed that the effective distance of a station depends on the height of the antenna, both at the sending station and at the receiving station; and in particular for the case of umbrella stations that the height which is effective is the height of the extremities of the ribs of the umbrella above the conducting surface, thus showing that the radiation is emitted and received at the extremities of the ribs.

An important problem in this connection is the determination of the wave length of the oscillations which are most effective for transmitting signals. When it is remembered that each signal occupies a time which is very great compared with the period of an oscillation, it is clear that the production of a signal requires a train of waves containing very many oscillations, and, that being so, the problem to be solved is approximately the problem of radiation from the sending station when the radiation is steady; and, as has been seen, this problem admits of solution in certain cases. For example, in the case of the simple vertical antenna, the required wave length is that belonging to a straight conducting rod radiating steadily, the length of the rod being double the height of the antenna, so that the fundamental wave length in this case is five times the height of the antenna. This result agrees with observation. The solution of the problem in more complicated arrangements has not so far been solved, but the same method with the necessary slight modifications would apply.

Difficulties have arisen in connection with the measurements of the effect at a distance from the sending station. It has usually been assumed by observers that the observed disturbance is expressible in terms of the square of the amplitude of the oscillation. The reason for this assumption is not clear; but it may have been suggested by the expression for the intensity of sound or the expectation that it depended simply on the energy. The essential feature of any detecting arrangement would appear to be that the resistance in a portion of a circuit is not constant, but that the resistance diminishes when the electric force increases above a certain magnitude. Accurate information is not available to enable the problem of any particular arrangement being stated in accurate terms, so as to be submitted to analysis; but it is comparatively easy to state a mathematical

problem which involves the essential fact that the resistance is not constant and which admits of solution. For example, assuming that the resistance is constant when the electric force is less than E_1, while, when it is greater than E_1, the change in the conductivity is proportional to its excess over E_1 when the electric force is in one direction, with corresponding conditions when the electric force is in the other direction, the electric force for which the conductivity begins to vary in this case being E_2, a relation can be obtained between the resultant current in the detecting circuit and E the amplitude of the electric force in the incident waves. When E is less than E_1 and E_2 the resultant current is proportional to E, and when E is greater than E_1 and E_2, the expression for the resultant current tends to the form $aE+bE^2$, ultimately tending to bE^2.

It follows that the relation between the amplitude of the oscillations and the current in the receiving telephone is in general not a simple relation, although it may be expected that, as in the above formula when E is small, as it will be at a considerable distance, the current in the telephone circuit is approximately proportionally to the amplitude of the electric force. This agrees with the result of observation.

The remaining problems are those connected with the transmission of signals to a distance around the earth's surface. The primary problem in this connection is to select the simplest arrangement which possesses the essential features; and when the portion of the earth's surface that intervenes between the two stations is covered by the sea, the problem is at once simplified by assuming the surface to be perfectly conducting. Further when it is remembered that the electric force is everywhere perpendicular to a perfectly conducting surface, it is clear that the essential features of the problem are preserved if the source of the waves is taken to be a simple oscillator whose axis is perpendicular to the surface. This problem admits of solution, and it has been shown that the results obtained agree with the observed results at a considerable distance. The explanation provided by this theory, known as the diffraction theory, accounts for the most important features of the transmission of signals. Probably the reluctance to adopt it owes in some measure its origin to comparison with optical phenomena; but it should be observed that as the ratio of the wave length used in wireless telegraphy to the earth's radius is of the order 10^{-3}, the size of the corresponding sphere in the case of light is indefinitely small, and the observed results in the case of optical phenomena do not provide a true analogy. The remaining problems connected with the transmission of signals are first the effect of imperfect conduction and second the effect of the atmosphere.

In the case of the transmission over the surface of the sea, the effect

of imperfect conduction is, as might have been expected, negligible, and not greater than the possible errors of observations at the distances involved. With regard to the effect of the atmosphere, there is not at present sufficient detailed observation to enable the problem to be submitted to accurate mathematical analysis. The main question to be answered would appear to be the effect of change of atmospheric conditions at sending stations or at the receiving stations, that is whether the intensity of the radiation from a station depends on the atmospheric conditions at that station. Further there is the question as to whether there is reflection of the waves from the upper atmosphere, and whether there may not be absorption under certain conditions. There are indications that under certain circumstances reflection does take place in the atmosphere, but until more information is available as to the conditions obtaining in the atmosphere at different heights above the earth's surface, an approximate estimate of the effect to be expected cannot be obtained.

THE THREE-BAR SEXTIC CURVE

By G. T. Bennett.

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1. In a recent paper* Col. R. L. Hippisley gives a theorem for a three-bar curve which may be stated thus :—If P is any point on the curve, and if PD, PE, PF are the perpendiculars from P on to the sides BC, CA, AB of the triangle of foci, then D, E, F are at constant distances from a variable point T. He further proposes a linkage involving nine Peaucellier cells (and needing 69 moving links in all) in order to reproduce the sextic in a mechanical manner by use of the theorem. He finds that P', the image of P in T, is another point on the sextic, and, on the basis of certain equations, he credits the locus of T with degree 24 (subject to a tentative reduction to 12).

The first purpose of the following notes is to point out a simple but unregarded property of the triple-generation mechanism of Roberts which furnishes a *raison d'être* for the pedal theorem in question, and for the association of the points P, P' and T: to indicate what appear other ways, more natural and economical, in which the geometrical theorem may assume a mechanical aspect: and to show that the locus of T is nothing but a cubic.

The further and more extensive purpose of the notes is to put into a concise and symmetrical form some of the cardinal results due to Cayley,† to bring the analysis of Darboux ‡ into intimate relation with the sextic curve, and to add to the known theorems some noteworthy properties of the figure. So fundamental is the three-bar movement in the theory of mechanisms that any pains are well spent that may tend to develop the significant features of the associated geometry; and the intention has here been to exclude all gratuitous additions and to set forth only such material as seems inevitably bound up with the simple mechanism itself.

* *Proc. London Math. Soc.*, Ser. 2, Vol. 18 (1919), pp. 136–140.

† *Proc. London Math. Soc.*, Ser. 1, Vol. 7 (1876), pp. 136–166 and 166–172.

‡ *Bulletin des Sciences Mathématiques*, 1879, pp. 109–128.

2. The triple-generation mechanism consists (Fig. 1) of three directly similar triangular plates $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, with the vertices A_1,

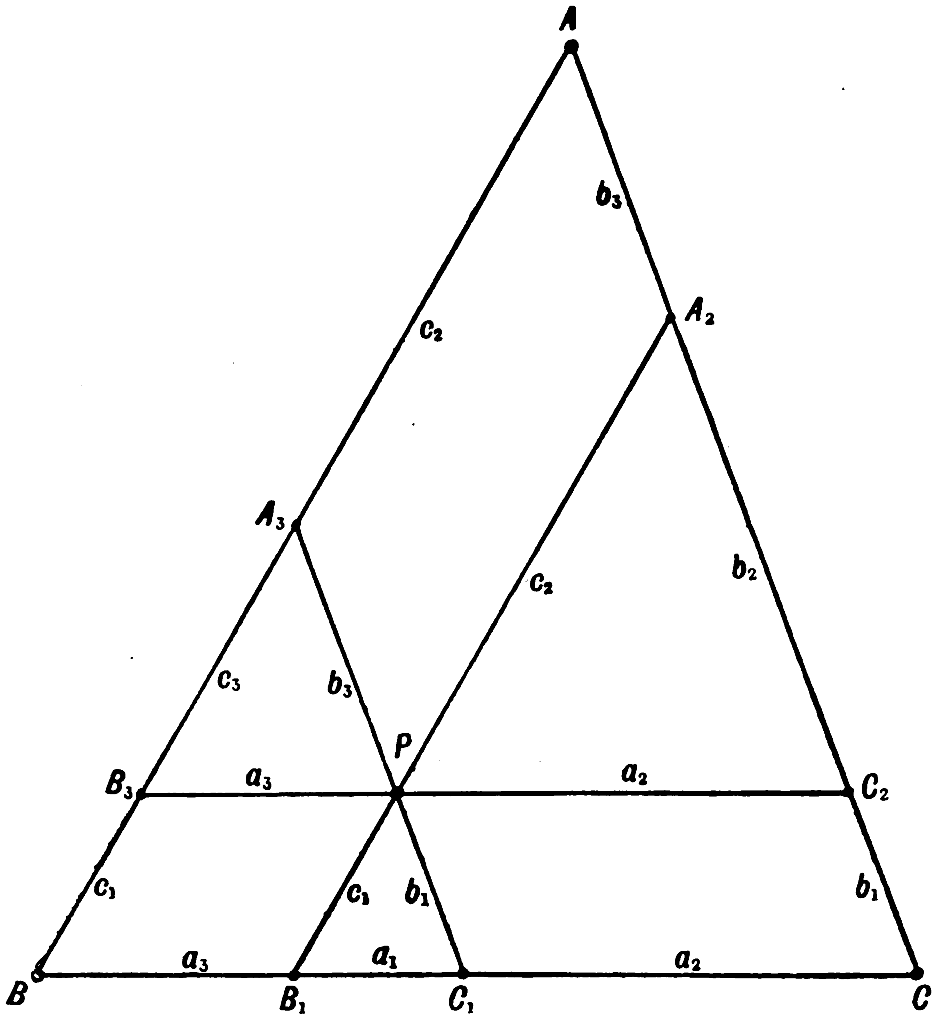

Fig. 1.—The triple-generation mechanism in its zero form ($\theta_1 = 0$, $\theta_2 = 0$, $\theta_3 = 0$) showing notation.

B_2, C_3 united at P, together with three pairs of links completing the parallelograms A_2PA_3A, B_3PB_1B, C_1PC_2C. For all variations of the angles of the parallelograms the triangle ABC remains directly similar to each of the triangular plates; and if ABC is itself kept fixed and invariable there arises the triple-generation of the locus of P (Fig. 2). The ratios

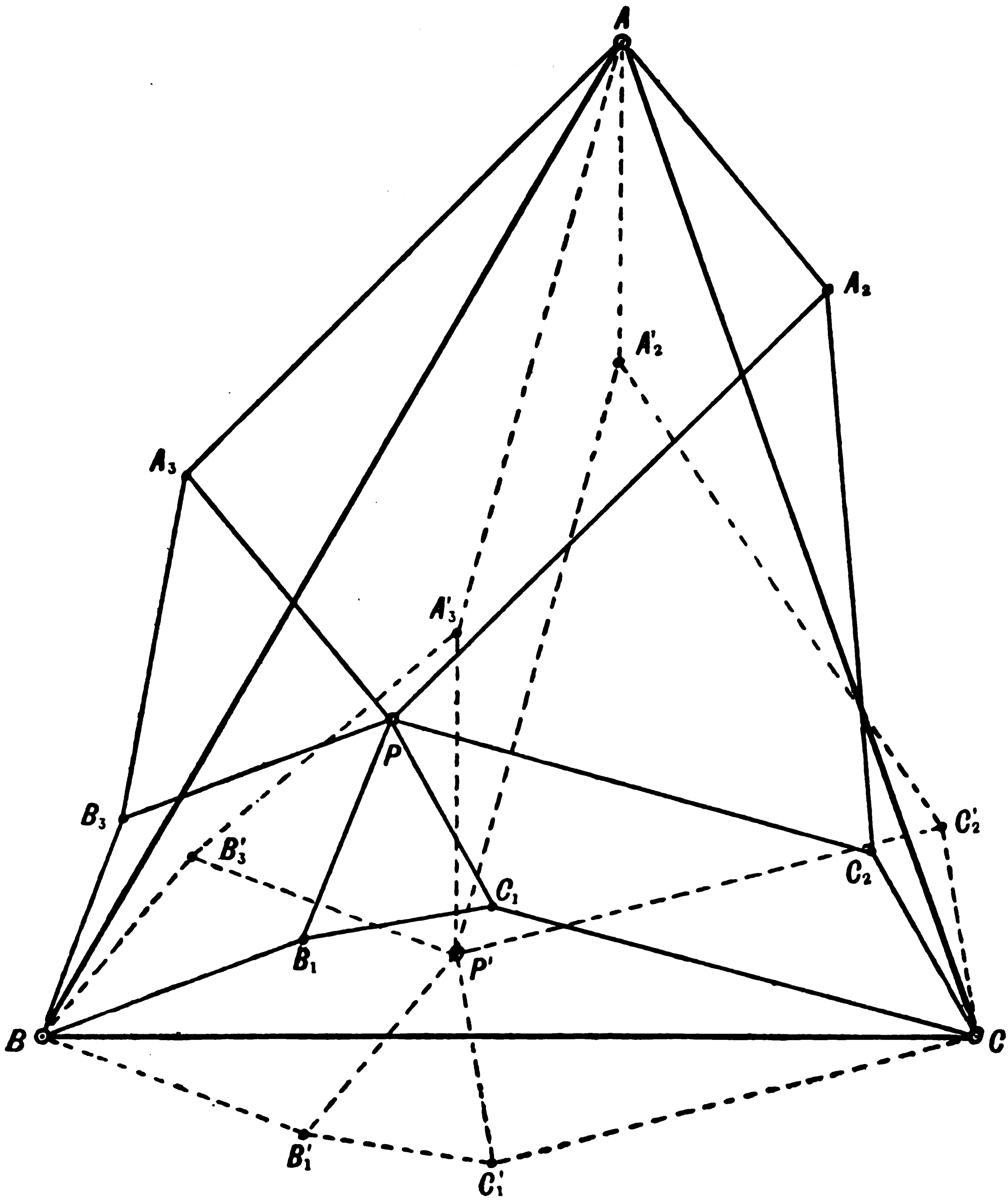

FIG. 2.—The triple-generation mechanism in a pair of corresponding positions (θ_1, θ_2, θ_3 and $-\theta_1$, $-\theta_2$, $-\theta_3$).

of the sides of the plates to the corresponding sides of the triangle ABC are then constant quantities k_1, k_2, k_3; and the corresponding angles of inclination are variable angles θ_1, θ_2, θ_3. The angles are so related that the sum of the vectors (k_1, θ_1), (k_2, θ_2), (k_3, θ_3) is a vector (1, 0) of unit length with direction-angle zero. The quadrilateral linkworks BB_1C_1C, CC_2A_2A, AA_3B_3B represent each the same vector summation, but with linear (scalar) multipliers BC, CA, AB respectively.

It is apparent therefore that, to any position of the mechanism, there corresponds another position for which $\theta_1\theta_2\theta_3$ have their signs all reversed; and that these pairs of positions give pairs of points P and P' on the three-bar sextic curve. The points P and P' may be called "corresponding points," and the line PP' may be called a "principal chord" of the sextic. If the two corresponding forms of the mechanism are shown in the one figure (Fig. 2) the three three-bar linkages $BB_1'C_1'C$, $CC_2'A_2'A$ and $AA_3'B_3'B$ are images of BB_1C_1C, CC_2A_2A, AA_3B_3B in BC, CA, AB respectively. But the triangles $P'B_1'C_1'$, PB_1C_1 are directly and not reversely equal: and hence Q, the image of P in B_1C_1, which is a point of the plate PB_1C_1, coincides with the image of P' in BC. (Fig. 3.) Thus P and Q, mutually images in B_1C_1, describe reversely equal sextics which are images in BC; and the points P, Q "correspond" as P and P' when either sextic is reflected to coincide with the other. Similarly, if Q' is the image of P' in $B_1'C_1'$ it is also the image of P in BC; so that BC perpendicularly bisects PQ' at D and $P'Q$ at D' (Fig. 3). If T is the middle point of the principal chord PP' of the sextic, TD is parallel to and equal to half of $P'Q'$, and so equal to PD_1, which is constant. Similar results hold for CA and AB. Thus $TD = PD_1$, $TE = PE_2$, $TF = PF_3$, all constant lengths, and the image relationships take the form of the pedal theorem, that T is at constant distances from the feet of the perpendiculars PD, PE, PF. The corresponding point P' gives rise to the same point T and the same constant lengths. In its simplest mechanical aspect the theorem involves simply the constancy of PQ, and the description by P and Q, images in B_1C_1, of reversely equal sextics, images in BC.

3. If the figure of the mechanism associated with P' and T' (coincident with T) receives a translational displacement such as to bring P' to T and T' to P, then the two figures exhibit a number of simple relations. Notably the points $D_1'E_2'F_3'D'E'F'$, feet of the perpendiculars from P' on to the sides, are brought to the positions $DEFD_1E_2F_3$; so that $P'D_1'$, $P'E_2'$, $P'F_3'$ coincide with TD, TE, TF, conformably with the pedal theorem. Reciprocally the perpendiculars PD_1, PE_2, PF_3 coincide with $T'D'$, $T'E'$, $T'F'$; so that the feet of the perpendiculars of the triangular

plates of either figure lie on the sides of the focal triangle of the other figure.

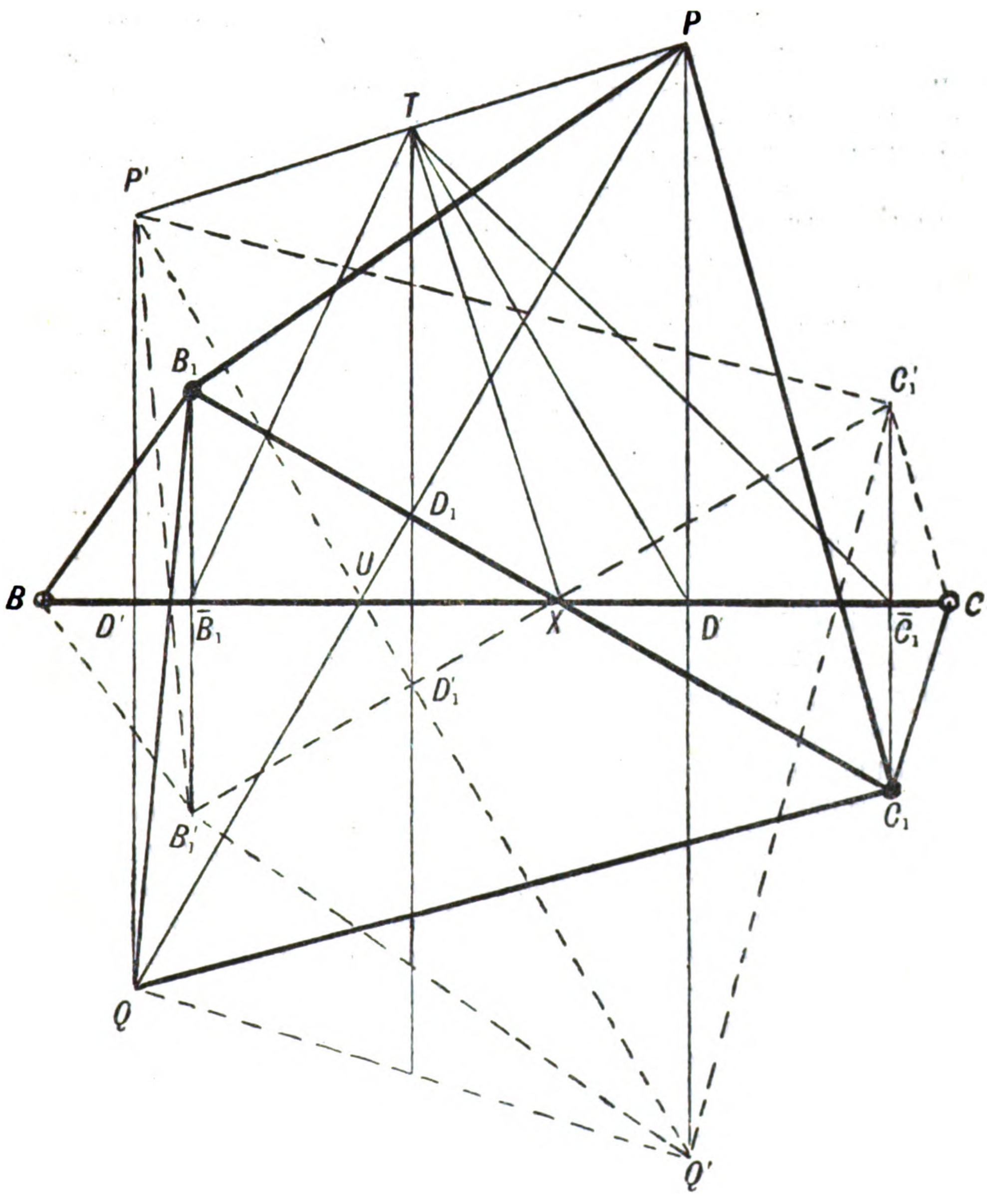

FIG. 3.—A three-bar mechanism in a pair of corresponding positions (θ_1, θ_2, θ_3 and $-\theta_1$, $-\theta_2$, $-\theta_3$).

4. Another mechanical aspect may be given to the geometry by supposing that an inextensible thread passes geodesically from P to P' *viâ* BC. If PQ and $P'Q'$ cut BC in U (Fig. 3), then the thread lies along PU and UP' and has constant length equal to PQ, that is $2PD_1$. If the two points PP' are connected by three such threads PUP', PVP', PWP', all constant and geodesic and each visiting one side of the triangle ABC, then P and P' describe each the same three-bar curve, and on this curve P and P' are corresponding points. The theorem associates itself with those of Graves and Staude for confocal conics and quadrics, and others of this special geodesic type. It may be noticed that the effect of the threads is to cause P and P' to be the foci of three different conics, each with a given length for its major axis, and each touching one side of the triangle ABC. (It is plain that the thread representation is limited to the case of ellipses and the summation of lengths; hyperbolas and differences precluding mechanical simplicity.)

5. The locus of T may now be found. Let B_1C_1 and $B_1'C_1'$, images in BC, have $\bar{B}_1\bar{C}_1$ as their projection on BC, and let them meet BC in X. (Fig. 3.) Then with X, the centre of the circle $PP'QQ'$, as centre of rotation, and $2\theta_1$ as angle of rotation, the triangle $P'B_1'C_1'$ can be brought to coincide with PB_1C_1. The points T, $\bar{B}_1$, $\bar{C}_1$ bisect the displacements $P'P$, $B_1'B_1$, $C_1'C_1$ of the vertices; hence $T\bar{B}_1\bar{C}_1$ is similar to each triangle and therefore also to ABC. Hence $T\bar{B}_1$ and $T\bar{C}_1$ are parallel to AB and AC. With other like results included it follows that each line through T parallel to a side of the triangle ABC meets the other two sides in points which are the orthogonal projections of vertices (other than P) of the triangular plates: $\bar{B}_3\bar{C}_2$, $\bar{C}_1\bar{A}_3$, $\bar{A}_2\bar{B}_1$, that is, are parallel to BC, CA, AB and meet in T. The ranges given by these six variable points are similar in pairs and such that a (2, 2)-correspondence holds for any two of the others, with the infinity element self-corresponding on each side. The concurrent sets of parallels $\bar{B}_3\bar{C}_2$, $\bar{C}_1\bar{A}_3$, $\bar{A}_2\bar{B}_1$ have the same property; and so it follows that the locus of T is a cubic curve having its asymptotes parallel to the sides BC, CA, AB.

Another property of the point T may be noticed. As D_1T is parallel to (and half of) $P'Q$ it is normal to BC. So lines perpendicular to BC, CA, AB through D_1, E_2, F_3 (the moving feet of the perpendiculars PD_1, PE_2, PF_3 of the triangular plates) remain always concurrent, and meet in the point T.

6. It may be noticed incidentally that the figure presented by the

points P and P', with Q, R, S the images of P' in BC, CA, AB, gives rise to a three-position theorem. Two rigid configurations, that is, are supposed to occupy in turn three different relative positions: the triangle ABC with the point P represents one plate, and the images of P' and of this plate in BC, CA, AB represent the three positions of the other plate. So PQ, PR, PS are the distances apart of the selfsame points, one of each plate, on the three occasions. If these distances are given, then the points P, P' (as has been seen) are corresponding points of a three-bar sextic; and hence the points, one in each plate, lie on reversely equal sextics, at points which "correspond" (but do not coincide) on reflection of the sextics into coincidence.

7. Some properties of the mechanism are unnecessarily restricted by the supposition that ABC is a fixed triangle, and remain true with two degrees of internal freedom given to the mechanism in place of one. The pedal theorem is a case in point: for the lengths TD, TE, TF remain constant and equal to PD_1, PE_2, PF_3 not merely for one sextic locus of P with ABC fixed, but for all sizes of the triangle ABC as well. From this enlarged point of view a locus for T does not arise.

8. The notes now following, put in a moderately condensed form sufficient for their purpose, will give an analytical account of the geometry of the three-bar curve in terms of areal coordinates having the focal triangle as triangle of reference.

It is convenient to use quantities λ, μ, ν defined by equations

$$a^2 = \mu+\nu, \quad b^2 = \nu+\lambda, \quad c^2 = \lambda+\mu, \tag{1}$$

so that $\cos A = \lambda/bc, \quad \sin A = \delta/bc, \quad \delta^2 = \Sigma\mu\nu, \quad \delta^2\rho = -\lambda\mu\nu,$ (2)

where $\frac{1}{2}\delta$ is the area of the triangle, and ρ is the squared radius of the polar circle. The squared distance between two points is then $\Sigma\lambda(x-x')^2$, and the circular points at infinity have equation

$$\Omega \equiv \Sigma\lambda(m-n)^2 = 0. \tag{3}$$

The angles θ_1, θ_2, θ_3 are reckoned in the sense of circulation of the points ABC. Being the directions of vectors of lengths k_1, k_2, k_3 with vector sum unity in direction zero, they satisfy the equations

$$k_1 \cos\theta_1 + k_2 \cos\theta_2 + k_3 \cos\theta_3 = 1, \tag{4}$$

$$k_1 \sin\theta_1 + k_2 \sin\theta_2 + k_3 \sin\theta_3 = 0. \tag{5}$$

With these may be associated other forms of equation derived from them

$$k_1 + k_2 \cos \theta_{12} + k_3 \cos \theta_{31} = \cos \theta_1, \tag{6}$$

$$k_2 \sin \theta_{12} - k_3 \sin \theta_{31} = \sin \theta_1, \tag{7}$$

$$k_1 \cos \theta_1 + k_2 k_3 \cos \theta_{23} = \tfrac{1}{2}(1 + k_1^2 - k_2^2 - k_3^2), \tag{8}$$

and

$$\Sigma\, k_2 k_3 \cos \theta_{23} = \tfrac{1}{2}(1 - \Sigma k_1^2), \tag{9}$$

where $\theta_{23} \equiv \theta_2 - \theta_3$, etc.

The coordinates of P take the form

$$x = k_1 \cos \theta_1 - (\mu k_2 \sin \theta_2 - \nu k_3 \sin \theta_3)/\delta, \text{ etc.}, \tag{10}$$

and, changing the signs of θ_1, θ_2, θ_3 for P',

$$x' = k_1 \cos \theta_1 + (\mu k_2 \sin \theta_2 - \nu k_3 \sin \theta_3)/\delta. \tag{11}$$

The point T midway between P and P' is therefore

$$x = k_1 \cos \theta_1, \quad y = k_2 \cos \theta_2, \quad z = k_3 \cos \theta_3, \tag{12}$$

giving $x + y + z = 1$ in virtue of (4); and, from (5),

$$(k_1^2 - x^2)^{\frac{1}{2}} + (k_2^2 - y^2)^{\frac{1}{2}} + (k_3^2 - z^2)^{\frac{1}{2}} = 0 \tag{13}$$

is the equation of its locus.* In rationalized form the resulting quartic

$$\Sigma\,(x^2 - k_1^2)^2 - 2\Sigma\,(y^2 - k_2^2)(z^2 - k_3^2) = 0 \tag{14}$$

has the line infinity as a factor, and the remaining cubic locus has equation

$$(-x+y+z)(x-y+z)(x+y-z) - 2\Sigma k_1^2 . \Sigma x^2 + 4\Sigma k_1^2 x^2$$
$$+ (k_1+k_2+k_3)(-k_1+k_2+k_3)(k_1-k_2+k_3)(k_1+k_2-k_3) = 0. \tag{15}$$

The asymptotes are

$$x = \tfrac{1}{2}(1 + k_1^2), \quad y = \tfrac{1}{2}(1 + k_2^2), \quad z = \tfrac{1}{2}(1 + k_3^2), \tag{16}$$

parallel to the sides of the focal triangle.

The cubic (15) cuts the sides of the focal triangle at points corresponding to a value $\frac{1}{2}\pi$ (mod π) for any one of the three angles θ_1, θ_2, θ_3; *i.e.* when one link of each three-bar linkage is perpendicular to the fixed base-link.

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* This, and any later equation not homogeneous in x, y, z, may be rendered homogeneous by use of the unit factor $x + y + z \equiv t$.

9. Calculation of the length PP' from (10)–(11) gives

$$\tfrac{1}{4}PP'^2 = TP^2 = TP'^2 = \Sigma\lambda k_1^2 \sin^2\theta_1. \tag{17}$$

If O is orthocentre of ABC with coordinates $(-\rho/\lambda,\ -\rho/\mu,\ -\rho/\nu)$,

$$OT^2 = \Sigma\lambda k_1^2 \cos^2\theta_1 + \rho. \tag{18}$$

So $$OT^2 + TP^2 = \sigma + \rho \text{ a constant,} \tag{19}$$

where $$\sigma \equiv \Sigma\lambda k_1^2. \tag{20}$$

It follows that a fixed circle having O as centre and $\sigma+\rho$ as squared radius, cuts the circle on PP' as diameter at the ends of another diameter; or cuts orthogonally the circle centre T and radius iTP. The equation of this fixed circle may be written in the form

$$\lambda(x^2-k_1^2)+\mu(y^2-k_2^2)+\nu(z^2-k_3^2) = 0. \tag{21}$$

10. The line PP' has equation

$$\lambda k_1 \sin\theta_1 \,.\, x + \mu k_2 \sin\theta_2 \,.\, y + \nu k_3 \sin\theta_3 \,.\, z = \Sigma\lambda k_1^2 \sin\theta_1 \cos\theta_1. \tag{22}$$

The parallel through O is

$$\lambda k_1 \sin\theta_1 \,.\, x + \mu k_2 \sin\theta_2 \,.\, y + \nu k_3 \sin\theta_3 \,.\, z = 0, \tag{23}$$

and the area of the triangle POP' is $\Sigma\lambda k_1^2 \sin\theta_1 \cos\theta_1$. The line $TXYZ$, say Λ, perpendicularly bisecting PP' is

$$(\sin\theta_{23}/k_1)\,x + (\sin\theta_{31}/k_2)\,y + (\sin\theta_{12}/k_3)\,z = 0, \tag{24}$$

passing through T (12) and meeting the line infinity at the point

$$k_1 \sin\theta_1 \,.\, l + k_2 \sin\theta_2 \,.\, m + k_3 \sin\theta_3 \,.\, n = 0. \tag{25}$$

Any point on this line Λ has coordinates of the form

$$x = k_1(\cos\theta_1 + r\sin\theta_1), \text{ etc.} \tag{26}$$

where r is arbitrary. If $r = -\cot\theta_1$ the point X is obtained as $(0,\ k_2\sin\theta_{12}/\sin\theta_1,\ -k_3\sin\theta_{31}/\sin\theta_1)$, and similarly for Y and Z.

The squared distance of the point (26) from T is equal to $r^2\Sigma\lambda k_1^2 \sin^2\theta_1$; and hence (17), if $r^2 = -1$ the antipoints Π, Π' of P, P' are obtained, with coordinates

$$\xi = k_1 e^{i\theta_1}, \quad \eta = k_2 e^{i\theta_2}, \quad \zeta = k_3 e^{i\theta_3}, \tag{27}$$

and $$\xi' = k_1 e^{-i\theta_1}, \quad \eta' = k_2 e^{-i\theta_2}, \quad \zeta' = k_3 e^{-i\theta_3}. \tag{28}$$

The envelope equation of these two points is

$$\Sigma k_1^2 l^2 + 2\Sigma k_2 k_3 \cos\theta_{23}\,.\,mn = 0. \tag{29}$$

The pairs of points P, P' and Π, Π' and the circular points at infinity ω, ω' are the pairs of vertices of a quadrilateral; and Π, Π' are imaginary when P, P' are real and conversely. The coordinates of either Π or Π' satisfy the equations

$$\left.\begin{array}{rl} & x+y+z = 1 \\ \text{and} & k_1^2/x+k_2^2/y+k_3^2/z = 1 \end{array}\right\}. \tag{30}$$

Hence the locus of the points Π, Π' is the cubic curve

$$H \equiv xyz - k_1^2 yz - k_2^2 zx - k_3^2 xy = 0. \tag{31}$$

It passes through the vertices of the focal triangle and through the points at infinity on the sides. The asymptotes are

$$x-k_1^2 = 0, \quad y-k_2^2 = 0, \quad z-k_3^2 = 0, \tag{32}$$

parallel to the sides of the focal triangle.

The asymptotes (16) of the cubic (15) are midway between the asymptotes (32) of the cubic H (31) and the opposite vertices of the triangle.

This cubic H is the Hessian of the cubic

$$U \equiv x^3/k_1^2 + y^3/k_2^2 + z^3/k_3^2 - 1 = 0, \tag{33}$$

and is also the Jacobian of the three pairs of parallels

$$x^2-k_1^2 = 0, \quad y^2-k_2^2 = 0, \quad z^2-k_3^2 = 0, \tag{34}$$

which are polar conics of the vertices of the focal triangle.

11. It thus appears that the theory of the three-bar sextic is co-extensive with the metrical geometry of a unique cubic curve U, which may be called "the representative cubic", from which the sextic curve is derivable. The cubic U determines pairs of corresponding points Π, Π' on the Hessian of U, each point having as polar conic with regard to U a line-pair through the other point; and P, P', the antipoints of Π, Π' in regard to the circular points at infinity, generate the sextic. Darboux* has used expressions and equations equivalent to (27) and (30), thus resting the two relations among the angles θ_1, θ_2, θ_3 upon an auxiliary cubic

* *Loc. cit.*, pp. 109, 110.

curve. But here the cubic H appears in explicit and intimate relation with the sextic curve itself; and a cubic U, one of the three cubics having H as Hessian, stands as parent of all the curves and of the correspondence of the point-pairs. An arbitrary cubic curve involves nine parameters; and the three-bar sextic, as here specified, needs six parameters for the focal triangle and three for the values of k_1, k_2, k_3.

Alternatively, the polar conics of the cubic U form a net of conics which generate the figure without use of the cubic. If a line Λ cuts the conics in an involution of points, the double points of the involution are Π, Π', and the antipoints P, P' of Π, Π' generate the sextic. In other terms, if Λ cuts the conics in an elliptic involution, then the circles having the chords of the conics on Λ as diameters have P, P' as common points. Any three conics serve to determine the net, and most simply the three pairs of parallels (34); and the corresponding circles have X, Y, Z as centres. As two parallel lines involve three parameters the determination of the system again depends on nine parameters.

12. The special types of conic of the net may now be examined in their relation to the sextic.

The polar conic of the cubic U (33) for the point $(x_0 y_0 z_0)$ is

$$\frac{x_0}{k_1^2}(x^2-k_1^2)+\frac{y_0}{k_2^2}(y^2-k_2^2)+\frac{z_0}{k_3^2}(z^2-k_3^2) = 0, \tag{35}$$

with $(k_1^2/x_0,\ k_2^2/y_0,\ k_3^2/z_0)$ proportional to the coordinates of the centre.

This conic becomes a line-pair when its Hessian is zero; and hence when $(x_0 y_0 z_0)$ lies on the cubic H (31), so that

$$k_1^2/x_0+k_2^2/y_0+k_3^2/z_0 = 1. \tag{36}$$

This equation and $x_0+y_0+z_0 = 1$ hold simultaneously (and exchangeably) for the point $\Pi(x_0\, y_0\, z_0)$ on H and for the corresponding point $\Pi'(k_1^2/x_0\ k_2^2/y_0\ k_3^2/z_0)$ through which pass the line-pair polar of Π.

The line-pairs which are polars of the points at infinity on the sides of the focal triangle ABC are

$$y^2/k_2^2-z^2/k_3^2 = 0, \quad z^2/k_3^2-x^2/k_1^2 = 0, \quad x^2/k_1^2-y^2/k_2^2 = 0. \tag{37}$$

These are pairs of sides of the quadrangle of points

$$x^2/k_1^2 = y^2/k_2^2 = z^2/k_3^2, \tag{38}$$

through which points pass the polar conics of all points on the line infinity.

The polar conics of A, B, C are the line-pairs (34).

Each pair of these parallels has a side of the focal triangle midway between them.

13. One conic of the net (35) is a circle. Its equation is

$$\lambda(x^2-k_1^2)+\mu(y^2-k_2^2)+\nu(z^2-k_3^2) = 0, \tag{39}$$

which has already appeared (21). It is concentric with the polar circle

$$\lambda x^2+\mu y^2+\nu z^2 = 0. \tag{40}$$

The conjugacy of Π and Π' makes the circle on $\Pi\Pi'$ as diameter orthogonal to the net circle; and hence, for the antipoints, the circle on PP' (any principal chord of the sextic) as diameter is cut at the ends of another diameter by the net circle centred at the orthocentre of the focal triangle.

The net circle (39) cuts the tricircular sextic three times at each circular point at infinity, and hence in six finite points. For each of these six points the corresponding point must also lie on the net circle; hence the net circle cuts the sextic at the ends of three principal chords. The lines Λ corresponding to these three principal chords pass all through the orthocentre O.

14. Among the net of conics (35) occur a single infinity of parabolas. They are the polar conics of points on the conic

$$k_1^2yz+k_2^2zx+k_3^2xy = 0, \tag{41}$$

which itself is the polar conic of the point

$$K \equiv k_1^2l+k_2^2m+k_3^2n = 0, \tag{42}$$

with regard to the triangle $xyz = 0$.

So any one of the parabolas is given by equation (35) subject to the condition

$$k_1^2/x_0+k_2^2/y_0+k_3^2/z_0 = 0. \tag{43}$$

Among the parabolas occur the parallels (34), polars of the vertices of the focal triangle.

The envelope of the parabolas is

$$\Sigma(x^2-k_1^2)^2-2\Sigma(y^2-k_2^2)(z^2-k_3^2) = 0, \tag{44}$$

which (14) consists of the line infinity and the cubic locus of the middle point T of principal chords PP'. Each parabola touches the cubic at

three points. The parabola touching at T (12) is

$$\Sigma\,(x^2-k_1^2)/k_1 \sin\theta_1 = 0, \tag{45}$$

the tangent being $\Sigma(x\cos\theta_1-k_1)/\sin\theta_1 = 0,$ (46)

and the line Λ, with infinity point

$$k_1 \sin\theta_1 \,.\, l+k_2\sin\theta_2 \,.\, m+k_3\sin\theta_3 \,.\, n = 0 \tag{47}$$

on the parabola, is the diameter through T.

The chord joining the other two contact-points is

$$\Sigma\,(\sin\theta_{23}/k_1)^2\, x-\tfrac{1}{2}\Sigma\sin\theta_{23}/k_1 \,.\, \Sigma\,(\sin\theta_{23}/k_1)\, x = 0, \tag{48}$$

which cuts Λ (24) on the line

$$\Sigma\,(\sin\theta_{23}/k_1)^2\, x = 0, \tag{49}$$

at the point ϖ, say. The point ϖ has coordinates

$$x = -k_1^2 \sin\theta_{31} \sin\theta_{12}/\sin\theta_2 \sin\theta_3, \text{ etc.}, \tag{50}$$

and lies on the cubic H, being the third point of the curve on the line $\Pi\Pi'$. The corresponding point on the cubic has coordinates

$$x = -\sin\theta_2 \sin\theta_3/\sin\theta_{31} \sin\theta_{12}, \text{ etc.}, \tag{51}$$

and is the point of intersection of tangents at Π and Π'.

Further a unique parabola touches the sides of the focal triangle and the line Λ. Its equation is

$$\sin\theta_1 \sin\theta_{23}\, mn+\sin\theta_2 \sin\theta_{31}\, nl+\sin\theta_3 \sin\theta_{12}\, lm = 0, \tag{52}$$

and it touches the line Λ at the same point ϖ.

Moreover, as two triangles whose sides touch a conic have their vertices on another conic, it follows that the parabola circumscribing ABC and having Λ as diameter passes through ϖ. Its equation is

$$k_1^2 \sin^2\theta_1 \,.\, yz+k_2^2\sin^2\theta_2 \,.\, zx+k_3^2\sin^2\theta_3 \,.\, xy = 0, \tag{53}$$

and it has as tangent at ϖ the line (49).

15. Among the net of conics (35) occur a single infinity of rectangular hyperbolas. The apolarity of (35) and the circular points $\Omega = 0$ gives

$$(a^2/k_1^2)\, x_0+(b^2/k_2^2)\, y_0+(c^2/k_3^2)\, z_0 = 0, \tag{54}$$

so that the rectangular hyperbolas are polar conics of points on the line

$$(a^2/k_1^2)\, x+(b^2/k_2^2)\, y+(c^2/k_3^2)\, z = 0, \tag{55}$$

and their centres lie on the circumcircle

$$a^2yz + b^2zx + c^2xy = 0. \tag{56}$$

The hyperbolas pass through four common points

$$(x^2 - k_1^2)/a^2 = (y^2 - k_2^2)/b^2 = (z^2 - k_3^2)/c^2, \tag{57}$$

forming a quadrangle of orthocentric points, say $I_0 I_1 I_2 I_3$, with diagonal triangle $N_1 N_2 N_3$, of which I_0 is the centre of the inscribed circle and $I_1 I_2 I_3$ the centres of the escribed circles.

Three of the rectangular hyperbolas degenerate into line-pairs and become the pairs of sides of the quadrangle such as $N_1 I_1 I_0$ and $N_1 I_2 I_3$, the angle-bisectors of the triangle $N_1 N_2 N_3$ at N. The conic (35) becomes two lines when $(x_0 y_0 z_0)$ and the centre of the conic are both on the cubic H. Hence the line (55) cuts the cubic H in three points whose polars are the orthogonal line-pairs through the corresponding points $N_1 N_2 N_3$. The points $N_1 N_2 N_3$, along with the centres of all the other rectangular hyperbolas, lie on the focal circle (56).

The point $$k_1^2 \rho_1 l + k_2^2 \rho_2 m + k_3^2 \rho_3 n = 0 \tag{58}$$

at infinity on the line (55) has as its polar conic the special hyperbola

$$\rho_1 x^2 + \rho_2 y^2 + \rho_3 z^2 = 0, \tag{59}$$

where $$\rho_1 = b_3^2 - c_2^2, \quad \rho_2 = c_1^2 - a_3^2, \quad \rho_3 = a_2^2 - b_1^2, \tag{60}$$

with the identities $$\Sigma a^2 \rho_1 \equiv 0, \quad \Sigma k_1^2 \rho_1 \equiv 0. \tag{61}$$

These quantities ρ_1, ρ_2, ρ_3 are named the "moduli" of the mechanism by Cayley. With regard to this unique rectangular hyperbola of the net the focal triangle is self-polar as well as the triangle $N_1 N_2 N_3$. It may be called "the principal rectangular hyperbola" of the net. It passes through the centres

$$x^2/a^2 = y^2/b^2 = z^2/c^2 \tag{62}$$

of the inscribed and escribed circles of the focal triangle (as well as through those of the triangle $N_1 N_2 N_3$). It passes also through the four points

$$x^2/k_1^2 = y^2/k_2^2 = z^2/k_3^2. \tag{63}$$

Its centre is the point $$J \equiv l/\rho_1 + m/\rho_2 + n/\rho_3 = 0 \tag{64}$$

on the circumcircle.

The polar conics of the points in which (55) meets the sides of the

focal triangle are the three rectangular hyperbolas

$$c^2y^2-b^2z^2+\rho_1 = 0, \text{ etc.} \tag{65}$$

The first of these has A as centre, the angle-bisectors at A as asymptotes, and circumscribes the parallelogram of line-pairs (34)

$$y^2-k_2^2 = 0, \quad z^2-k_3^2 = 0,$$

and similarly for the other two.

16. As Π, Π' are conjugate with regard to every conic of the net, they are so with regard to all the rectangular hyperbolas of the net. So also are the circular points at infinity; and hence the points P, P', the third pair of vertices of the imaginary quadrilateral, are also conjugate. Hence not only the pairs of points Π, Π' are isogonal conjugates with regard to the triangle $N_1N_2N_3$ but P and P' are isogonal conjugates also.

The isogonal conjugacy of two points with regard to the triangle $N_1N_2N_3$ corresponds to a transformation such that the points are conjugate with regard to every conic of the pencil given by (35) and (54). Hence the correspondence is given by

$$(xx'-k_1^2)/a^2 = (yy'-k_2^2)/b^2 = (zz'-k_3^2)/c^2. \tag{66}$$

Solving these equations and putting

$$S \equiv a^2yz+b^2zx+c^2xy, \tag{67}$$

$$H \equiv xyz-k_1^2yz-k_2^2zx-k_3^2xy, \tag{68}$$

the common value of the fractions (66) is H/S, and the coordinates of P' are

$$x' = S_1/S, \quad y' = S_2/S, \quad z' = S_3/S, \tag{69}$$

where

$$\left.\begin{aligned} S_1 &= a^2yz+\rho_2 y-\rho_3 z \\ S_2 &= b^2zx+\rho_3 z-\rho_1 x \\ S_3 &= c^2xy+\rho_1 x-\rho_2 y \end{aligned}\right\}. \tag{70}$$

Identities occurring among these functions are

$$S_1+S_2+S_3 \equiv S, \tag{71}$$

$$\rho_1 xS_1+\rho_2 yS_2+\rho_3 zS_3 \equiv 0, \tag{72}$$

$$b^2zS_3-c^2yS_2 \equiv \rho_1 S, \tag{73}$$

$$xS_1-k_1^2S \equiv a^2H, \tag{74}$$

and these exhibit the equations (69) as equivalent to the reciprocal equations

$$x = S_1'/S', \quad y = S_2'/S', \quad z = S_3'/S'. \tag{75}$$

The conics $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ are the loci of points isogonal to points on the sides of the focal triangle. Each is a hyperbola circumscribing the triangle $N_1 N_2 N_3$ and having its asymptotes parallel to two of the sides of the focal triangle.

Each conic S_1, S_2, S_3 cuts the circle S in the points N_1, N_2, N_3 and in one of the foci A, B, C. The cubic H cuts the circle S in all six of these points.

17. The envelope equation of the isogonal pair of points is

$$\Phi \equiv (xl+ym+zn)(S_1 l+S_2 m+S_3 n) = 0. \tag{76}$$

The equation of the antipoints is

$$\Theta\Phi - \Theta'\Omega = 0, \tag{77}$$

where Θ is the invariant of degrees 1 and 2 in the coefficients of Φ and Ω respectively; and reversely for Θ'. Hence

$$\Theta = \delta^2 S \quad \text{and} \quad -4\Theta' = W \equiv \Sigma\lambda(xS - S_1)^2; \tag{78}$$

and hence the antipoints of the point-pair Φ are

$$4\delta^2 S\Phi + W\Omega = 0. \tag{79}$$

If now the point (xyz) is P, so that (76) is the pair PP', and (79) the pair $\Pi\Pi'$, it is only necessary to make the latter pair conjugate with regard to any conic of the net (other than the rectangular hyperbolas to which they are already conjugate) in order to get the equation of the locus of P and P'. Sufficiently and most simply the point-pair (79) may be made apolar to the line-pair $x^2 - k_1^2 = 0$ (34). The apolar invariant for this line-pair and Φ is $a^2 H$; and for the line-pair and Ω it is a^2. Hence the equation of the three-bar sextic is

$$4\delta^2 SH + W = 0. \tag{80}$$

If in this equation the suppressed line at infinity $t \equiv x+y+z$ is restored, the equation is

$$4\delta^2 tSH + W = 0, \tag{81}$$

where

$$H \equiv xyz - k_1^2 yzt - k_2^2 zxt - k_3^2 xyt, \tag{82}$$

$$W \equiv \Sigma\lambda(xS - tS_1)^2. \tag{83}$$

Of the factors in the first term, t is the line infinity, S is the circumcircle

of the foci, and H is the Jacobian cubic locus of point-pairs conjugate to the parallels (34). The equation $W = 0$ is the locus of point-pairs which are isogonal conjugates of the triangle $N_1N_2N_3$ and have a distance between them that is analytically zero. The points P, P' are collinear with one or other of the circular points ω, ω'. If $PP'\omega$ are collinear, the cubic locus of P and P' is

$$\begin{vmatrix} x & S_1 & \mu+\nu \\ y & S_2 & -\nu+i\delta \\ z & S_3 & -\mu-i\delta \end{vmatrix} = 0, \tag{84}$$

i.e.
$$i\delta(xS-S_1)-\mu(yS-S_2)+\nu(zS-S_3) = 0. \tag{85}$$

It passes through both ω and ω', and touches the line infinity at ω; it is an imaginary circular parabolic cubic. The companion cubic (with reverse sign for i) passes through ω and ω' and touches t at ω'. The nine common points of the two cubics consist of the three points $N_1N_2N_3$, the four points $I_0I_1I_2I_3$ (self-isogonal points) and the circular points ω, ω'. The product of the two cubics (85) is identically a^2W.

Further, each imaginary cubic is self-isogonal, so also is the cubic H, and t and S are mutually isogonal. So the imaginary cubic touching t at ω touches S at ω', and the other imaginary cubic touches t at ω' and S at ω. The six points in which each imaginary cubic cuts the circle S are the points $N_1N_2N_3$ and the circular points, one counted twice.

It is apparent now from the form of equation (81), since S and H and both imaginary cubics pass through $N_1N_2N_3$, that the sextic curve has double points at $N_1N_2N_3$. Thus the orthogonal line-pairs of the net intersect at the nodes of the three-bar sextic, and the triangle $N_1N_2N_3$, hitherto described merely as the self-polar triangle of the pencil of rectangular hyperbolas of the net, may now be called the nodal triangle of the three-bar sextic. Further, the sextic cuts the line infinity t in the same points as do the cubic-pair W, and so each of the circular points is a triple point of the sextic.

The equation of the sextic in the form (80) may be regarded as representing the constancy of the length $P'Q'$ from P' to the image of P in BC (§ 2), conjointly with the isogonality of the pair PP' in regard to the nodal triangle $N_1N_2N_3$. For if P is (xyz), then P' is $(S_1/S,\ S_2/S,\ S_3/S)$, and the image of P in BC is

$$Q'\,(-x,\ y+2\nu x/a^2,\ z+2\mu x/a^2).$$

Then
$$P'Q'^2 = [W+(4\delta^2/a^2)\,xSS_1]/S^2\,; \tag{86}$$

and putting for $P'Q'$ the proper value $2\delta k_1/a$ (§ 2), the equation (86) re-

duces to (80). The application of Ptolemy's theorem to the trapezium $PP'QQ'$ (Fig. 3) leads to the same result, by a slightly different route.

18. The sextic W (78) may be put into a fresh form by expansion, and use of the identity

$$\Sigma\lambda x S_1 \equiv \sigma t S + 2\delta^2 H, \tag{87}$$

derivable from (71), (74). The sextic equation (81) then takes the form

$$S^2[\Sigma\lambda x^2 - 2\sigma t^2] + t^2\Sigma\lambda S_1^2 = 0. \tag{88}$$

In this equation, besides S^2 and t^2, the squares of the focal circle and the line infinity, there appear two fresh loci. The equation

$$\Sigma\lambda x^2 - 2\sigma t^2 = 0 \tag{89}$$

gives a circle concentric with the polar circle of the focal triangle (40) and the net circle (39). The squared radius of the net circle is the mean-square of the radii of the other two; so that a circle drawn with any point on the net circle as centre to cut the polar circle orthogonally is cut at the ends of a diameter by the circle (89), or conversely. Or, as equivalent, the powers of any point on the net circle with regard to the other two circles are equal and opposite.

The equation

$$\Sigma\lambda S_1^2 = 0 \tag{90}$$

is the quartic curve generated by points isogonal, with regard to the nodal triangle, to points of the polar circle (40) of the focal triangle. It is a circular quartic curve and has a node at each vertex of the nodal triangle; it cuts the circumcircle twice at each node and once at each circular point.

The equation (88) may be written in the form

$$\Sigma\lambda(x^2 - k_1^2) + \Sigma\lambda(S_1^2/S^2 - k_1^2) = 0, \tag{91}$$

with the immediate meaning that the powers of isogonal points P and P' (66, 69) with regard to the net circle (39) are equal and opposite. This is equivalent to the equation

$$OP^2 + OP'^2 = 2(\sigma + \rho), \tag{92}$$

which has appeared in the form (19), and to the other geometrical interpretation there given. The three-bar sextic in the form (88) or (91) thus appears as the locus of a pair of points that are isogonal with regard to a given triangle, and that have equal and opposite powers with regard to a given circle. (The parameters involved are nine.)

19. Conics (written as envelopes) apolar to the net of conics (35) form a web of conics associated with the three-bar curve. If

$$(A, B, C, F, G, H \between lmn)^2 = 0 \tag{93}$$

is a conic of the web, it is only necessary that it should be apolar to any three conics of the net, and most simply to the parallels (34). Hence

$$A/k_1^2 = B/k_2^2 = C/k_3^2 = \Sigma(A+2F), \tag{94}$$

and the web conics are given by (93), (94). The point-pairs of the web are the points Π, Π' conjugate with respect to all conics of the net. The envelope of the lines $\Pi\Pi'$ is the Jacobian of the web conics, and is

$$\Sigma k_1^2 l(n-l)(l-m)+lmn = 0, \tag{95}$$

this being identical with the Cayleyan of the cubic U (33), the representative cubic of the net. The web conics, correlatively, are derivable as polar conics of a class-cubic. Its Hessian is (95) the envelope of the lines Λ. It may be calculated as being lineally related to its first and second Hessians, or as productive of the polar conics (93, 94); but most fundamentally from the correlative cubic U, directly, as the eliminant of x, y, z from the ten equations

$$x\frac{\partial U}{\partial x} = 0, \text{ etc.,} \quad y\frac{\partial U}{\partial z} = 0, \text{ etc.,} \quad \text{and} \quad (xl+ym+zn)^3 = 0, \tag{96}$$

as being linear in the ten cubic products x^3, xyz, yz^2, y^2z, etc. It may be put in the form

$$[8K_1K_2K_3-(\Sigma K_1)^2]\,\Sigma K_1(-l+m+n)^3+[\Sigma K_1(-l+m+n)]^3 = 0, \tag{97}$$

where $$1/K_1 \equiv 1+k_1^2-k_2^2-k_3^2, \text{ etc.}$$

The Cayleyan and Hessian of (97) are respectively the Hessian and Cayleyan of U (33).

20. Among the conics of the web occur four circles. The orthogonal line-pairs of the net are conjugate with regard to each circle; the centres of the circles are the orthocentric points $I_0 I_1 I_2 I_3$. Each circle, associated with any conic of the net, has an infinity of circumscribed triangles self-polar with regard to the net conic. Association specially with the net circle having centre O shows the squared radius of the web circle having centre I_0 to be $\frac{1}{2}(OI_0^2-\rho-\sigma)$; where $\rho+\sigma$ is the squared radius of the net circle; and similarly for the web circles having centres I_1, I_2, I_3. Three arbitrary circles (nine parameters) might be taken as web circles definitive of the whole figure.

21. Among the conics of the web are a sheaf of parabolas, given by equations

$$Fmn+Gnl+Hlm = 0, \tag{98}$$

$$F+G+H = 0, \tag{99}$$

consistently with (94). These are the parabolas inscribed in the focal triangle. The focus

$$(a^2/F)\, l+(b^2/G)\, m+(c^2/H)\, n = 0 \tag{100}$$

lies on the circumcircle. Specially the parabola

$$a^2\rho_1 mn+b^2\rho_2 nl+c^2\rho_3 lm = 0 \tag{101}$$

has focus

$$J \equiv l/\rho_1+m/\rho_2+n/\rho_3 = 0, \tag{102}$$

coincident with the centre of the principal rectangular hyperbola (64). This is the parabola, examined by Cayley, which is inscribed, as will presently be seen (106), in the nodal triangle as well as in the focal triangle.

Among the parabolas are to be included the point-pairs

$$l\,(m-n) = 0, \quad m\,(n-l) = 0, \quad n\,(l-m) = 0, \tag{103}$$

each consisting of a vertex, A, B or C, and the point at infinity on the opposite side. The sides of the focal triangle and the line at infinity form a quadrilateral whose pairs of vertices are corresponding points of the cubic H.

Each pair of these points are isogonal conjugates with regard to the nodal triangle. All points at infinity have isogonal conjugates on the circumcircle; but for the focal triangle the points at infinity on the sides have the opposite vertices themselves as isogonal conjugates. These three conditions are equivalent to a single one, namely, the known property, for the six points on the circle, that the sum of the central vectorial angles for the foci is the same (mod 2π) as that of the nodes.

22. The general equation of all conics inscribed in the nodal triangle (whose sides and vertices individually have irrational equations) may be obtained by taking any two of the rectangular hyperbolas of the net, written as envelopes, together with the equation of an arbitrary point, and forming their Jacobian. Among conics so obtained are three of the form

$$\Psi_1 \equiv l\,(m-n)-(\rho_1/b^2c^2)\,\Omega = 0, \tag{104}$$

and three of the form

$$\Xi_1 \equiv \rho_3 m^2+(\rho_2-\rho_3+a^2)\,mn-\rho_2 n^2+(\rho_2\rho_3/b^2c^2)\,\Omega = 0. \tag{105}$$

The first (104) is a parabola having A as focus and axis parallel to BC. The equations of the three parabolas have a zero sum, and they belong to the sheaf of parabolas inscribed in the nodal triangle. One parabola of the sheaf is

$$\Sigma\,(k_1^2/a^2)\,l(m-n) = 0, \tag{106}$$

which is identical with (101). Hence this parabola, which may be called "the principal parabola" of the web, is inscribed in the nodal triangle as well as in the focal triangle. This parabola and the circumcircle of the focal triangle are polar reciprocals with regard to the rectangular hyperbola (59) of the net. As the nodal triangle and the focal triangle are both self-polar with regard to the hyperbola the results are consistent and equivalent.

The conic (105) has a pair of foci on BC given by the equation

$$\rho_3 m^2 + (\rho_2 - \rho_3 + a^2)\,mn - \rho_2 n^2 = 0, \tag{107}$$

or by

$$x = 0, \quad \rho_2 y^2 + (\rho_2 - \rho_3 + a^2)\,yz - \rho_3 z^2 = 0, \tag{108}$$

which is the Jacobian of

$$a^2z^2 + \rho_2(y+z)^2 = 0 \quad \text{and} \quad a^2y^2 - \rho_3(y+z)^2 = 0. \tag{109}$$

Hence the foci of the conic are the limiting points of the circles

$$(-\rho_2) \equiv c^2x^2 + 2\mu xz + a^2z^2 + \rho_2 = 0, \tag{110}$$

$$(+\rho_3) \equiv b^2x^2 + 2\nu xy + a^2y^2 - \rho_3 = 0, \tag{111}$$

with B and C as centres and squared radii $-\rho_2$ and $+\rho_3$. There are two of these circles associated with each vertex. The ends of the links AA_2, AA_3 describe two circles, say (A_2), (A_3), with centre A and radii b_3, c_2; and the circles $(+\rho_1)$ and $(-\rho_1)$ with centre A are such that the circle (A_2) is the orthoptic locus of (A_3) and $(+\rho_1)$, and (A_3) is the orthoptic locus of (A_2) and $(-\rho_1)$. The three pairs of circles $(+\rho_1)$, $(-\rho_1)$, etc., may be called the "modular circles". The two modular circles, with centres B and C, whose limiting points are the foci of the conic Ξ_1, have a symmetric (and not a skew) relation to B and C. They have as orthoptic circles, when associated with the circles described by B_3 and C_2, the circles described by B_1 and C_1 (Fig. 2).

The limiting points, being foci of a conic inscribed in the nodal triangle, are also isogonal conjugates with respect to the nodal triangle; and the conic S_1 (70) cuts BC in the same points (108).

23. If the equations (105) are added after multiplication by ρ_1^2, ρ_2^2, ρ_3^2, the conic

$$\Sigma a^2\rho_1^2 mn + \Sigma \rho_2\rho_3 l\,.\,\Sigma \rho_1(m-n) = 0 \tag{112}$$

appears as one of those inscribed in the nodal triangle. In this equation

$$\Sigma \rho_2 \rho_3 l = 0 \tag{113}$$

is the point J on the circle S, focus of the principal parabola (101), (102).

The point
$$\Sigma \rho_1 (m-n) = 0 \tag{114}$$

is the point at infinity I on the line

$$\rho_1 x + \rho_2 y + \rho_3 z = 0, \tag{115}$$

and this is the polar of J with regard to the focal triangle (regarded as a cubic). It passes through the symmedian point Γ of the focal triangle

$$a^2 l + b^2 m + c^2 n = 0, \tag{116}$$

as do the triangular polars of all points on the circle S. On the line is also the point
$$K \equiv k_1^2 l + k_2^2 m + k_3^2 n = 0. \tag{117}$$

The conic
$$\Sigma a^2 \rho_1^2 mn = 0 \tag{118}$$

is inscribed in the focal triangle and touches the line (115) at the symmedian point Γ. And this conic and the conic (112) inscribed in the nodal triangle have the same tangents from I and J. Consequently, regarding K as an arbitrary point determined by the ratios (only) of the parameters $k_1 k_2 k_3$, the line $K\Gamma$ joining it to the symmedian point of the focal triangle has triangular pole J on the circumcircle S. A unique conic may be inscribed in ABC to touch $K\Gamma$ at Γ; and then a conic is found to touch seven lines, namely, $K\Gamma$ and the tangent parallel thereto, the two tangents from J, and the three sides of the nodal triangle.

The point J also lies on the conic (41), the polar conic of K with regard to the triangle.

24. If triradial coordinates are used for P, say $AP = u$, $BP = v$, $CP = w$, then
$$\cos BPC = (v^2 + w^2 - a^2)/2vw, \tag{119}$$

and as three such angles have the sum 2π the identical relation among the coordinates may be written

$$\Sigma u^2 (v^2 + w^2 - a^2)^2 - \Pi (v^2 + w^2 - a^2) - 4u^2 v^2 w^2 = 0, \tag{120}$$

where Σ and Π are sum- and product-symbols.

Similarly (Fig. 2)

$$\cos (A - \theta_2 + \theta_3) = (u^2 - b_3^2 - c_2^2)/2b_3 c_2, \tag{121}$$

and as three such angles have sum π the triradial equation of the sextic is

$$\Sigma a_1^2(u^2-b_3^2-c_2^2)+\Pi(u^2-b_3^2-c_2^2)-4a_1^2b_2^2c_3^2=0. \tag{122}$$

It should be noticed that this equation connecting PA, PB, PC holds good not only for the single infinity of configurations with ABC fixed, but for the complete double infinity arising when the triangle ABC varies in size.

The focal circle is given by the equation

$$Q \equiv au+bv+cw=0, \tag{123}$$

expressing Ptolemy's theorem algebraically. The chords $u=AP$, $a=BC$, etc., have their signs determined by attributing a cyclic sense to the circle and placing an arbitrary barrier point on the curve; the sign of the chord being taken to agree with that of the arc that does not pass the barrier point.

The identity (120) when used conjointly with (123) may be replaced by

$$avw+bwu+cuv+abc=0, \tag{124}$$

representing the zero sum of the areas of the triangles of the quadrangle $ABCP$.

The points in which the sextic cuts the circle (123) may be got by putting (122) into the identically equivalent form

$$(uvw+\Sigma b_1c_1u)^2+Q\Sigma(2au-Q)(k_1^2u^2+k_2^2k_3^2a^2)=0, \tag{125}$$

so that for each of the nodes of the sextic

$$uvw+b_1c_1u+c_2a_2v+a_3b_3w=0. \tag{126}$$

The same equation is also directly obtainable from the equations (67), (68). For when P is on the circle S,

$$x=-vw/bc, \quad y=-wu/ca, \quad z=-uv/ab. \tag{127}$$

With these relations between u, v, w and x, y, z,

$x+y+z=1$ transforms into (124),

$S \equiv a^2yz+b^2zx+c^2xy=0$ transforms into (123),

and $H \equiv xyz-k_1^2yz-k_2^2zx-k_3^2xy=0$ transforms into (126),

with the additional factor uvw giving the foci. Each of the conics S_1, S_2, S_3 (70) converts into the same nodal equation (126) with the additional factors u, v, w respectively.

25. The nodal equation may be put into trigonometric form,

$$\sin(\phi-a)\sin(\phi-\beta)\sin(\phi-\gamma)+\Sigma k_1^2\sin(\gamma-a)\sin(a-\beta)\sin(\phi-a) = 0, \tag{128}$$

where chords from an arbitrary point of the circle to A, B, C and any one node have directions a, β, γ, ϕ. As a cubic in $\tan\phi$ it is of the form

$$(\tan\phi-\tan a)(\tan\phi-\tan\beta)(\tan\phi-\tan\gamma)+(\tan^2\phi+1)(L\tan\phi+M) = 0, \tag{129}$$

so that
$$\phi_1+\phi_2+\phi_3 \equiv a+\beta+\gamma \pmod{\pi}, \tag{130}$$

the well-known relation (§ 21) connecting the positions of the vertices of the focal and nodal triangles.

Further relations may be derived analytically from (128), but may be attached more immediately to the geometry of the parabolas. If any parabola is inscribed in the triangle of reference, with focus at (xyz) on the circumcircle of diameter D, its parameter (quarter latus-rectum) is equal to $(\delta/D)(-xyz)^{\frac{1}{3}}$; and if u, v, w are the radial coordinates of the focus, then (127) the parameter is uvw/D^2.* The principal parabola (101) is inscribed in both the focal and nodal triangles; and hence for its focus J,

$$JA\,.\,JB\,.\,JC = JN_1\,.\,JN_2\,.\,JN_3. \tag{131}$$

The parabola Ψ_1 (104) is inscribed in the nodal triangle, and has A for focus, and axis parallel to BC. Its parameter equals $\delta\rho_1/Dbc$, and hence three equations of the form

$$AN_1\,.\,AN_2\,.\,AN_3 = a\rho_1. \tag{132}$$

[Equations (131) and (132) are given by Cayley, but not (126) nor (128).]

If the equations (132) are written

$$AN_1\,.\,AN_2\,.\,AN_3 = \pm BC\,.\,\rho_1, \ \&c., \tag{133}$$

then they all hold algebraically for the upper signs if the barrier point is taken on any one of three alternate arcs of the six into which the circle is divided by the foci and nodes; and if the barrier point is placed on one of the other three arcs the lower signs are to be taken. The signs are collectively and not individually ambiguous.

As alternative to the parabolas, any conic through the inscribed and escribed centres of triangle ABC is a rectangular hyperbola with centre

* Cf. S. Roberts, *Quarterly Journal of Mathematics*, Vol. 15 (1878), pp. 52–55.

(xyz) on the circumcircle, and the square of its semi-axis is $\delta(-xyz)^{\frac{1}{3}}$, which is equal to uvw/D. The principal rectangular hyperbola (59) passes through these points, and hence gives the result (131). And similarly the net hyperbola (65) with centre A has a squared semi-axis equal to $a\rho_1/D$, and hence gives equation (132).

26. If the concyclic foci and nodes are given, with the condition (130) observed, there remains one parameter undetermined for the three-bar curve. The values of the moduli ρ_1, ρ_2, ρ_3 are determined (133), but the values of k_1^2, k_2^2, k_3^2 have one degree of indeterminacy. Each may be increased by any multiple of a^2, b^2, and c^2, respectively. This adds to (126) a multiple of (123) only; and adds to (128) a multiple of the zero $\Sigma \sin(\beta-\gamma)\sin(\phi-a)$. For the same change the point K (117), representing the ratios of k_1^2, k_2^2, k_3^2, moves along the fixed line ΓI (115), which is determined by the ratios of ρ_1, ρ_2, ρ_3.

If the foci A, B, C are given and only the ratios of ρ_1, ρ_2, ρ_3, with the condition $\Sigma a^2\rho_1 = 0$ (seven parameters), the principal parabola (101) is determined and the nodal triangle is any one of the single infinity which. like the focal triangle, are inscribed in the circle and circumscribed to the principal parabola; and with each of these nodal triangles the point K may be taken arbitrarily on the fixed line ΓI.

27. Four vectors with a zero sum have a certain simple property in regard to the ranges got by projecting any one of the three closed quadrilaterals on to any line. If the projection is made by lines parallel to one of the sides, four different triads of points are thus got; and it may be shown that there exists a range of four points which, by threes, are similar to the four three-point ranges. If the vectors are (k_1, θ_1), (k_2, θ_2), (k_3, θ_3), (k_4, θ_4), with the conditions

$$\Sigma k_1 \cos\theta_1 = 0 \quad \text{and} \quad \Sigma k_1 \sin\theta_1 = 0, \tag{134}$$

the segments of the three-point ranges represent the terms of the equations

$$k_1 \sin\theta_{14} + k_2 \sin\theta_{24} + k_3 \sin\theta_{34} = 0, \text{ etc.}, \tag{135}$$

and a range of four points R_1, R_2, R_3, R_4 may be taken on a line so that

$$R_1R_4 = k_2k_3 \sin\theta_{23}, \quad R_2R_3 = k_1k_4 \sin\theta_{14}, \text{ etc.} \tag{136}$$

The range $R_1R_2R_3R_4$ may be called a "similitude range" of the vectors. If the vectors are placed to form a pencil, the range has the same cross-ratio as the pencil, and may be obtained from a transversal of the pencil uniquely determinable in direction. Such a transversal, cutting the pencil

in a similitude range, may be called a "principal transversal" of the pencil. If O is the vertex of the pencil, the four vectors are then proportional to $OR_1 \div R_1R_2 . R_1R_3 . R_1R_4$, etc.

As the triple-generation mechanism depends intrinsically on the zero sum of four vectors it is natural that the above property should appear in the figure. A bare indication may suffice :—

(i) The lines joining P to X, Y, Z and to the infinity point on Λ form a pencil giving the angles θ_1, θ_2, θ_3, 0 for the vectors $(k_1\theta_1)$, $(k_2\theta_2)$, $(k_3\theta_3)$, $(-1, 0)$ of the mechanism.

(ii) Parallel lines through A, B, C and T, all with the direction of the line Λ, are cut by any line in a similitude range.

(iii) The principal transversals of the pencil (i), cutting the pencil in similitude ranges, are parallel to $P\varpi$; where ϖ is the point already distinguished by various properties (49), (52), (53).

(iv) The points of the similitude range (ii) on PP', joined to the corresponding points of the similitude range on Λ, give tangents to a rectangular hyperbola with ϖ as centre and Λ as one asymptote.

ON THE USE OF A PROPERTY OF JACOBIANS TO DETERMINE THE CHARACTER OF ANY SOLUTION OF AN ORDINARY DIFFERENTIAL EQUATION OF THE FIRST ORDER OR OF A LINEAR PARTIAL DIFFERENTIAL EQUATION OF THE FIRST ORDER

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Introduction.

1. In two recent papers* I have employed the two simplest cases of the property of Jacobians discussed in this paper to determine the character of the solutions of an ordinary differential equation of the first order, and of a linear partial differential equation of the first order, with one dependent and two independent variables.

The property here proved is as follows:—

Let there be $n+1$ dependent variables $u_1, u_2, \ldots, u_n, u$ and $n+1$ independent variables $x_1, x_2, \ldots, x_n, x$; and let J be the Jacobian

$$\frac{D(u_1, u_2, \ldots, u_n, u)}{D(x_1, x_2, \ldots, x_n, x)};$$

then, if J vanish when x is such a function of $x_1, x_2, \ldots, x_n$ as to make u vanish, it can be proved that a function of the remaining dependent variables $u_1, u_2, \ldots, u_n$ exists which will vanish when u vanishes.

In applying this property to determine the character of an integral $u = 0$ of a differential equation it is supposed that $u_1 = a_1$, $u_2 = a_2$, ..., $u_n = a_n$ (where $a_1, a_2, \ldots, a_n$ are arbitrary constants) are n independent ordinary integrals of the equation, and that $u_1, u_2, \ldots, u_n$ are taken in such a form that none of them are infinite when $u = 0$.

* "On the Classification of the Integrals of Linear Partial Differential Equations of the First Order," *Proc. London Math. Soc.*, Ser. 2, Vol. 16 (1917), p. 219; "On the Singular Solutions of Ordinary Differential Equations of the First Order with Transcendental Coefficients," *Proc. London Math. Soc.*, Ser. 2, Vol. 17 (1918), p. 149.

There are then three cases:—

(i) If J vanish identically, then u is a function of $u_1, u_2, \ldots, u_n$, and $u = 0$ is an *ordinary* integral.

(ii) If J vanish when u vanishes, then some function of $u_1, u_2, \ldots, u_n$ exists, say $\phi(u_1, u_2, \ldots, u_n)$, such that the equation

$$\phi(u_1, u_2, \ldots, u_n) = 0$$

is satisfied when x is such a function of $x_1, x_2, \ldots, x_n$ as to make u vanish.

In this case although u is not a function of $u_1, u_2, \ldots, u_n$, the integral $u = 0$ is included in the integral

$$\phi(u_1, u_2, \ldots, u_n) = 0,$$

and is to be regarded as a *particular* integral.

(iii) If J does not vanish when u vanishes, then no function of the form $\phi(u_1, u_2, \ldots, u_n)$ exists such that

$$\phi(u_1, u_2, \ldots, u_n) = 0$$

includes the integral $u = 0$.

In this case the integral $u = 0$ is regarded in this paper as a *Singular* Integral, although it does not satisfy the definition which Darboux gives of a singular integral in his memoir "Sur les solutions singulières des équations aux dérivées partielles du premier ordre" (*Mémoires présentés par divers savants à l'Académie des Sciences de l'Institut de France*, Tome 27, Deuxième Série, No. 2, 1883). No partial differential equation of the first order, which is linear, can satisfy Darboux's definition. My reasons for regarding the solutions discussed in this paper under this third heading as singular are (i) that they possess the envelope property, and (ii) that they are not in general obtainable from the ordinary integrals by giving special values to the arbitrary constants involved in the ordinary integrals.

2. To prove the above-mentioned property of the Jacobian observe that

$$J = \frac{D(u_1, \ldots, u_n, u)}{D(x_1, \ldots, x_n, x)}$$

is a function of $x_1, \ldots, x_n, x$.

Now let x be determined as a function of $x_1, \ldots, x_n$ which makes u vanish, and suppose that when this value of x is inserted in $u_1, \ldots, u_n$, they become $\bar{u}_1, \ldots, \bar{u}_n$ respectively.

Let ∂ denote partial differentiation with regard to $x_1, \ldots, x_n$; whilst D denotes partial differentiation with regard to $x_1, \ldots, x_n, x$.

Then it will be proved that

$$\frac{D(u_1, \ldots, u_n, u)}{D(x_1, \ldots, x_n, x)} = \frac{\partial(\bar{u}_1, \ldots, \bar{u}_n)}{\partial(x_1, \ldots, x_n)} \cdot \frac{Du}{Dx}.$$

Since $\dfrac{\partial \bar{u}_r}{\partial x_s} = \dfrac{Du_r}{Dx_s} + \dfrac{Du_r}{Dx}\dfrac{\partial x}{\partial x_s}$ for all integral values of r and s between 1 and n inclusive, and since

$$0 = \frac{Du}{Dx_s} + \frac{Du}{Dx}\frac{\partial x}{\partial x_s}$$

for all integral values of s between 1 and n inclusive, we see that if we take

$$J = \begin{vmatrix} \dfrac{Du_1}{Dx_1}, & \ldots, & \dfrac{Du_1}{Dx_n}, & \dfrac{Du_1}{Dx} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{Du_n}{Dx_1}, & \ldots, & \dfrac{Du_n}{Dx_n}, & \dfrac{Du_n}{Dx} \\ \dfrac{Du}{Dx_1}, & \ldots, & \dfrac{Du}{Dx_n}, & \dfrac{Du}{Dx} \end{vmatrix},$$

and multiply the constituents in the last column by $\partial x/\partial x_s$ and add to the corresponding constituents in the s-th column, and if we do this for all values of s from 1 to n inclusive, then J takes the form

$$\begin{vmatrix} \dfrac{\partial \bar{u}_1}{\partial x_1}, & \ldots, & \dfrac{\partial \bar{u}_1}{\partial x_n}, & \dfrac{Du_1}{Dx} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial \bar{u}_n}{\partial x_1}, & \ldots, & \dfrac{\partial \bar{u}_n}{\partial x_n}, & \dfrac{Du_n}{Dx} \\ 0, & \ldots, & 0, & \dfrac{Du}{Dx} \end{vmatrix} = \frac{\partial(\bar{u}_1, \ldots, \bar{u}_n)}{\partial(x_1, \ldots, x_n)} \cdot \frac{Du}{Dx}.$$

3. Since by the hypothesis the equation $u = 0$ determines x as a function of $x_1, \ldots, x_n$, it follows that Du/Dx does not vanish identically.

Hence, if J vanishes when u vanishes,

$$\frac{\partial(\bar{u}_1, \ldots, \bar{u}_n)}{\partial(x_1, \ldots, x_n)} = 0,$$

when x is such a function of $x_1, \ldots, x_n$ as to make u vanish. But $\bar{u}_1, \ldots, \bar{u}_n$ do not contain x.

Hence $\dfrac{\partial(\bar{u}_1, \ldots, \bar{u}_n)}{\partial(x_1, \ldots, x_n)}$ vanishes identically.

Hence some relation of the form

$$\phi(\bar{u}_1, \ldots, \bar{u}_n) = 0$$

exists.

Take now the function $\phi(u_1, \ldots, u_n)$, and suppose that x is taken to be such a function of $x_1, \ldots, x_n$ as to make u vanish; then $\phi(u_1, \ldots, u_n)$ becomes $\phi(\bar{u}_1, \ldots, \bar{u}_n) = 0$. Hence the integral $u = 0$ is *included* in the equation

$$\phi(u_1, \ldots, u_n) = 0,$$

but u is not itself of the form $\phi(u_1, \ldots, u_n)$. Consequently $u = 0$ is not an *ordinary* integral of the differential equation, but it is a *particular* integral.

4. If J do not vanish when x is such a function of $x_1, \ldots, x_n$ as to make u vanish, it follows that

$$\frac{\partial(\bar{u}_1, \ldots, \bar{u}_n)}{\partial(x_1, \ldots, x_n)}$$

does not then vanish.

Hence no relation of the form

$$\phi(\bar{u}_1, \ldots, \bar{u}_n) = 0$$

exists, and therefore no equation of the form

$$\phi(u_1, \ldots, u_n) = 0$$

exists which would be satisfied if x were chosen such a function of $x_1, \ldots, x_n$ as to make u vanish. The integral $u = 0$ is not therefore included in the form

$$\phi(u_1, \ldots, u_n) = 0.$$

It is not therefore an *ordinary* integral. It will be regarded in this paper as a *singular* integral.

The following examples will illustrate the theory.

Example I.

5. Consider the differential equation

$$(x+bx_1)^3\left(\frac{dx}{dx_1}+a\right)-2x_1(x+ax_1)(x+bx_1)+2x_1^2(x+ax_1)\left(\frac{dx}{dx_1}+b\right) = 0,$$

where a, b are fixed constants.

Its complete primitive is

$$(x+ax_1)\exp\left[-x_1^2/(x+bx_1)^2\right] = \text{const.}$$

Also $x+ax_1 = 0$ is an integral of the equation. Taking

$$u = x+ax_1,$$

$$u_1 = (x+ax_1)\exp\left[-x_1^2/(x+bx_1)^2\right],$$

it follows that

$$J = -2x_1(x+ax_1)^2(x+bx_1)^{-3}\exp\left[-x_1^2/(x+bx_1)^2\right].$$

Hence $J = 0$ when $u = 0$.

In this case $u = 0$ is a particular case of the primitive $u_1 = \text{const.}$, viz.: it is obtained by taking the arbitrary constant to vanish.

In this case the quantity $\bar{u}_1$ (obtained by choosing a relation between x and x_1 which will make u vanish) is identically zero.

The relation corresponding to

$$\phi(u_1, u_2, \ldots, u_n) = 0$$

is now $u_1 = 0$, and this is satisfied when $u = 0$.

Example II.

6. Consider the differential equation

$$x_2^2(x-x_1-x_2)^{\frac{1}{2}}\left(\frac{\partial x}{\partial x_1}-1\right)+(x_2^2-2x_1x_2)\left(\frac{\partial x}{\partial x_2}-1\right)+4(x_2-x_1)(x-x_1-x_2) = 0.$$

The ordinary integrals are

$$u_1 = x_1+x_2(x-x_1-x_2)^{\frac{1}{2}} = a_1,$$

$$u_2 = x_1^2+x_2^2(x-x_1-x_2)^{\frac{1}{2}} = a_2.$$

Another integral is $\quad u = x-x_1-x_2 = 0,$

$$J = \frac{D(u_1, u_2, u)}{D(x_1, x_2, x)} = 2(x_2-x_1)(x-x_1-x_2)^{\frac{1}{2}}.$$

Hence J vanishes when u vanishes.

Now u is not itself expressible as a function of u_1 and u_2. But, by § 3, some function of u_1 and u_2 exists which when equated to zero includes $u = 0$.

In this case $\qquad \bar{u}_1 = x_1, \quad \bar{u}_2 = x_1^2.$

Therefore $\qquad \bar{u}_2-\bar{u}_1^2 = 0.$

The relation in question is therefore

$$u_2 - u_1^2 = 0.$$

This gives $\quad x_2(x-x_1-x_2)^{\frac{1}{2}}[x_2-2x_1-x_2(x-x_1-x_2)^{\frac{1}{2}}] = 0.$

The first factor, giving $x_2 = 0$, does not lead to a relation determining x as a function of x_1 and x_2.

The next factor gives $\quad x - x_1 - x_2 = 0,$

i.e. $$u = 0.$$

Another solution is given by the remaining factor, which gives

$$(x-x_1-x_2)^{\frac{1}{2}} = 1-(2x_1/x_2).$$

Putting $\quad u = (x-x_1-x_2)^{\frac{1}{2}} - 1 + (2x_1/x_2),$

it follows that $J = -x_1(x-x_1-x_2)^{-\frac{1}{2}}[(x-x_1-x_2)^{\frac{1}{2}} - 1 - (2x_1/x_2)],$

and so J vanishes if u vanishes.

Hence in accordance with § 3, $u = 0$ is a particular integral.

Example III.

7. Consider the differential equation

$$\frac{\partial x}{\partial x_1} + \{1+(x-x_1-x_2)^{\frac{1}{2}}\}\left(\frac{\partial x}{\partial x_2} - 2\right) = 0.$$

In this case take
$$u_1 = x - 2x_2 = a_1,$$
$$u_2 = x_1 - 2(x-x_1-x_2)^{\frac{1}{2}} = a_2,$$
$$u = x - x_1 - x_2 = 0.$$

And now $J = 1$.

No function of u_1 and u_2 exists from which the relation $u = 0$ can be derived.

In this case
$$\frac{\partial u_2}{\partial x_1} = 1 + (x-x_1-x_2)^{-\frac{1}{2}},$$
$$\frac{\partial u_2}{\partial x_2} = (x-x_1-x_2)^{-\frac{1}{2}},$$
$$\frac{\partial u_2}{\partial x} = -(x-x_1-x_2)^{-\frac{1}{2}}.$$

The parts of these differential coefficients, which are infinite on the

locus $x-x_1-x_2=0$, *i.e.* $u=0$, are proportional to $\frac{\partial u}{\partial x_1}$, $\frac{\partial u}{\partial x_2}$ and $\frac{\partial u}{\partial x}$. *i.e.* to -1, -1, 1.

Hence the surfaces $u_2=a_2$ touch the surface $u=0$ where they meet it.

In the Jacobian the constituents in one row are proportional to the constituents of another row *at points on the envelope*, yet the Jacobian does not vanish.

8. It will now be proved that, if $u_1=a_1$ represents a family of surfaces which satisfy the differential equation, then $\frac{Du_1}{Dx_1}$, $\frac{Du_1}{Dx_2}$, $\frac{Du_1}{Dx}$ are infinite where the surfaces meet the envelope of the family.

Take the surfaces in the form

$$f(x_1, x_2, x, a_1)=0.$$

If this be equivalent to $u_1=a_1$, then u_1 is determined by the equation

$$f(x_1, x_2, x, u_1)=0;$$

therefore
$$\frac{\delta f}{\delta x_1}+\frac{\delta f}{\delta u_1}\frac{Du_1}{Dx_1}=0,$$

$$\frac{\delta f}{\delta x_2}+\frac{\delta f}{\delta u_1}\frac{Du_1}{Dx_2}=0,$$

$$\frac{\delta f}{\delta x}+\frac{\delta f}{\delta u_1}\frac{Du_1}{Dx}=0,$$

where δ denotes partial differentiation with regard to x_1, x_2, x and u_1.

Since at a point on the envelope $\frac{\delta f}{\delta u_1}=0$, it follows that $\frac{Du_1}{Dx_1}$, $\frac{Du_1}{Dx_2}$, $\frac{Du_1}{Dx}$ are in general all infinite at points where the surfaces $u_1=a_1$ meet the envelope.

9. The transformation of the Jacobian given in § 2 is a particular case of the following:—

Let there be $i+k$ independent variables

$$x_1, \ldots, x_i, \quad y_1, \ldots, y_k,$$

and $i+k$ dependent variables

$$u_1, \ldots, u_i, \quad v_1, \ldots, v_k.$$

Choose $y_1, \ldots, y_k$ such functions of $x_1, \ldots, x_i$ as to make $v_1, \ldots, v_k$ all vanish, and suppose that when these values of $y_1, \ldots, y_k$ have been substituted in $u_1, \ldots, u_i$ they become $\bar{u}_1, \ldots, \bar{u}_i$ respectively, then, if D denote partial differentiation with regard to $x_1, \ldots, x_i$, $y_1, \ldots, y_k$, and if ∂ denote partial differentiation with regard to $x_1, \ldots, x_i$, it may be shown that

$$J = \frac{D(u_1, \ldots, u_i,\ v_1, \ldots, v_k)}{D(x_1, \ldots, x_i,\ y_1, \ldots, y_k)} = \frac{\partial(\bar{u}_1, \ldots, \bar{u}_i)}{\partial(x_1, \ldots, x_i)} \cdot \frac{D(v_1, \ldots, v_k)}{D(y_1, \ldots, y_k)},$$

from which the conclusion may be drawn that, if J vanish when $y_1, y_2, \ldots, y_k$ are such functions of $x_1, x_2, \ldots, x_i$ as to make $v_1, v_2, \ldots, v_k$ all vanish, then some function of $u_1, u_2, \ldots, u_i$ exists which will vanish when $y_1, y_2, \ldots, y_k$ are replaced by the above-mentioned functions of $x_1, x_2, \ldots, x_i$.

The determinant $\dfrac{D(v_1, \ldots, v_k)}{D(y_1, \ldots, y_k)}$ does not vanish identically, for if it did, then the equations $v_1 = 0$, $v_2 = 0$, $\ldots$, $v_k = 0$ would not determine $y_1, y_2, \ldots, y_k$ as functions of $x_1, x_2, \ldots, x_i$.

The demonstration is similar to that given in § 2, so that it is not worth while to set it out.

ON PLANE CURVES OF DEGREE n WITH A MULTIPLE POINT OF ORDER $n-1$ AND A CONIC OF $2n$-POINT CONTACT

By HAROLD HILTON.

1. We have considered elsewhere the properties of a plane algebraic curve of degree n (an n-ic) with tangents of n-point contact (*Messenger of Mathematics*, 1920). The case of an n-ic with a conic or conics of $2n$-point contact at once suggests itself. It will be found immediately that an n-ic meeting $y = x^2$ in $2n$-points coinciding with the origin (which is not a double point) has an equation of the form $(y-x^2)\,u_{n-2} = y^n$, where $u_{n-2} = 0$ is some $(n-2)$-ic. Then, by a change of axes, we have the result that an n-ic meeting the conic $u_2 = 0$ $2n$ times at the point of contact of its tangent $u_1 = 0$ has the equation

$$u_2 u_{n-2} = u_1^n.$$

2. In this paper we confine ourselves to the case of an n-ic with a multiple point B of order $n-1$ [an $(n-1)$-ple point] meeting a conic Σ $2n$ times at C.

Let the polar of B with respect to Σ meet the tangent at C in A, and meet BC in O. Let BC meet Σ again at D.

There are two cases to consider. In §§ 3–8 we discuss the case in which B is outside Σ, and in § 9 the case in which B is inside Σ.

3. Let B be outside Σ.

We shall find the following notation useful later on.

Let U, V be points on BC conjugate with respect to Σ, and such that the cross-ratio $UB\,.\,CD/UD\,.\,CB$ of the range $(UBCD)$ is $2/(n+1)$, and therefore the cross-ratio of $(VBCD)$ is $2n/(n+1)$. Let AO meet Σ in E, F, and let AV meet the tangents BE, BF from B to Σ in I, J. Let W, X be the points on BC such that the cross-ratio of $(WBCD)$ is $8n/(2n+1)^2$, and the cross-ratio of $(XBCD)$ is $4n/(2n+1)$. Let (BO, XY) be a harmonic range.

It is possible to project Σ into the circle $x^2+y^2 = y$, A being the

point $(\infty, 0)$, B being $(0, \infty)$, and C the origin; while the axes of reference CA, CB are perpendicular.

Then E, F become $(\pm\frac{1}{2}, \frac{1}{2})$; I, J become $[\pm\frac{1}{2}, (n-1)/2n]$; D becomes $(0, 1)$; O becomes $(0, \frac{1}{2})$; V becomes $[0, (n-1)/2n]$; U becomes $[0, -\frac{1}{2}(n-1)]$; W becomes $[0, -(2n-1)^2/8n]$; X becomes $[0, (2n-1)/4n]$; and Y becomes $[0, (2n+1)/4n]$.

By § 1 the n-ic becomes

$$(x^2+y^2-y)\,u_{n-2} = y^n;$$

and since B is an $(n-1)$-ple point, the coefficients of $y^2, y^3, \ldots, y^n$ are zero in this equation. From this fact the coefficients in u_{n-2} may be calculated, and the equation of the n-ic is found to be

$$y(1-{}^{n-2}C_1x^2+{}^{n-3}C_2x^4-\ldots) = x^2(1-{}^{n-3}C_1x^2+{}^{n-4}C_2x^4-\ldots). \qquad \text{(i)}$$

The equation may also be written

$$y = \frac{x^2}{1-}\,\frac{x^2}{1-}\,\frac{x^2}{1-\ldots}, \qquad \text{(ii)}$$

$n-1$ convergents of the continued fraction being taken.

For an indirect but less laborious method of obtaining this result, see § 12.

From (ii) we get, when $4x^2 \geqslant 1$,

$$x = \tfrac{1}{2}\sec\phi, \quad y = \tfrac{1}{2}\sin(n-1)\phi\,\operatorname{cosec} n\phi\,\sec\phi, \qquad \text{(iii)}$$

and, when $4x^2 \leqslant 1$,

$$x = \pm\tfrac{1}{2}\operatorname{sech}\phi, \quad y = \tfrac{1}{2}\sinh(n-1)\phi\,\operatorname{cosech} n\phi\,\operatorname{sech}\phi. \qquad \text{(iv)}$$

If $\phi = 0$, these points coincide with I or J.

Many geometrical properties of the n-ic follow from the fact that (ii) is symmetrical about $x = 0$. Other properties are given below.

The tangents at B are all real, being given by $\phi = s\pi/n$,

$$s = 1, 2, \ldots, (n-1)$$

in (iii) when n is odd, and by

$$s = 1, 2, \ldots, \tfrac{1}{2}(n-2), \tfrac{1}{2}(n+2), \ldots, (n-1)$$

when n is even, while in the latter case AB is also a tangent at B.

When n is odd, AU touches the n-ic at A.

Every line meets the curve in n or $(n-2)$ real points.

The curve (i) consists of a single circuit with $n-1$ branches when n is even, and n branches when n is odd.

The values of ϕ giving the intersections of the curve with AC, AD, AO are at once obtained; and the same is true for the intersections with the conics ($y = x^2$ and $y = 2x^2$) through B having four-point contact with Σ at C, and through B, E, F, C touching AC at C. For instance, the intersections with AD are given by $\phi = s\pi/(n+1)$, $s = 1, 2, \ldots, n$.

4. The condition that the tangent to the n-ic at the point of § 3 (iii) passes through $(0, c)$ is

$$(2n+2c-1)\sin\phi = \sin 2n\phi\cos\phi + (2c-1)\cos 2n\phi\sin\phi. \tag{i}$$

Eliminating ϕ and $n\phi$ from this and § 3 (iii), we get

$$2(n+2c-1)x^2+2ny^2 = (2n-1)y+c. \tag{ii}$$

Hence the points of contact of the $2(n-1)$ tangents to the n-ic from a point on BC lie on a conic through E, F, I, J.

In particular, the tangents to the n-ic at its intersections with EF and IJ (other than I, J) all pass through U. Similarly the tangents to the n-ic at its intersections with XI, XJ (other than (I, J) all pass through W.

5. The $3(n-2)$ inflexions of § 3 (iii) are given by

$$4n^2\cot n\phi\sin^2\phi = n\sin 2\phi+\sin 2n\phi. \tag{i}$$

Of these $n-2$ are real. If n is odd, one of these real inflexions is at A.

Eliminating ϕ and $n\phi$ from (i) and § 3 (iii), we get

$$x^2(1-2y) = n\{2n(1-2y)-1\}(x^2+y^2-y). \tag{ii}$$

Hence all the inflexions of the n-ic lie on the 3-ic (ii) which touches the n-ic at C, I, J. This 3-ic touches the conic Σ at C, D, and cuts it at E, F. It touches AX at X and passes through the intersections of AY with BI, BJ.

If $\lambda x+\mu y+1 = 0$ is the tangent to the n-ic at the point of § 3 (iii)

$$\frac{\lambda}{2(\sin 2n\phi-n\sin 2\phi)} = \frac{\mu}{4\sin\phi\sin^2 n\phi} = \frac{1}{(2n-1)\sin\phi-\sin(2n-1)\phi}. \tag{iii}$$

Eliminating ϕ and $n\phi$ from (i) and (iii), we have

$$\lambda^2\{(2n-1)^2\mu-8n\} = 16n(\mu+1)\{(n-1)\mu-2\}. \tag{iv}$$

Hence all the inflexional tangents touch the curve (iv) of class 3 and degree 6. It touches AC, AD, AU at C, D, U. It also touches the given

n-ic at I, J. It passes through E and F, the tangents at these points being $\pm 4nx+2y+2n-1=0$. It touches BI and BJ where $y=(8n+1)/16n$. The line BC is a cuspidal tangent at the cusp W.

6. If the point of § 3 (iii) lies on $y=(p-1)/2p$, $\tan n\phi = p \tan\phi$. Eliminating ϕ and $n\phi$ from this and § 5 (iii), we get

$$\lambda^2 \{(1-p)(p-n-np)\mu-2p^2\} = \{(n-1)\mu-2\}\{(1-p)\mu-2p\}^2. \quad \text{(i)}$$

We conclude that, in general :—

The tangents to the n-ic at its intersections with a line through A all touch a curve of class 3 and degree 4, having the line as bitangent. The curve touches BI, BJ and touches AU at U.

The following special cases may be noted :—

If the line is AV ($p=n$) or AO ($p=\infty$), the curve degenerates into three points, as is evident from § 4.

If the line is AC ($p=1$) or AD ($p=-1$), the tangents touch a conic.

If the line is AU ($np=1$), the tangents touch a cuspidal cubic with cusp at U.

7. We may show similarly that, in general, the tangents to the n-ic at its intersections with a conic touching AB and AC at B and C (other than the tangents at B and C) all touch a curve of class 4.

But if the conic goes through I and J, the tangents (other than the tangents at B, C, I, J) all touch a conic touching AB and AC at B and C.

Another similar result is :—

The tangents to the n-ic at the points given by $\tan n\phi = c$, c being any given constant, all touch a curve of class 4. The curve degenerates if $c=0$ or ∞.

Or, again :—

The tangents to the n-ic at its intersections with a line through O all touch a curve of class 6.

8. If x', y', x, y are connected by the birational relations

$$x'=x, \quad y'=1-x^2/y,$$

then § 3 (iii) gives

$$x'=\tfrac{1}{2}\sec\phi, \qquad y'=\tfrac{1}{2}\sin(n-2)\phi \operatorname{cosec}(n-1)\phi \sec\phi.$$

Hence (x', y') traces out the same curve as (x, y) but with $n-1$ instead of n. This enables us to derive properties of the n-ic we are considering from properties of the corresponding $(n-1)$-ic.

For instance, we proved in § 4 that the tangents to the $(n-1)$-ic at its intersections with $y=\frac{1}{2}$, $y=(n-2)/(2n-2)$ all pass through $(0, 1-\frac{1}{2}n)$. We deduce that:—

If conics are drawn through B touching the n-ic at C, and at an intersection of the n-ic with either of the conics which touch the n-ic at C and pass through B, E, F or B, I, J, then they all osculate at C.

9. Now suppose B lies inside the conic Σ.

As in § 3 we may project Σ into $x^2-y^2=y$, and we find that the equation of the n-ic is that given in § 3 (i) or (ii) with all the *minus* signs replaced by *plus*.

If n is odd, the curve is the locus of

$$x=\tfrac{1}{2}\operatorname{cosech}\phi, \qquad y=\tfrac{1}{2}\sinh(n-1)\phi\operatorname{sech}n\phi\operatorname{cosech}\phi.$$

If n is even, the curve is the locus of

$$x=\tfrac{1}{2}\operatorname{cosech}\phi, \qquad y=\tfrac{1}{2}\cosh(n-1)\phi\operatorname{cosech}n\phi\operatorname{cosech}\phi.$$

If n is odd, the curve consists of the isolated $(n-1)$-ple point B and a single branch having three real collinear inflexions, one of which is A. Every line (not through B) meets the branch in one or three real points.

If n is even, the curve consists of a single inflexionless branch touching AB at B. Every line meets the curve in two real points or none.

If a line through A meets the n-ic in real points P and Q, and meets BC in H, (PQ, AH) is a harmonic range.

If we project Σ into a circle and B into its centre, the equation of the n-ic becomes in polar coordinates

$$(-1)^n\tan^{2n}\tfrac{1}{2}\theta=(a-r)/(a+r).$$

The n-ic has $2n$-point contact with the circle $r=a$.

10. As in § 1, we may show that an n-ic touching the conic $u_2=0$ n times at each intersection with the line $u_1=0$ has an equation of the form $u_2u_{n-2}=u_1^n$.

Suppose the n-ic has an $(n-1)$-ple point B, and has n-point contact with the conic Σ at two points H, K. Let the polar of B with respect to Σ meet HK at A, and let (AC, HK) be a harmonic range. Then pro-

jecting AB to infinity and taking B on the axis of y, the equation of Σ can be put in the form

$$(x^2-y)\pm(y^2-k)=0. \tag{i}$$

Taking the upper sign, the equation of the n-ic becomes

$$(x^2+y^2-y-k)\,u_{n-2}=y^n.$$

Choosing the coefficients of u_{n-2} so that no terms involving $y^2, y^3, \ldots, y^n$ occur in this equation, we find for the equation of the curve

$$yf_{n-1}=(x^2-k)f_{n-2}, \tag{ii}$$

where (supposing ${}^qC_0\equiv 1$ for all values of q)

$$f_r=v_{r,0}+kv_{r-1,1}+k^2v_{r-2,2}+k^3v_{r-3,3}+\ldots, \tag{iii}$$

$$v_{r,t}={}^rC_t\,{}^rC_r-{}^{r-2}C_t\,{}^{r-1}C_1x^2+{}^{r-4}C_t\,{}^{r-2}C_2x^4-{}^{r-6}C_t\,{}^{r-3}C_3x^6+\ldots. \tag{iv}$$

This is the standard curve into which may be projected any n-ic with an $(n-1)$-ple point B having n-point contact with a conic Σ (not enclosing B) at two points (see also § 12).

Taking $k=0$, we have the curve of §§ 3–8.

The equation (ii) may also be written

$$y=\frac{x^2-k}{1}\,\frac{}{-}\,\frac{x^2-k}{1}\,\frac{}{-}\,\frac{x^2-k}{1}\,\frac{}{-}\ldots, \tag{v}$$

$n-1$ convergents of the continued fraction being taken.

Any point on the curve is

$x=(k+\frac{1}{4}\sec^2\phi)^{\frac{1}{2}}, \qquad y=\frac{1}{2}\sin(n-1)\phi\,\mathrm{cosec}\,n\phi\sec\phi,$

if $\qquad x^2>k+\frac{1}{4}$;

$x=(k+\frac{1}{4}\,\mathrm{sech}^2\phi)^{\frac{1}{2}}, \qquad y=\frac{1}{2}\sinh(n-1)\phi\,\mathrm{cosech}\,n\phi\,\mathrm{sech}\,\phi,$

if $\qquad k+\frac{1}{4}>x^2>k$;

$x=(k-\frac{1}{4}\,\mathrm{cosech}^2\phi)^{\frac{1}{2}}, \qquad y=-\frac{1}{2}\cosh(n-1)\phi\,\mathrm{cosech}\,n\phi\,\mathrm{cosech}\,\phi$ for n even,

$\qquad y=-\frac{1}{2}\sinh(n-1)\phi\,\mathrm{sech}\,n\phi\,\mathrm{cosech}\,\phi$ for n odd,

if $\qquad k>x^2$.

An alternative is to put

$$x=\tfrac{1}{2}(1+4k)^{\frac{1}{2}}\cos\phi,\ \ \&c.$$

Results similar to those of §§ 3 to 7, but less simple, may be obtained

for the curve (ii). For instance, the conic of § 4 (ii) is replaced by the quartic

$$\{(n+2c-1)x^2-(2c-1)k\}(4x^2-4k-1)+(4k+1)(x^2-k)(2y-1)$$
$$+nx^2(2y-1)^2=0.$$

If $k=-\frac{1}{4}$, the n-ic is of the type discussed elsewhere (*Messenger of Mathematics*, 1920), having an $(n-1)$-ple point and two tangents of n-point contact.

The transformation of § 8 is to be replaced by

$$x'=x, \quad y'=1-(x^2-k)/y.$$

11. If B is inside the conic Σ, we take the lower sign in § 10 (i), and prove similarly that the equation of the n-ic is

$$y=\frac{x^2+k}{1}+\frac{x^2+k}{1}+\frac{x^2+k}{1}+\ldots, \tag{i}$$

to $n-1$ convergents. This is the same as

$$yf_{n-1}=(x^2+k)f_{n-2},$$

the *minus* signs in § 10 (iv), being replaced by *plus*.

If we project Σ into the circle $r=a$ and B into its centre, the equation of the curve in polar coordinates becomes

$$\frac{a-r}{a+r}=\left(\frac{\cos\theta-(1+4k)^{\frac{1}{2}}}{\cos\theta+(1+4k)^{\frac{1}{2}}}\right)^n.$$

Since B lies outside Σ, $1+4k>0$.

More generally, the n-ic with an $(n-1)$-ple point at the pole meeting $r=a$ at its intersections with $r\cos(\theta-a_i)=k_ia$ $(i=1, 2, \ldots, n)$ is

$$\frac{a-r}{a+r}=\Pi\left(\frac{\cos(\theta-a_i)-k_i}{\cos(\theta-a_i)+k_i}\right).$$

12. Another method of obtaining the equations of § 3 (i) and (ii), § 10 (ii) and (v), § 11 (i), is the following.

First, we notice that the number of conditions which the curves discussed have to satisfy is just enough to determine them uniquely, so we need only verify that the curves given by these equations have the properties stated.

This follows from the result that, if f and ϕ are polynomials in x,

$q_n y = fq_{n-1}$ meets the curve $y(\phi+y)=f$ only where $y=0$; fq_{n-1}/q_n being the n-th convergent of the continued fraction

$$\frac{f}{\phi+}\frac{f}{\phi+}\frac{f}{\phi+}\cdots,$$

so that $q_n \equiv \phi^n + {}^{n-1}C_1\phi^{n-2}f + {}^{n-2}C_2\phi^{n-4}f^2 + {}^{n-3}C_3\phi^{n-6}f^3 + \ldots.$

In fact, since $$q_n = \phi q_{n-1} + fq_{n-2},$$

$$q_n y - fq_{n-1} \equiv q_{n-1}\{y(\phi+y)-f\} - y(q_{n-1}y - fq_{n-2});$$

from which the result follows at once by induction.

Taking $\phi\equiv 1$ and $f\equiv x^2+k$, we get the result of § **11** (i), and similarly in the other cases.

The reader will find the case $\phi\equiv 1$, $f\equiv x$ of interest.

RELATION BETWEEN APOLARITY AND THE PIPPIAN-QUIPPIAN SYZYGETIC PENCIL

By WILLIAM P. MILNE *and* D. G. TAYLOR.

With a Note on Apolarity by H. W. RICHMOND.

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CAYLEY in his "Third Memoir on Quantics" [*Philosophical Transactions of the Royal Society of London*, Vol. 146 (1856), pp. 627–647] obtained two contravariants of the cubic curve, which he denoted by

$$P \equiv k(l^3+m^3+n^3)+(1-4k^3)lmn,$$

$$Q \equiv (1-10k^3)(l^3+m^3+n^3)-6k^2(5+4k^3)lmn,$$

for the case of the canonical equation

$$U \equiv x^3+y^3+z^3+6kxyz = 0,$$

and which he called respectively the "Pippian" and the "Quippian." In a subsequent paper entitled "Memoir on Curves of the Third Order" [*Philosophical Transactions of the Royal Society of London*, Vol. 147 (1857), pp. 415–446], Cayley investigated the geometrical interpretation of the concomitants he had obtained, and makes the following statement which we quote verbatim :—

"I have not succeeded in obtaining any good geometrical definition of the Quippian and the following is only given for want of something better. The curve

$$T.PU\{P6H(aU+6\beta HU)\}$$
$$-P(6HU)\{T(aU+6\beta HU).P(aU+6\beta HU)\} = 0,$$

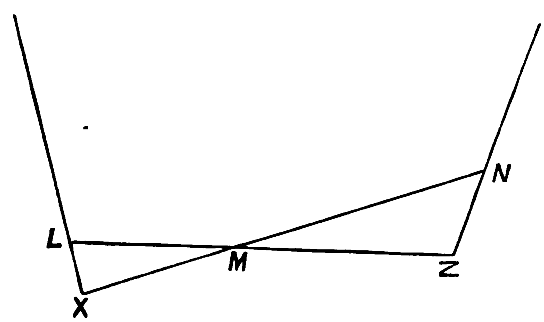
which is derived in what may be taken to be a known manner from the cubic, is in general a curve of the sixth class. But if the syzygetic cubic $aU+6\beta HU = 0$ be properly selected, viz. if this curve be such that its

Hessian breaks up into three lines, then both the Pippian of the cubic $aU+6\beta HU=0$, and the Pippian of its Hessian will break up into the same three points, which will be a portion of the curve of the sixth class, and discarding these three points the curve will sink down to one of the third class, and will in fact be the Quippian of the cubic."

Subsequent writers have to a large extent discarded the term "Pippian" and use the name "Cayleyan" instead. Salmon and Elliott define P and Q as the first evectants of the invariants S and T of the cubic, but neither of them give convenient geometrical definitions. Clebsch defines the Quippian as the envelope of lines whose polar-conics with respect to the Cayleyan are apolar to their polo-conics with respect to the original cubic (the polo-conic of a line being defined as the locus of points whose polar-conics with respect to the original cubic touch that line).

We have not been able to find, however, in previous papers dealing with the cubic curve any geometrical definition of the Pippian-Quippian pencil of class-cubics, which derives this system simply and concisely, member by member, from the fundamental syzygetic pencil of cubics through the intersections of a given cubic and its Hessian. This desideratum is furnished, however, by the extension of the results of our joint-paper on "The Significance of Apolar Triangles in Elliptic Function Theory" (*Proc. London Math. Soc.*, Ser. 2, Vol. 18, pp. 375–384) to the case when the apolar triangle becomes a collinear triad of points.

In the above paper we showed that if LMN be a triangle inscribed in a cubic curve, one, and only one, member of the pencil of curves through the points of inflexion is apolar to LMN; and that if L, M, N be projected through any point T of the cubic curve on to the curve again, so that the points L', M', N' are obtained, the triangle $L'M'N'$ remains



apolar to the same member of the syzygetic pencil. We proceed to consider this property when the triad of points L, M, N are collinear.

Let L, M, N be three points nearly collinear on a cubic curve U, and let U' be that member of the flex-pencil to which the triad L, M, N is apolar. Let the line LM meet the curve in Z. Then it is a known property that the bipolar (or mixed polar) line of L, M with respect to U' passes through Z. This bipolar line also passes through N since the triad L, M, N is apolar to U'. Hence proceeding to the limit when N moves up to coincidence with Z, we see that the bipolar line of L, M with respect to U' is the tangent at N with respect to U. Similarly it may be shown that the bipolar line of M, N with respect to U' touches U at L, and that the bipolar line of L, N with respect to U' touches U at M.

If now we take as lines of reference the line LMN, and the tangents to the polo-conic of LMN with respect to U at the points where this conic meets LMN, the equation to the cubic U reduces to the form

$$U \equiv ax^3+by^3+cz^3+3c_1z^2x+3c_2z^2y+6kxyz = 0.$$

The equation to the cubic U' passing through the points of inflexion of U, and such that the bipolar-line of any two of the points L, M, N with respect to U' touches U at the third, is easily found to be

$$U' \equiv \lambda U+H = 0,$$

where H is the Hessian whose equation is given in expanded form by Salmon in his *Higher Plane Curves*, and

$$\lambda = \frac{10k^3-abc}{18k}.$$

If we compute the equation to the Cayleyan or Pippian (*i.e.* the first evectant of the Invariant S) we see that the coefficient of n^3 is abk, using l, m, n as tangential coordinates.

Also the coefficient of n^3 in the case of the Quippian (*i.e.* the first evectant of the Invariant T) is

$$2ab(abc-10k^3) \equiv -36abk\lambda.$$

Hence plainly the contravariant curve $36\lambda P+Q = 0$ touches the line LMN.

We have therefore the following general result:—

If L, M, N be three collinear points on a cubic curve U constituting a triad apolar to $\lambda U+H = 0$, a member of the flex-pencil of U, the line

LMN envelops the member $36\lambda P+Q=0$ of the Pippian-Quippian syzygetic pencil of class-cubics.

The following particular cases of the above general theorem are of great importance:—

If L, M, N be three collinear points on a cubic curve constituting a triad apolar to the curve itself, the line LMN envelops the Cayleyan (Pippian).

This result had already been obtained from an entirely different standpoint in the *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 235–243.

If L, M, N be three collinear points on a cubic curve constituting a triad apolar to the Hessian, the line LMN envelops the Quippian.

The foregoing results are also easily obtained by means of Elliptic Functions, and thus serve to throw additional light on the significance of Apolarity in Elliptic Function theory.

Let
$$U_{\alpha\beta\gamma} \equiv 12\wp(\alpha)\,\wp(\beta)\,\wp(\gamma)-\Sigma\wp'(\beta)\,\wp'(\gamma)-g_2\Sigma\wp(\alpha)-3g_3,$$
$$H_{\alpha\beta\gamma} \equiv \Sigma\wp(\alpha)\,\wp'(\beta)\,\wp'(\gamma)-g_2\Sigma\wp(\beta)\,\wp(\gamma)-3g_3\Sigma\wp(\alpha)-\tfrac{1}{4}g_2^2.$$

If α, β, γ be collinear both the above expressions vanish identically, and hence the condition for apolarity with respect to $\lambda U+H=0$, has to be satisfied by either $\alpha+\delta\alpha$, β, γ or α, $\beta+\delta\beta$, γ or α, β, $\gamma+\delta\gamma$, from which we deduce at once by addition the required condition in a symmetrical form, viz.

$$\lambda\left(\frac{\partial U_{\alpha\beta\gamma}}{\partial\alpha}+\frac{\partial U_{\alpha\beta\gamma}}{\partial\beta}+\frac{\partial U_{\alpha\beta\gamma}}{\partial\gamma}\right)+\left(\frac{\partial H_{\alpha\beta\gamma}}{\partial\alpha}+\frac{\partial H_{\alpha\beta\gamma}}{\partial\beta}+\frac{\partial H_{\alpha\beta\gamma}}{\partial\gamma}\right)=0.$$

It is thereafter easy to show that this expression is equal to $36\lambda P+Q$, where P and Q are the tangential forms for the Pippian and Quippian respectively in Weierstrassian canonical form. The required result follows at once.

We also obtain the following property:—

If A, B, C be any three points on a cubic curve, and if these three points be projected through a point of the curve on to the curve again so that a collinear triad is obtained, the lines of the nine collinear triads thus found all touch the same member of the Pippian-Quippian pencil.

For all the nine lines cut the original cubic U in triads of points

apolar to the same member $\lambda U+H$ to which ABC is apolar, and hence touch $36\lambda P+Q$.

A Note on Apolarity by H. W. Richmond.

[Received September 25th, 1920.]

An Apolar Triad of points (P, Q, R) on a cubic curve (C) is a set of three points apolar with respect to one of the pencil of cubics based upon C and its Hessian. The relation which connects the coordinates of three such points

(1) is symmetrical;

(2) is linear in the coordinates of each point, so that when two (P, Q) of the three points are given, R is constrained to lie on a definite straight line;

(3) is such that the line passes through the point N of C collinear with P and Q. Thus, if P and Q are given, there are two positions of R on C (N being now excluded).

Recently, Prof. Milne and Dr. D. G. Taylor have shown that when the coordinates of points on C are expressed by elliptic functions of a parameter, the relation between the parameters of P, Q, R depends only on the *differences* of the parameters. I wish to point out that this is a necessary consequence of the facts (1), (2), (3), stated above.

Let u, v, w be the parameters of P, Q, R. Let PQ cut C in N (parameter $-u-v$), and let NR cut C in R' (parameter $u+v-w$). Then, as stated above, P, Q, R' form another apolar triad, their parameters being u, v, and w', where

$$w' = u+v-w. \tag{1}$$

By the same reasoning v, w' and $v+w'-u$, *i.e.*

$$u+z, \quad v+z, \quad w+z, \quad \text{where } z = v-w, \tag{2}$$

also form an Apolar Triad.

The process can be repeated, and it follows that if u, v, w form an Apolar Triad, so also do the points $u+z$, $v+z$, $w+z$, where z has any of the values

$$r(v-w)+s(u-w),$$

r, s being integers positive or negative. Except when both $v-w$ and $u-w$ are in a rational ratio to a period, z has an infinite number of values. Since two of u, v, w can be chosen arbitrarily, the condition of apolarity must be satisfied generally, and therefore universally, by *all* values of z.

By including the result (1), we can assert that if the points u, v, w form an Apolar Triad, so also do the points

$$z+u, \quad z+v, \quad z+w,$$

and

$$z-u, \quad z-v, \quad z-w,$$

whatever be the value of z.

RELATION BETWEEN APOLARITY AND A CERTAIN PORISM OF THE CUBIC CURVE

By Prof. W. P. MILNE.

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1. *Introduction.*

In two papers in *Liouville's Journal*—"Mémoire sur les courbes du troisième ordre" (tome ix, 1844) and "Nouvelles remarques sur les courbes du troisième ordre" (tome x, 1845)—Cayley developed at considerable length the properties of "corresponding points" on a cubic curve, the definition of "corresponding points" being that the tangents thereat should meet on the curve.

Since then much attention has been given to the study of "corresponding points on the Hessian," regarded as conjugate points with respect to all the polar conics of a cubic curve. This problem has in later years been generalised to include the case of conics apolar to all the polar conics of a cubic curve. The object of the present communication is to study in greater detail generalisations of the properties of "corresponding points on the Hessian," regarded as degenerate conics apolar to the net of polar conics of a given cubic. It will be found that in the case of many fundamental properties, conics subjected to the condition of being apolar to the polar-conic net cannot be regarded as direct generalisations of "corresponding points on the Hessian," but must in essence be regarded as satisfying two further conditions (see § 7 below).

Incidentally, the paper throws a good deal of light on the properties of the pencil of cubic curves through the nine points of intersection of the sides of two triangles. Caporali discusses the general properties and covariant loci of any pencil of cubic curves in a communication, "Teoremi sui fasci di curve del terzo ordine," reprinted on p. 52 of his *Memorie di Geometria,* while Salmon deals with the particular case of the pencil defined by two triangles (as explained above) in his treatise on *Higher Plane Curves.* He devotes considerable attention to the "critical

centres," *i.e.* the nodes of the rational members of the pencil. He does not discuss, however, the case where both triangles are circumscribed to the same conic. This problem is investigated in the present communication, and was suggested by the attempt to extend and generalise the results obtained in my paper on "Determinantal Systems of Co-Apolar Triads on a Cubic Curve" (*Proc. London Math. Soc.*, Ser. 2, Vol. 18, pp. 274–279). It is inevitable to consult also in this connexion the papers on the porismatic character of polygrams circumscribed to a conic and inscribed in other curves by Darboux ("Sur une classe remarquable de courbes et de surfaces algébriques") and by Clifford ("On the Transformation of Elliptic Functions," *Proc. London Math. Soc.*, Vol. 7). There is also an important paper "On Polygons Circumscribed about a Conic and Inscribed in a Cubic" (*Proc. London Math. Soc.*, Ser. 2, Vol. 17, pp. 158–171), by R. A. Roberts, who develops the subject from the algebraic standpoint and obtains several of the results obtained by the methods of synthetic geometry in the present paper. In particular, a complete generalisation is obtained of a theorem due to Dr. William L. Marr (*Proc. Edin. Math. Soc.*, Vol. 37, p. 72), viz. that if a conic touches the line of flexes of a nodal cubic and also the three inflexional tangents, corresponding pairs of triangles can be circumscribed to the conic, the nine intersections of whose sides lie on the cubic.

The two chief results obtained in the present paper are, however, the following :—

The conditions that a conic must satisfy in order that corresponding pairs of triangles may be found circumscribed to the conic and having the nine intersections of their sides lying on a given cubic, can be expressed simply in terms of apolar properties of the cubic curve.

If two straight lines cut a cubic curve and if the points of intersection be joined, two and two, by straight lines cutting the cubic again, it is well known that these further points of intersection lie three by three on six straight lines. The condition that these six straight lines shall touch a conic can be expressed very simply in terms of apolar properties of the cubic curve.

It will be found that the configuration of the two triangles mentioned above, and with which we shall have principally to deal, is a particular case, possessing very special and fundamental properties, of the configuration discussed by Dr. W. Franz Meyer (*Apolarität und Rationale Curven*, p. 222, § 26), consisting of the complete hexagon circumscribed to a conic and having the fifteen triangles formed by the intersections of

its sides, taken three by three, so that no two lie on the same side of the hexagon, apolar to a cubic curve.

2. *Preliminary Definitions and Properties.*

We shall frequently use the following definition :—

If a conic-envelope be apolar to every member of the net of polar-conics of a given cubic, the conic-envelope is said to be apolar to the given cubic.

The subjoined results were established in the present volume of the *Proc. London Math. Soc.*, pp. 101–104, and will often be required :

(1) *If a straight line cut a cubic curve S in three points L, M, N, such that the bipolar (or mixed polar) line of M, N with respect to a member U of the pencil of cubics through the flexes of S touches S at L, this property is symmetrical with respect to L, M, N.*

(2) *The triad L, M, N is said to be a collinear triad apolar to U, and possesses all the properties of a non-collinear triad apolar to U* (*Proc. London Math. Soc.*, Ser. 2, Vol. 18, pp. 375–384).

(3) *If U be the curve S itself, the line LMN envelopes the Pippian (Cayleyan).*

If U be the Hessian of S, the line LMN envelopes the Quippian (a contravariant of the third degree of the cubic curve, discovered by Cayley and so designated ; see Salmon's *Higher Plane Curves* or Elliott's *Invariants*).

If U be any given member of the syzygetic flex-pencil, the line LMN envelopes a corresponding member Ψ of the Pippian-Quippian pencil of class-cubics.

Consider now Fig. 1 on page 110.

Let ABC and DEF be two triangles, the nine intersections of whose sides are typified by the points P, Q, R, as shown above. We shall refer to the above nine intersections as a "Determinantal System of points" (see *Proc. London Math. Soc.*, Ser. 2, Vol. 18, pp. 274–279), and we shall denote them by the determinantal form

$$\begin{matrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{matrix}$$

or more briefly by $$|PQR|.$$

We shall mainly be concerned with the properties of the triads of points denoted by the six terms of the expansion of the above determinantal form, and inasmuch as the triads defined by the positive terms of the expansion differ in properties from the triads defined by the negative, we shall refer to them as "positive and negative" triads respectively.

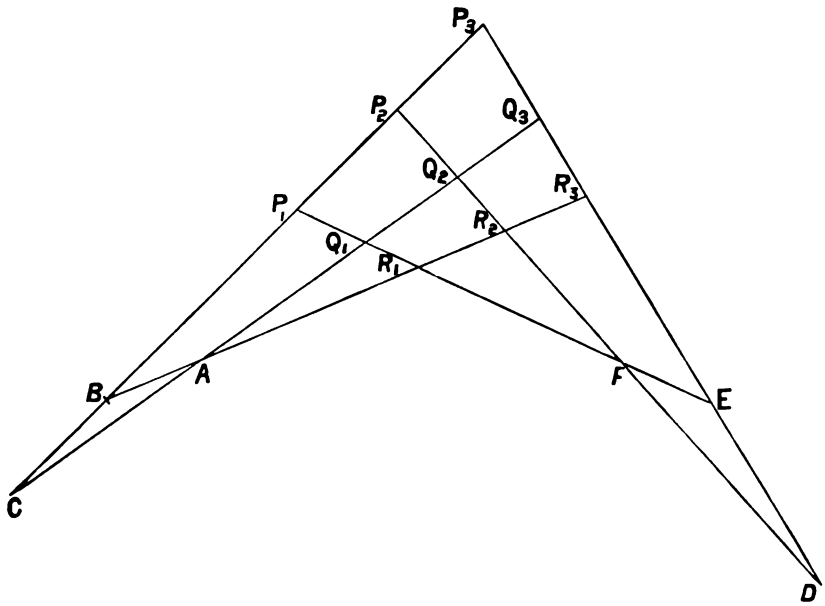

Fig. 1.

We proceed to establish the following theorem :—

If S_1 and S_2 denote the sides of the triangles ABC and DEF, and if $\lambda_1 S_1 + \lambda_2 S_2$ be any member of the pencil of cubics through the nine points $|PQR|$ of the intersections of their sides, the three "positive triads" are each apolar to the same member U of the pencil of cubics through the flexes of $\lambda_1 S_1 + \lambda_2 S_2$, and similarly the "negative triads" are each apolar to another member U′ of the syzygetic pencil.

Consider Fig. 2 below, and let the cubic $\lambda_1 S_1 + \lambda_2 S_2 \equiv S$, and let U be that cubic through its points of inflexion which is apolar to the triad $P_1 Q_2 R_3$. Then the triad obtained by projecting $P_1 Q_2 R_3$ through the point P_2 on to the cubic S again remains apolar to U (*Proc. London Math. Soc.*, Ser. 2, Vol. 18, pp. 375–384). Thus $P_3 L R_2$ is apolar to U where the line $P_2 R_3$ meets the curve again in L, and hence the bipolar line of P_3, R_2 with respect to U passes through L. Let the line $P_3 R_2$ meet the cubic again in M. Then we know that the bipolar line of P_3, R_2 passes through

M, and hence is LM. Consider now the four points P_2, P_3, R_3, R_2. One conic through them consists of the two lines P_2R_2, P_3R_3, and cuts the cubic again in the two points Q_2, Q_3. The chord Q_2Q_3 intersects the curve in the third point Q_1. Hence the conic consisting of the two lines P_2R_3, P_3R_2 cuts the cubic again in a chord which passes through Q_1. Thus LM, the bipolar line of P_3, R_2 with respect to U, passes through Q_1. Hence $P_3Q_1R_2$ is a triad apolar to U, and similarly with regard to $P_2Q_3R_1$. In precisely the same way it can be proved that $P_1Q_3R_2$, $P_3Q_2R_1$, $P_2Q_1R_3$ are each apolar to another cubic U' of the syzygetic pencil of S.

We note that if S_1 and S_2 each consist of three concurrent lines whose points of concurrency are H and K respectively, then all the triads of $|PQR|$ are apolar to every cubic $\lambda_1 S_1+\lambda_2 S_2$, and the degenerate class-conic H, K is apolar to every cubic of the pencil $\lambda_1 S_1+\lambda_2 S_2$ (*Proc. London Math. Soc.*, Ser. 2, Vol. 9). We proceed to generalise this theorem.

3. *Fundamental Theorem.*

If, as before, $\lambda_1 S_1+\lambda_2 S_2 \equiv S$ be a member of the pencil of cubics through the nine points of intersection of the sides of the triangles $S_1 \equiv ABC$ and $S_2 \equiv DEF$, and if both the "positive" and "negative" triads of the system of points $|PQR|$ be apolar to the same member U of the syzygetic pencil of S, this property is poristic in that every member of the pencil $\lambda_1 S_1+\lambda_2 S_2$ possesses the same property, and the necessary and sufficient condition is that the six sides of the triangles ABC and DEF shall all touch the same conic.

We shall give two proofs, one by the methods of analytical geometry, and one by the use of elliptic functions.

The locus of the points of inflexion of the cubics of the pencil $\lambda_1 S_1+\lambda_2 S_2$ is easily found to be the sextic curve

$$S_1 U_2+S_2 U_1 = \Delta_0 S_1 S_2,$$

where Δ_0 is the invariant whose vanishing denotes that the sides of S_1 and S_2 shall touch the same conic, and where U_1, U_2 are the equianharmonic cubics having the triangles S_1, S_2 as base triangles, and cutting the sides of S_1 and S_2 in triads apolar to $P_1P_2P_3$, $Q_1Q_2Q_3$, $R_1R_2R_3$, $P_1Q_1R_1$, $P_2Q_2R_2$, $P_3Q_3R_3$. This sextic may be considered as being generated by corresponding members of the cubic-pencils

$$\lambda_1 S_1+\lambda_2 S_2 = 0,$$

$$\lambda_2 U_2-\lambda_1 U_1 = \lambda_2 \Delta_0 S_2.$$

Hence the general equation to the pencil of cubics through the flexes of $\lambda_1 S_1 + \lambda_2 S_2 = 0$ is

$$(\lambda_2 U_2 - \lambda_1 U_1 - \lambda_2 \Delta_0 S_2) + \rho (\lambda_1 S_1 + \lambda_2 S_2) = 0,$$

i.e.
$$(\lambda_2 U_2 - \lambda_1 U_1) + \rho\lambda_1 S_1 + \lambda_2 (\rho - \Delta_0) S_2 = 0. \tag{1}$$

Since U_1, U_2 can be expressed as the sum of the cubes of the sides of S_1, S_2 respectively, and since the six triads of points of $| PQR |$ can each be regarded as a degenerate class-cubic inscribed in S_1 and S_2, we see that the lineo-linear invariants of $P_1 Q_2 R_3$, &c., and U_1, U_2 vanish identically. Furthermore, the condition that $P_1 Q_2 R_3$ shall be apolar to S_1, S_2, being the vanishing of the discriminants Δ_1, Δ_2 respectively, we see from (1) that the conditions that the "positive" and "negative triads" shall be each apolar to the cubic (1) are respectively

$$\rho\lambda_1 \Delta_1 + \lambda_2 (\rho - \Delta_0) \Delta_2 = 0,$$

$$\rho\lambda_1 \Delta_1 - \lambda_2 (\rho - \Delta_0) \Delta_2 = 0,$$

from which we deduce at once that $\rho = 0$ and $\Delta_0 = 0$.

The fundamental proposition is thus established; but we can deduce it in another way, by means of elliptic functions, for which I am indebted to Mr. C. W. Gilham.

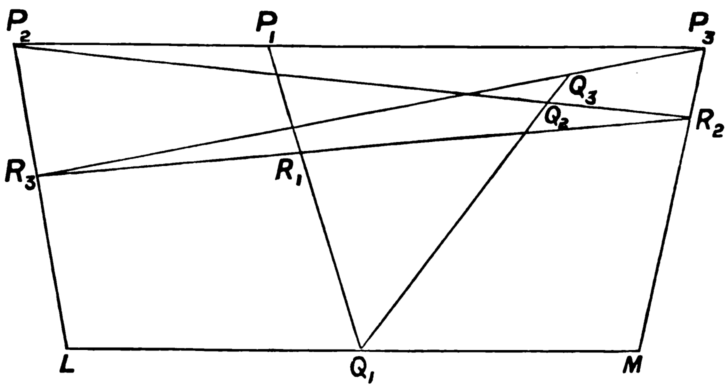

Fig. 2.

Let us use the above figure described in § 2.

The necessary and sufficient condition that the six lines $P_2 P_3$, $R_2 R_3$, $P_2 R_2$, $P_3 R_3$, $P_1 Q_1$, $Q_3 Q_1$ shall all touch the same conic is that the three pairs of lines $Q_1 R_3$, $Q_1 P_2$; $Q_1 R_2$, $Q_1 P_3$; $Q_1 P_1$, $Q_1 Q_3$ shall form an involution. This condition is satisfied if LM be the bipolar line of P_2, R_3 and P_3, R_2 with respect to the same member U of the Hessian pencil of S.

For, in the Weierstrassian notation, $x = \wp(u)$, $y = \wp'(u)$, if $P_2 \equiv a$, $R_3 \equiv a'$, $P_3 \equiv \beta$, $R_2 \equiv \beta'$, the condition that the lines

$$Q_1 [R_3, P_2 : P_1, Q_2 : P_3, R_2]$$

shall be corresponding pairs in an involution is

$$\sigma(a+2\beta)\,\sigma(\beta+2a')\,\sigma(a'+2\beta')\,\sigma(\beta'+2a)$$
$$= \sigma(2a+\beta)\,\sigma(2\beta+a')\,\sigma(2a'+\beta')\,\sigma(2\beta'+a). \quad (2)$$

Again, the condition that $P_2 Q_1 R_3$ and $P_3 Q_1 R_2$ shall be each apolar to the same member U of the Hessian-pencil of S is (see *Proc. London Math. Soc.*, Ser. 2, Vol. 18, p. 378)

$$\frac{\wp(a'+\beta+\beta')\,\wp'(a+\beta+\beta')+\wp(a+\beta+\beta')\,\wp'(a'+\beta+\beta')}{\wp'(a'+\beta+\beta')+\wp'(a+\beta+\beta')}$$
$$= \frac{\wp(a+a'+\beta')\,\wp'(a+a'+\beta)+\wp(a+a'+\beta)\,\wp'(a+a'+\beta')}{\wp'(a+a'+\beta')+\wp'(a+a'+\beta)}. \quad (3)$$

It is easy to show that the conditions (2) and (3) are equivalent, which establishes the required result.

We therefore see that the sides of the two triangles S_1 and S_2 must all touch the same conic Σ, and that the property is poristic.

We next proceed to show that *the conic Σ is apolar to the cubic*

$$U \equiv \lambda_1 U_1 - \lambda_2 U_2.$$

Since the degenerate class-conics P_2, R_3 and P_3, R_2 are each apolar to the polar-conic of Q_1 with respect to U, therefore the conic Σ which belongs to the pencil $(P_2, R_3)+\mu\,(P_3, R_2) = 0$ must also be apolar to the polar conic of Q_1. Similarly Σ is apolar to the polar-conics of P_2 and R_3, and must therefore be apolar to the cubic U. We therefore have the following result:—

If S be a cubic through the nine points of intersection $|PQR|$ of the sides of the triangles ABC and DEF, and if U be a member of its Hessian-pencil such that the triads defined by the terms in the expansion of the determinantal form $|PQR|$ are each apolar to U, the sides of the triangles ABC and DEF must all touch the same conic Σ, and Σ will be a conic apolar to the cubic U. The property is poristic inasmuch as if one cubic S of the pencil exist so that all the six triads of $|PQR|$ are apolar to a member U of its Hessian-pencil, then all the members S

possess the property that each has a corresponding cubic U belonging to its Hessian-pencil to which all the triads of $|PQR|$ *are apolar. In general, if two triangles ABC and DEF be given arbitrarily no cubic S of the pencil defined by their points of intersection possesses this property. The necessary and sufficient condition for the porism to exist is that the sides of the two given triangles shall all touch the same conic.*

4. *Theorem on Two Chords of a Cubic Curve.*

It is an elementary property of the cubic curve that if the points of intersection of two lines x and y with the curve be joined in pairs so as to cut the curve again in three points, these three points lie on a straight line, and that if this be done in every possible way, six straight lines in all are obtained. We proceed to find the condition that these six lines shall touch a conic.

Let the lines x and y cut the cubic S in the points L, M, N and L', M', N' respectively. Let the third point of intersection of the chord LL' be denoted by (LL') and so on. We thus obtain the following determinantal configuration of points:—

$$\begin{array}{ccc} (LL'), & (MM'), & (NN'), \\ (MN'), & (NL'), & (LM'), \\ (NM'), & (LN'), & (ML'). \end{array}$$

The rows represent one set of three lines and the columns the other set of three lines.

Since the above six lines touch a conic, the six triads of the determinantal system of points are each apolar to the same member U of the Hessian-pencil. Hence the three points (LL'), (NL'), (ML') constitute a triad apolar to U. But if these three points be projected through L' on to the cubic S again, the three points L, M, N also constitute a collinear triad apolar to U by § 2. Similarly, the three points L', M', N' form a collinear triad apolar to U. Hence the two lines x and y on which these triads lie must each touch the same member of the Pippian-Quippian pencil of class-cubics. We therefore have the following result.

If x and y be two lines cutting the cubic curve S in points which, joined two and two, cut the curve S again in nine points lying three by three on six lines that touch a conic, the necessary and sufficient condition

is that x and y shall each touch the same member of the Pippian-Quippian pencil of class-cubics.

This is a direct extension of the results obtained in my paper on "Determinantal Systems of Co-Apolar Triads on a Cubic Curve" (*Proc. London Math. Soc.*, Ser. 2, Vol. 18).

We proceed to investigate further properties of the conic Σ.

5. *The Intersections of S and Σ.*

Let l_1, l_2, l_3 and m_1, m_2, m_3 be the sides of two triangles touching the conic Σ and having their nine points of intersection on the cubic S, and let U be that cubic through the points of inflexion of S to which Σ is apolar. Then it is known that the above configuration is poristic. In fact, if we take any arbitrary tangent l_1' of the conic Σ, and from the points of intersection of l_1' with S draw the tangents m_1', m_2', m_3' to Σ, the remaining six points of intersection of m_1', m_2', m_3' with S will lie on two other tangents l_2', l_3' of the conic Σ. We thus see that *S and Σ are so related that an infinite number of corresponding pairs of triangles can be found touching Σ and having the nine points of intersection of their sides lying on the cubic S. The property is a porism inasmuch as if one pair of such triangles can be found, an infinite number exist. The six triads of any of the determinantal systems of points thus defined possess the property of being apolar to a fixed member U of the Hessian-pencil of S, namely, that member U to which Σ is apolar.*

Consider now the tangent to Σ at one of the points of intersection of the conic with S. If we express S in the form

$$l_1 l_2 l_3 + k m_1 m_2 m_3 = 0,$$

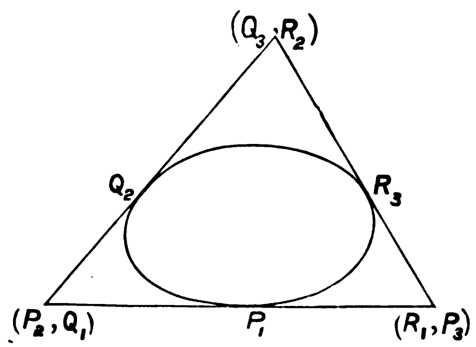

FIG. 3.

it is easy to see that the tangents drawn from the remaining two points of intersection of this line with S touch Σ at two further points of intersection of S with Σ. We thus see that S cuts Σ in two triads of points, each possessing the property that the tangents to Σ at the points of each triad of intersection form two triangles whose vertices lie on the given cubic S.

Let, for example, P_1 be one of the points of intersection of S and Σ, in which case the two tangents from P_1 to Σ become coincident. Let us suppose that Q_1 coincides with P_2 and R_1 with P_3, it being known that $P_1P_2P_3$ and $P_1Q_1R_1$ are the two tangents from P_1 to Σ. Then the tangent from P_2 to Σ, other than $P_1P_2P_3$ is $P_2Q_2R_2$, and the tangent from Q_1 to Σ other than $P_1Q_1R_1$ is $Q_1Q_2Q_3$. Hence we see that the tangent from the point of coincidence of P_2 and Q_1 to Σ touches at Q_2 and meets the cubic again at R_2. Similarly with R_1, P_3. The cubic curve therefore passes through the vertices of the above triangle (Q_3, R_2), (R_1, P_3), (P_2, Q_1) and also through the three points where the sides are touched by Σ. But it is known from the theory of conics that the straight lines joining the vertices of a triangle to the points of contact of the opposite sides with an inscribed conic are concurrent. Also it is known from the theory of cubic curves that if a cubic circumscribe a triangle and cut the opposite sides in three points which, joined respectively to the opposite vertices, are concurrent, then the tangents at the vertices of the triangle to the cubic are concurrent. Hence the tangents to S at the points (Q_3, R_2), (R_1, P_3), (P_2, Q_1) are concurrent. We therefore have the following results:—

If any of the three tangents from an arbitrary point O to a cubic curve S be taken, and their three points of contact (Q_3, R_2), (R_1, P_3), (P_2, Q_1) joined so as to cut the curve again in the three points P_1, Q_2, R_3 the triangles (Q_3R_2), (R_1P_3), (P_2Q_1), and $(P_1Q_2R_3)$ are each apolar to the same member U of the syzygetic pencil of cubics through the flexes of S. Also, if X, Y, Z be the tangential points of (Q_3, R_2), (R_1, P_3), (P_2, Q_1) with respect to S, the lines XP_1, YQ_2, ZR_3 are the polar-lines of (Q_3, R_2), (R_1, P_3), (P_2, Q_1) respectively with respect to U. The conic Σ which touches the sides of the triangle (Q_3, R_2), (R_1, P_3), (P_2, Q_1) at P_1, Q_2, R_3 is also apolar to U and cuts S in three further points $P_1'Q_2'R_3'$, which form another triad apolar to U and possess the property that the tangents at $P_1'Q_2'R_3'$ to Σ form a triangle inscribed in S, the tangents at whose vertices are concurrent.

Also, *let any three tangents from an arbitrary point O to S be taken, and their three points of contact joined so as to cut the curve again in*

the three points P_1, Q_2, R_3. Let the conic Σ be drawn touching at these points the sides of the triangle whose vertices are the points of contact. Σ possesses the property that corresponding pairs of triangles can be found circumscribed to Σ, and such that their nine points of intersection lie on the cubic S. These nine points form a determinantal system whose six triads are each apolar to that cubic U, of the Hessian-pencil of S, to which Σ is apolar.

The main result of the above enunciation can easily be verified as follows. Let Σ be taken in the form $x = t^2$, $y = 1$, $z = 2t$, and let us consider the triangle formed by the tangents to Σ at the points whose parameters are given by $t^3+1 = 0$. The general equation to any cubic S through the vertices of the triangle formed by the tangents, and also through their points of contact with Σ, is

$$S \equiv 2ax^3+2by^3+(a+b)\,z^3-6px^2y+3qx^2z+3py^2z-6qy^2x+3pz^2x$$
$$+3qz^2y-3(a+b)\,xyz = 0.$$

The point $[\theta\phi,\ 1,\ (\theta+\phi)]$ will lie on this cubic S if

$$(\theta^3+1)(a\phi^3+3q\phi^2+3p\phi+b)+(\phi^3+1)(a\theta^3+3q\theta^2+3p\theta+b) = 0,$$

showing that a singly-infinite system of corresponding pairs of triangles circumscribing Σ can be found whose nine points of intersection lie on the cubic S.

It will thus be seen that Σ is a direct generalisation of the degenerate conic formed by H, K, where H and K are corresponding points on the Hessian. The conic-locus which is the reciprocal of the conic envelope H, K is the line HK taken twice over. In this case we start from a point O on the Hessian, the points of contact (Q_3, R_2), (R_1, P_3), (P_2, Q_1) of three of the tangents from which are collinear and lie on the line HK. It is plain from Fig. 3 and the nature of the construction that in this case P_1 coincides with (Q_3, R_2), Q_2 with (R_1, P_3), and R_3 with (P_2, Q_1). Also in this case the three further points of intersection $P_1' Q_2' R_3'$ of the conic-locus $\Sigma \equiv (HK)^2$ and the cubic S become coincident with P_1, Q_2, R_3 respectively.

6. *The Common Tangents of S and Σ.*

Suppose next that in Fig. 1 the line $P_1Q_1R_1$ is one of the common tangents of S and Σ; let P_1 coincide with Q_1 in the limit and hence be

the point of contact of the line $P_1Q_1R_1$ with S. Then plainly the two lines $P_1P_2P_3$ and $Q_1Q_2Q_3$ coincide, and the lines $P_1Q_1R_1$, $P_2Q_2R_2$, $P_3Q_3R_3$ become the tangents to S at the points (P_1, Q_1), (P_2, Q_2), (P_3, Q_3) respectively. Also $R_1R_2R_3$ becomes the "satellite" of the line (P_1, Q_1), (P_2, Q_2), (P_3, Q_3) according to the usual definition. The bipolar line of P_1, Q_2 with respect to U passes through R_3, and hence in this particular case, when the line P_1Q_2, *i.e.* P_1P_2 cuts S in the third point P_3, is the line P_3R_3, *i.e.* the tangent at P_3 to S. Hence the collinear triad of points $P_1P_2P_3$ must be regarded as being apolar to U, and the line $P_1P_2P_3$ must touch the corresponding member Ψ of the Pippian-Quippian pencil. The following result is now evident:—

If any line l be taken cutting the cubic S in the three points $P_1P_2P_3$ and if l' be its "satellite-line," then the conic Σ which touches the five lines l, l' and the three tangents to S at $P_1P_2P_3$ possesses the property of being apolar to a certain member U of the syzygetic pencil of cubics through the flexes of S. The bipolar lines of P_2P_3, P_3P_1, P_1P_2 with respect to U are respectively the tangents to S at $P_1P_2P_3$. Also, corresponding pairs of triangles can be found circumscribed to Σ, such that their nine points $|PQR|$ of intersection lie on S, and possessing the property that the six triads of points corresponding to the expansion of the determinantal form $|PQR|$ are each apolar to U.

Furthermore, when the two lines $P_1P_2P_3$ and $Q_1Q_2Q_3$ become coincident, their points of contact with the conic Σ become coincident, and hence the line must be regarded as touching Σ at one of the points of the Jacobian-tetrad of the involution of triads defined by the points of contact of the triangles circumscribed to Σ and having their nine points of intersection lying on S. Now the cubic S and the conic Σ have twelve tangents in common, and it is plain from the above that these common tangents have their points of contact with S lying three by three on four lines which touch the conic Σ at the Jacobian-points of the given involution of circumscribing triangles. Furthermore, these four Jacobian-tangents each cut S in collinear triads of points apolar to U, and hence the four Jacobian-tangents each touch the same member Ψ of the Pippian-Quippian pencil.

It will be once again evident that Σ is a direct generalisation of the conic formed by H, K, two corresponding points on the Hessian. For the common tangents to the conic H, K and the cubic S are the six tangents drawn from H and K to S, whose points of contact lie three by three on four lines passing through K and H respectively, and all these

four lines are known to touch the same member of the syzygetic pencil of class-cubics, viz. the Cayleyan (Pippian).

If we combine the results obtained in the present article with those of § 5, we obtain the following properties:—

If a triangle be inscribed to S such that the tangents to S at its vertices are concurrent, and if the conic Σ be described touching the sides of this triangle at their third points of intersection with S, the points of contact with S of the twelve common tangents of S and Σ lie three by three on four lines, each touching Σ and the same member Ψ of the Pippian-Quippian pencil.

7. *Conditions for the Porism.*

We have already obtained in several forms the necessary and sufficient conditions that the conic Σ must satisfy in order that corresponding pairs of triangles can be found, circumscribed to Σ and having the nine points of intersection of their sides lying on the cubic S. The following are the two most important forms which these conditions assume:—

(I) A triangle is taken with its vertices on the cubic S, and such that the tangents to S at the vertices are concurrent. Σ touches the sides of this triangle at the points where they cut the cubic again.

(II) l is any line and l' is its satellite with respect to the cubic S. Σ touches l, l' and the tangents to S at its points of intersection with l.

We proceed now to express these conditions in terms of apolarity.

If Σ satisfies the conditions (II), we have seen that Σ is apolar to that member U of the Hessian-pencil to which the three points of intersection of l and S constitute a collinear apolar triad. Conversely:

The necessary and sufficient conditions that a conic Σ must satisfy in order that corresponding pairs of triangles can be found circumscribing Σ and having their nine points of intersection on a cubic S are: (1) *that Σ shall be apolar to a member U of the Hessian-pencil of S;* (2) *that Σ shall touch a line l and its satellite l', where l cuts S in a collinear triad of points apolar to U.*

We wish to show that these conditions are independent and therefore uniquely define a conic Σ. Consider the pencil of class-conics apolar to

U and touching l. Let Fig. 4 represent the four lines touched by all the members of the pencil. Then X, X'; Y, Y'; Z, Z' are the three de-

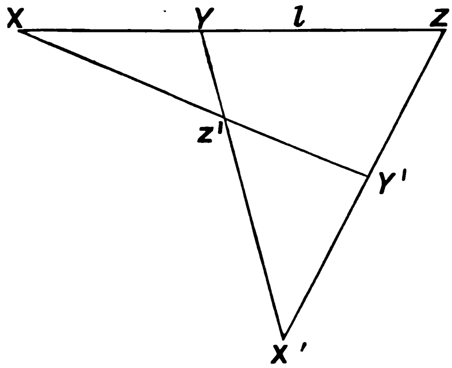

FIG. 4.

generate members of the pencil, and hence the points X, X', Y, Y', Z, Z' lie on the Hessian of U. Now the satellite l' of l is not identical with any of the four lines of Fig. 4; for, if possible, let it be identical with $Y'Z'$ and hence pass through X. But we have seen that X lies on the Hessian of U, which is a curve of the form $S+\rho H = 0$, H being the Hessian of S. Hence, on the above supposition, l must intersect its satellite l' in a point lying on the curve $S+\rho H$. But Cayley has proved in "A Memoir on Curves of the Third Order" (*Philosophical Transactions of the Royal Society of London*, Vol. 147 (1857), pp. 415–446) that if a line l intersects its satellite-line l' on the curve $S+\rho H$, l must envelop the curve $F+\rho P^2 = 0$, where F is the tangential equation to S and P is the tangential equation to the Cayleyan (Pippian). But l satisfies no condition except that of intersecting S in a collinear triad of points apolar to U, and hence touching a definite member Ψ of the Pippian-Quippian pencil. l' is therefore not identical in general with any one of the four lines of Fig. 4, and hence a unique conic Σ can be found apolar to U and touching the lines l and l'. Consider next the conic Σ' touching l, l' and the tangents to S at its points of intersection with l. It has already been proved that Σ' is apolar to U as well as touching l and l'. Hence the conics Σ and Σ' are identical. But by (II) of this article, Σ' possesses the property that pairs of triangles can be circumscribed to Σ' whose points of intersection lie on S. Hence Σ being identical with Σ' satisfies the necessary and sufficient conditions for the porism.

The following is an immediate corollary to (II) above, and is an ex-

tension of a particular case established by Dr. W. L. Marr in connection with the nodal cubic (*Proc. Edin. Math. Soc.*, Vol. 37, Session 1918–19, p. 72):

All the members of the pencil of class-conics touching the line joining three collinear points of inflexion on a cubic curve, and the tangents at these points of inflexion, possess the property that corresponding pairs of triangles can be found circumscribing the conic and having their nine points of intersection lying on the cubic, since in this case the line and its satellite are coincident.

Many particular theorems as to the nature of the intersections of a cubic and a conic touching a line of flexes, and the corresponding inflexional tangents, follow at once from the general properties investigated in this paper.

8. *Critical Centres of the Pencil* $\lambda_1 S_1 + \lambda_2 S_2 = 0$.

Salmon in his *Higher Plane Curves* has discussed the "critical centres" (*i.e.* the nodal points) of the pencil of cubics through the nine points of intersection of the sides of two triangles, and Cayley has studied exhaustively the case when two of the sides of one triangle become coincident in a paper in Vol. 11 (1864) of the *Transactions of the Cambridge Philosophical Society*. The foregoing theory leads naturally to the case when the sides of the two triangles touch the same conic Σ. Let the Jacobian-tetrad of the involution of triads defined by the points of contact of the sides of the triangles S_1 and S_2 with Σ be denoted by J_1, J_2, J_3, J_4, and let j_1, j_2, j_3, j_4 be the respective tangents to Σ at these points. Let J_{12} denote the point of intersection of j_1 and j_2, and so on. Consider the cubic $S \equiv \lambda_1 S_1 + \lambda_2 S_2$ which pass through the point J_{12}. Then, by § 6, the line j_1 meets S in three points, the tangents from which to Σ intersect S in two coincident points at each of the intersections of j_1 with S. Hence j_2 meets S in two coincident points at J_{12}. Similarly, j_1 meets S in two coincident points at J_{12}. Hence J_{12} is a node of the curve S. We therefore have the following result:—

If two triangles are circumscribed to a conic, the nodal cubics through the nine points of intersection of the sides of these triangles have their nodes at the meeting-points of the tangents to the conic at the Jacobian-tetrad of points relative to the involution of triads defined by the two triads of points at which the sides of the two triangles touch the conic.

In other words, if two triangles are circumscribed to the same conic, their "critical centres" lie three by three on four lines that touch the conic.

NOTE I.—*The nodal tangents at J_{12} harmonically separate j_1 and j_2.*

NOTE II.—*Let Σ, Σ' be respectively the conics inscribed, circumscribed to the above two triangles, and let Σ_0 be the conic to which they are each self-conjugate. Let the lines j_1, j_2, j_3, j_4 reciprocate with respect to Σ_0 into J'_1, J'_2, J'_3, J'_4 respectively. Then the common polar-lines, with respect to the given triangles, of the "critical centres" J_{12}, J_{34}, J_{13}, J_{42}, J_{14}, J_{23} are respectively the lines $J'_3J'_4$, $J'_1J'_2$, $J'_4J'_2$, $J'_1J'_3$, $J'_2J'_3$, $J'_1J'_4$.*

AN EXAMPLE OF A THOROUGHLY DIVERGENT ORTHOGONAL DEVELOPMENT

By H. STEINHAUS.

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No instance of a thoroughly divergent orthogonal development has yet been given. A simple example will be constructed in this note. In other words, an orthogonal, normalised and complete sequence of functions $\{\phi_n(x)\}$, integrable (L), together with their squares, in (a, b), will be defined, and a suitable integrable function $f(x)$ found, whose Fourier-like development

$$\sum_{i=1}^{\infty} \phi_i(x) \int_a^b f(t)\, \phi_i(t)\, dt \tag{1}$$

is divergent for every value of x in (a, b).

I. *The sequence* $\{\phi_n(x)\}$. Put for the sake of simplicity $a = 0$, $b = 1$. Let

$$\psi_1(x) = 1 \quad (0 \leqslant x \leqslant 1).$$

To define $\psi_2(x)$ we choose, among all curves passing through the points $(0, 2)$ and $(\frac{1}{2}, 2)$ of the (x, y) plane, situated above the line $y = \frac{1}{2}$, and symmetrical about the line $x = \frac{1}{4}$, a curve $y = \psi_2(x)$ fulfilling the condition

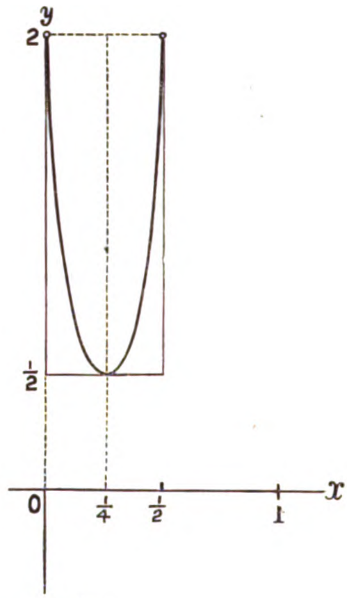
$$\int_0^{\frac{1}{2}} \psi_2^2(t)\, dt = \tfrac{1}{2}. \tag{2}$$

This choice is possible; in fact, when the curve $y = \psi_2(x)$ approaches the rectilinear form shown in the figure, the limit of the integral (2) is

$$\int_0^{\frac{1}{2}} (\tfrac{1}{2})^2\, dt = \tfrac{1}{8};$$

while when it approaches the straight line $y = 2$ the limit in question is

$$\int_0^{\frac{1}{2}} 2^2 dt = 2\,;$$



$y = \psi_2(x)$

so there must be an intermediate shape for which we have exactly the equation (2).

Let us define $\psi_2(x)$ for $\frac{1}{2} < x \leqslant 1$ by

(3) $$\psi_2(x) = -\psi_2(x - \tfrac{1}{2}) \quad (0 < x \leqslant \tfrac{1}{2}).$$

The equations

(4) $$\int_0^1 \psi_1^2(t)\,dt = 1, \quad \int_0^1 \psi_2^2(t)\,dt = 1, \quad \int_0^1 \psi_1(t)\,\psi_2(t)\,dt = 0$$

are immediate consequences of (1), (2), (3) and the symmetry of $\psi_2(x)$ about the line $x = \frac{1}{4}$. It can be also immediately seen that the maximum values of $|\psi_1(x)|$, $|\psi_2(x)|$ in (0, 1) are respectively 1 and 2, the minimum values of both functions being $\frac{1}{2}$.

We can repeat the same constructions in the case of $\psi_n(x)$. We divide (0, 1) into 2^{n-1} equal parts; and draw a curve $y = \psi_n(x)$ above the line $y = \frac{1}{2}$, passing through the points $(0, n)$, $(2^{-(n-1)}, n)$, giving the value $2^{-(n-1)}$ to the integral

$$\int_0^{2^{-(n-1)}} \psi_n^2(t)\,dt,$$

and symmetrical about the line $x = 2^{-n}$. We define $\psi_n(x)$ in the intervals $(2^{-(n-1)}, 2.2^{-(n-1)})$, $(2.2^{-(n-1)}, 3.2^{-(n-1)})$, ... by

$$\psi_n(x) = -\psi_n\left(x - \frac{1}{2^{n-1}}\right) \quad \left(\frac{1}{2^{n-1}} < x \leqq \frac{2}{2^{n-1}}\right),$$

$$\psi_n(x) = -\psi_n\left(x - \frac{1}{2^{n-1}}\right) \quad \left(\frac{2}{2^{n-1}} < x \leqq \frac{3}{2^{n-1}}\right),$$

and so forth.

The functions thus defined have the following properties

$$\int_0^1 \psi_n^2(t)\,dt = 1, \tag{5}$$

$$\int_0^1 \psi_i(t)\,\psi_k(t)\,dt = 0 \quad (i \neq k), \tag{6}$$

$$\underset{0 \leqq x \leqq 1}{\text{Max}}\ |\psi_n(x)| = n, \tag{7}$$

$$\underset{0 \leqq x \leqq 1}{\text{Min}}\ |\psi_n(x)| = \tfrac{1}{2}. \tag{8}$$

As to (7), we have to emphasize that the essential maximum value of $|\psi_n(x)|$ is n, *i.e.* that, for every $\epsilon > 0$, $|\psi_n(x)| \geqq n - \epsilon$ in a set of points x of positive measure. Thus (7) involves, by a well known theorem of Lebesgue,* the existence of an integrable function $f(x)$ with the property

$$\limsup_{n \to \infty} \int_0^1 f(t)\,\psi_n(t)\,dt = +\infty. \tag{9}$$

The sequence $\{\psi_n(x)\}$ being normalised and orthogonal, according to (5) and (6), a *complete* (or closed) sequence $\{\phi_n(x)\}$ can be found including all functions $\psi_n(x)$.

It obviously follows from (9) that

$$\limsup_{i \to \infty} \quad f(t)\,\phi_i(t)\,dt = +\infty, \tag{10}$$

and, as we have $\qquad \phi_{i_n}(x) = \psi_n(x),$

* H. Lebesgue, *Annales de Toulouse*, Ser. 3, Vol. 1 (1909), pp. 25–117: quoted (without proof) by Mr. Burton Camp in "Lebesgue Integrals containing a Parameter, with Applications," *Transactions of the American Mathematical Society*, Vol. 15, pp. 87–106. The theorem applied here is slightly different, and was proved by Mr. S. Banach [cf. H. Steinhaus, "Additive und stetige Funktionaloperationen," *Mathematische Zeitschrift*, Bd. 5, Heft 3/4 (1919), p. 219, Hilfssatz 4].

for suitable indices $i_1, i_2, \ldots, i_n, \ldots,$ we shall have, by (8),

$$|\phi_{i_n}(x)| \geqslant \tfrac{1}{2} \tag{11}$$

for all n and all x in (0, 1). But (10) and (11) imply

$$\limsup_{i \to \infty} \phi_i(x) \int_0^1 f(t)\, \phi_i(t)\, dt = +\infty \tag{12}$$

for every x in (0, 1), which shows immediately the divergence of (1) throughout the interval (0, 1). The argument can be applied to give an effective construction of the function $f(t)$.

APPROXIMATE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

By R. H FOWLER *and* C. N. H. LOCK.

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I. *Introduction.*

1. The asymptotic expansions of the solutions of linear differential equations for large values of a parameter have been discussed by various writers.* The general theory of such asymptotic forms has been practically completed by Schlesinger and Birkhoff (*loc. cit.*) for real values of the independent variable and complex values of the parameter, when the equation or system of equations is *homogeneous*. Except, however, for one very special case† the solutions of *non-homogeneous* linear differential equations have not been considered. In general therefore the behaviour of the complementary function is known, but the particular integral has not been studied.

In some recent investigations on the motion of a spinning projectile,‡ we have had occasion to make use of this asymptotic theory in forming approximate solutions of the system of differential equations which governs the motion. It was necessary to extend the theory to cover the case of the particular integral which in such an investigation is of not less importance than the complementary function.

Our primary object in this paper is therefore to complete this theory by extending it to include asymptotic expansions of particular integrals in the general case. This is done in Part III, and could be effected by regarding as known Schlesinger's or Birkhoff's results for the complementary function. It appears. however, that the whole problem of determining

* (1) J. Horn, *Math. Ann.*, Vol. 52 (1899), p. 271; (2) J. Horn, *Math. Ann.*, Vol. 52 (1899), p. 340; (3) L. Schlesinger, *Math. Ann.*, Vol. 63 (1907), p. 277; (4) G. D. Birkhoff, *Trans. Amer. Math. Soc.*, Vol. 9 (1908), p. 219.

† J. Horn, *loc. cit.* (2), p. 352.

‡ "The aerodynamics of a spinning shell," *Phil. Trans.*, A, Vol. 221 (1920), p. 295.

asymptotic forms for any solutions of linear equations really falls into two distinct parts. The first part is the determination of the dominant terms only in any *one* convenient fundamental set of solutions forming the complementary function. The second part is the use of this set of functions to establish the asymptotic expansions of any proposed solution of the system of equations, whether homogeneous or not. The methods of Schlesinger and Birkhoff do not seem to preserve this important distinction. Since we can also make slight extensions of these results, including the removal of the restriction to real values of the independent variable, we have ventured to include a discussion of the first part of the problem, before proceeding to obtain particular integrals. This forms Part II of the paper.

2. We note in passing two further points. One is the connection between the equations discussed here and *equations with nearly constant coefficients*. It is not difficult to see that, by suitable choice of the parameter, equations of the latter type can be put in a form to which the results of this paper are applicable. The asymptotic expansions, or at any rate their leading terms, may then form valuable approximate solutions of the equations with nearly constant coefficients. This has proved to be the case with the equations of motion of a spinning shell mentioned above.

The value of these expressions as approximate solutions will depend of course on the accuracy of the upper limit which can be assigned to the error term. This introduces our second point. The proper determination of the error term appears to be difficult,* but the need for its determination should be borne in mind at each step so that no unnecessary loss of accuracy occurs. For this reason it appears to be best to make explicit use of the adjoint system of equations.

3. *A summary of known results.*—Schlesinger (*loc. cit.*) has considered the system of differential equations

$$(1) \qquad y_i' = \sum_{\lambda=1}^{n} a_{\lambda i} y_\lambda \quad (i = 1, 2, \ldots, n),$$

where the coefficients $a_{\lambda i}$ are functions of x and μ expressible in series in

* See, for example, E. Cotton, (1) *Acta Math.*, Vol. 31 (1908), p. 107; (2) *Comptes Rendus*, Vol. 146 (1908), p. 274; Vol. 150 (1910), p. 511.

the form

$$(2) \qquad a_{\lambda i} = \mu\left({}_0a_{\lambda i} + \frac{{}_1a_{\lambda i}}{\mu} + \frac{{}_2a_{\lambda i}}{\mu^2} + \dots\right).$$

These series (2) are convergent when $|\mu| > R$ and $a \leqslant x \leqslant b$. The coefficients ${}_pa_{\lambda i}$ are functions of x only and possess continuous differential coefficients of all orders when x is in (a, b). He shows that the asymptotic form of the solutions is controlled by the determinantal equation

$$(3) \qquad \begin{vmatrix} {}_0a_{11} - \varpi & {}_0a_{21} & {}_0a_{31} & \dots & {}_0a_{n1} \\ {}_0a_{12} & {}_0a_{22} - \varpi & {}_0a_{32} & \dots & {}_0a_{n2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ {}_0a_{1n} & {}_0a_{2n} & {}_0a_{3n} & \dots & {}_0a_{nn} - \varpi \end{vmatrix} = 0.$$

This equation of the n-th degree in ϖ has n roots which are in general continuous functions of x, and essentially distinct when x is in (a, b).

In such a case he shows that the equations (1) can be linearly transformed into the system

$$(4) \qquad z_i' = \mu\omega_i z_i + \sum_{\lambda=1}^{n} b_{\lambda i} z_\lambda \quad (i = 1, 2, \dots, n),$$

where the coefficients $b_{\lambda i}$ are functions of x and μ of the form (2), except that

$$_0b_{\lambda i} = 0 \quad (\lambda, i = 1, 2, \dots, n),$$

so that

$$b_{\lambda i} = O(1),$$

as $|\mu| \to \infty$; the coefficients ω_i are the n distinct* functions of x which are the roots of equation (3). He shows, moreover, that n distinct solutions of the equations (4), and so of the equations (1), forming a fundamental set, can be expressed in the asymptotic form y_{ri}, where

$$(5) \qquad y_{ri} = e^{\mu\int_a^x \omega_r dx}\left({}_0u_{ri} + \frac{{}_1u_{ri}}{\mu} + \frac{{}_2u_{ri}}{\mu^2} \dots\right),\dagger$$

* *I.e.* it is never true that $\omega_i = \omega_j$ $(i \neq j)$.

† Owing to the large number of suffixes required, it is desirable to use a consistent notation. In any such expression the first suffix (usually r or s) specifies the particular solution out of the set of n, and the second (i or j) specifies the particular function of the n functions that form one solution. Thus y_{ri} is always one of the values of z_i.

provided $|\mu| \to \infty$ in such a way that am (μ) is constant, and*

$$\mathbf{R}(\mu\omega_1) > \mathbf{R}(\mu\omega_2) > \dots > \mathbf{R}(\mu\omega_n). \tag{6}$$

The coefficients $_0u_{ri}$ are functions of x alone which are easily determined according to given rules.

Birkhoff (*loc. cit.*) works with the differential equation

$$\frac{d^n z}{dx^n} + \mu a_{n-1} \frac{d^{n-1} z}{dx^{n-1}} + \dots + \mu^n a_0 z = 0, \tag{7}$$

where the a's are functions of x and μ of the form (2) but bounded as $|\mu| \to \infty$. He obtains solutions of the form (5), and establishes their asymptotic character under wider conditions than Schlesinger. He works, in fact, with a region S of the μ-plane, which is specified as the most general region of the plane such that the functions ω_i are distinct, the expansions (2) converge, and the following relations are satisfied, namely

$$\mathbf{R}(\mu\omega_1) \geqslant \mathbf{R}(\mu\omega_2) \geqslant \dots \geqslant \mathbf{R}(\mu\omega_n) \tag{8}$$

for all values of x in (a, b) and all values of μ in S.† Equations (1) or (4) are however somewhat more general than (7).

4. *Results of the present paper.*—It is necessary for applications to work with the wider conditions (8), for only these include the case in which ω_i is a pure imaginary and μ is real, which case is of the greatest practical importance. It is also more convenient in practice to discuss the equations in Schlesinger's form. We shall consider only the general case in which the functions ω_i are distinct, and shall therefore suppose that the system has already been reduced to the form (4), or rather to the form

$$z_i' = \mu\omega_i z_i + \sum_{\lambda=1}^{n} b_{\lambda i} z_\lambda + f_i e^{\mu\Omega} \quad (i = 1, 2, \dots, n), \tag{9}$$

where

$$f_i = {}_0f_i + \frac{{}_1f_i}{\mu} + \frac{{}_2f_i}{\mu^2} + \dots. \tag{10}$$

The series for f_i are convergent when μ is in S and x in (a, b). The coefficients $_pf_i$, Ω and $_pb_{\lambda i}$ are functions of x only, possessing differential

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* $\mathbf{R}(x)$ denotes the real part of x; am (x) the amplitude of x.

† It may be necessary to assume that a limited number of other inequalities of the form $|\mu| \geqslant R_i$ are satisfied in S.

coefficients of all orders in (a, b). Though we suppose in general that the independent variable x is real, we impose no further conditions of reality on any functions or coefficients.

In order to solve the first part of the problem in the most satisfactory way it is convenient (when necessary) to push the reduction one stage further and present the system of equations in the form

$$v_i' = (\mu\omega_i + b_{ii})\, v_i + \frac{1}{\mu} \sum_{\lambda=1}^{n} c_{\lambda i} v_\lambda \quad (i = 1, 2, \ldots, n), \tag{11}$$

where the b_{ii} are functions of x only, and

$$c_{\lambda i} = O(1) \quad (\lambda, i = 1, 2, \ldots, n),$$

when $|\mu| \to \infty$ in S and x is in (a, b). It is an easy extension of Schlesinger's work to show that the necessary linear transformation of the equations (1) can be carried out to put them into the form (11), provided that $\omega_i \neq \omega_j$ $(i \neq j)$, in (a, b) and $|\mu| \geqslant R_1$. The transformation can be carried still further if desired. It is important to observe, however, that this transformation is only required in the first part of the problem, for the construction of a special set of solutions.

The asymptotic expansions of solutions in general can be formed with or without previous transformation as may be desired.

In Part II, therefore, we construct a particular set of solutions of the equations (11), and the corresponding set of solutions of the equations adjoint to (11), namely*

$$v_i' = -(\mu\omega_i + b_{ii})\, v_i - \frac{1}{\mu} \sum_{\lambda=1}^{n} c_{i\lambda} v_\lambda \quad (i = 1, 2, \ldots, n). \tag{12}$$

We call all these solutions *the standard approximating set.* We conclude this part by showing how this standard set may be used to establish the asymptotic character of any set of solutions of equations (1), (4) or (9), whether obtained by Schlesinger's, Birkhoff's, or any other method of formation, and point out slight extensions of their results.

In Part III we obtain particular integrals of the equations (9) in the cases in which $\Omega' \neq \omega_i$ for any value of i or any x in (a, b), and $\Omega' \equiv \omega_j$ respectively. These cases correspond to the cases of forced oscillations of a vibrating system without and with resonance respectively.

* See, *e.g.* Goursat, *Cours d'Analyse*, Vol. 2, p. 481.

II. *The standard approximating set of solutions.*

5. *Particular integrals and the adjoint equations.*—We may suppose that a complete set of solutions of the equations (11) has been determined in some manner, and consists of the n^2 functions

$$g_{ri};$$

the meaning of the suffixes is defined in a footnote to § 3. We construct the determinant

$$\Delta = \begin{vmatrix} g_{11} & g_{21} & g_{31} & \cdots & g_{n1} \\ g_{12} & g_{22} & g_{32} & \cdots & g_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ g_{1n} & g_{2n} & g_{3n} & \cdots & g_{nn} \end{vmatrix};$$

then we know that

$$\Delta = \Delta(a) \exp \left\{ \int_a^x \left[\sum_1^n \left(\mu\omega_i + b_{ii} + \frac{c_{ii}}{\mu} \right) dx \right\} ; \tag{13}$$

where $\Delta(a)$ is independent of x. Let ΔG_{ri} be the co-factor of g_{ri} in Δ. Then a particular integral of the equations

$$v_i' = (\mu\omega_i + b_{ii}) v_i + \frac{1}{\mu} \sum_{\lambda=1}^{n} c_{\lambda i} v_\lambda + E_i \quad (i = 1, 2, \ldots, n), \tag{14}$$

may be expressed in the form

$$v_i = \sum_{j=1}^{n} \left\{ g_{1i} \int_a^x E_j G_{1j} dx + g_{2i} \int_a^x E_j G_{2j} dx + \ldots + g_{ni} \int_a^x E_j G_{nj} dx \right\} \tag{15}$$

$$(i = 1, 2, \ldots, n),$$

or

$$v_i = \sum_{r, j=1}^{n} g_{ri} \int_a^x E_j G_{rj} \, dx.$$

Any number of the integrals $\int_a^x$ may be replaced when required by the corresponding integrals $\int_b^x$. The n^2 functions G_{ri} form a set of solutions of the adjoint system of equations (12) whose initial values are determined by those of the functions g_{ri}.

If now we can determine suitable sets of functions g_{ri} and G_{ri}, with the proper dominant terms, we shall be able to use the solution (15) to establish the asymptotic character of any solution of the equations (4) and (9). We proceed to show in the following sections that we can

specify a standard set of solutions g_{ri} and G_{ri} such that

$$g_{ri} = e^{\mu \int_a^x \omega_r dx} \gamma_{ri}, \tag{16}$$

$$G_{ri} = e^{-\mu \int_a^x \omega_r dx} \Gamma_{ri}, \tag{17}$$

$$|\gamma_{ri}| \leqq m_{ri}, \tag{18}$$

$$|\Gamma_{ri}| \leqq M_{ri}, \tag{19}$$

where m_{ri} and M_{ri} are continuous functions of x independent of μ; a definite upper limit can thus be assigned to m_{ri} and M_{ri} when μ is in S and x in (a, b). These functions then will serve as the standard approximating set, and their construction constitutes the first part of the problem as distinguished in § 1.

6. *The standard set of solutions* g_{ri}.—The functions γ_{1i} form one solution of the equations

$$\left\{\begin{aligned} \zeta_1' &= b_{11}\zeta_1 + \frac{1}{\mu}\sum_{\lambda=1}^{n} c_{\lambda 1}\zeta_\lambda, \\ \zeta_i' &= [\mu(\omega_i-\omega_1)+b_{ii}]\zeta_i + \frac{1}{\mu}\sum_{\lambda=1}^{n} c_{\lambda i}\zeta_\lambda \quad (i = 2, 3, \ldots, n), \end{aligned}\right. \tag{20}$$

obtained from (11) by an obvious transformation. The solution of (20) with the initial conditions $\zeta_1 = 1$, $\zeta_i = 0$ $(i = 2, 3, \ldots, n)$, at $x = a$, which may be constructed by Picard's method,* has the properties required and will be taken to be γ_{1i}. For the equations (20) may be rewritten in the form

$$\frac{d}{dx}(\beta_1\zeta_1) = \frac{\beta_1}{\mu}\sum_{\lambda=1}^{n} c_{\lambda 1}\zeta_\lambda,$$

$$\frac{d}{dx}(a_i\beta_i\zeta_i) = \frac{a_i\beta_i}{\mu}\sum_{\lambda=1}^{n} c_{\lambda i}\zeta_\lambda \quad (i = 2, 3, \ldots, n),$$

where

$$a_i = e^{\mu \int_a^x (\omega_1-\omega_i)dx},$$

$$\beta_i = e^{-\mu \int_a^x b_{ii} dx}.$$

* The method of successive approximation. See, *e.g.*, Goursat, *loc. cit.*, p. 365.

The solution of these equations which we require is determined as the limits, as $p \to \infty$, of the sequence of functions $(\zeta_i)_p$, where

$$(21)\quad \begin{cases} (\zeta_1)_0 = 1, \\ (\zeta_i)_0 = 0 \quad (i = 2, 3, \ldots, n); \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ \beta_1(\zeta_1)_{p+1} = 1 + \dfrac{1}{\mu}\displaystyle\int_a^x \beta_1 \left(\sum_{\lambda=1}^n c_{\lambda 1}(\zeta_\lambda)_p\right) dx, \\ a_i\beta_i(\zeta_i)_{p+1} = \dfrac{1}{\mu}\displaystyle\int_a^x a_i\beta_i \left(\sum_{\lambda=1}^n c_{\lambda i}(\zeta_\lambda)_p\right) dx \quad (i = 2, 3, \ldots, n); \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{cases}$$

The limits required are known to exist, and to be unique. Moreover, by conditions (8),

$$\mathbf{R}\,\{\mu(\omega_1 - \omega_i)\} > 0 \quad (i = 2, 3, \ldots, n),$$

and therefore $|a_i|$ is an increasing function of x. Therefore

$$(22)\quad |\beta_i(\zeta_i)_{p+1}| \leqslant \frac{1}{|\mu|}\int_a^x \left|\beta_i\left(\sum_{\lambda=1}^n c_{\lambda i}(\zeta_\lambda)_p\right)\right| dx \quad (i = 2, 3, \ldots, n);$$

On referring to equation (11), we see that when μ is in S and x in (a, b), there exist inequalities of the form

$$(23)\quad |\beta_i c_{\lambda i}| \leqslant c^*_{\lambda i},$$

where $c^*_{\lambda i}$ are continuous functions of x independent of μ. We can therefore form a series of dominant functions $(Z_i)_p$ by the sequence of equations

$$\begin{gathered} (Z_1)_0 = 1, \\ (Z_i)_0 = 0 \quad (i = 2, 3, \ldots, n); \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ |\beta_1|(Z_1)_{p+1} = 1 + \frac{1}{|\mu|}\int_a^x \left(\sum_{\lambda=1}^n c^*_{\lambda 1}(Z_\lambda)_p\right) dx, \\ |\beta_i|(Z_i)_{p+1} = \frac{1}{|\mu|}\int_a^x \left(\sum_{\lambda=1}^n c^*_{\lambda i}(Z_\lambda)_p\right) dx \quad (i = 2, 3, \ldots, n); \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{gathered}$$

From the form of these equations, these sequences converge to limit functions which are less than functions m_{1i} depending on x, but independent of μ, when μ is in S; from the inequalities (22), (23) it follows that

$$|\gamma_{1i}| \leqslant m_{1i},$$

as we require. For the purpose of the actual determination of reasonable values of m_{1i}, it is important to notice* that the limit functions to which $(Z_i)_p$ converge as $p \to \infty$ are precisely the solution of the equations

$$\frac{d}{dx}(|\beta_i|Z_i) = \frac{1}{|\mu|}\sum_{\lambda=1}^{n} c^{*}_{\lambda i}Z_\lambda \quad (i = 1, 2, \ldots, n), \tag{24}$$

with the initial conditions $Z_1 = 1$, $Z_i = 0$ $(i = 2, 3, \ldots, n)$, at $x = a$.

We cannot in general obtain the second and succeeding integrals in just this manner, for in general $\mathbf{R}\{\mu(\omega_s - \omega_1)\} < 0$, and the arguments fail. But the difficulty is turned by replacing the range of integration (a, x) by (b, x), where it becomes necessary.† Consider the s-th solution $e^{\mu\int_a^x \omega_s dx}\gamma_{si}$, for which the functions γ_{si} form a solution of the equations

$$\frac{d}{dx}(\beta_s\zeta_s) = \frac{\beta_s}{\mu}\sum_{\lambda=1}^{n} c_{\lambda s}\zeta_\lambda,$$

$$\frac{d}{dx}(a_i\beta_i\zeta_i) = \frac{a_i\beta_i}{\mu}\sum_{\lambda=1}^{n} c_{\lambda i}\zeta_\lambda \quad (i \neq s),$$

where β_i has the same meaning as before, and a_i now denotes $e^{\mu\int_a^x(\omega_s-\omega_i)\,dx}$. We can attempt to construct a solution of the required form, with the terminal conditions

$$\begin{cases} \gamma_{si}(b) = 0 & (i < s), \\ \gamma_{ss}(a) = 1, \\ \gamma_{si}(a) = 0 & (i > s), \end{cases} \tag{25}$$

by Picard's sequence of operations

$$(\zeta_s)_0 = 1,$$

$$(\zeta_i)_0 = 0 \quad (i \neq s);$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$a_i\beta_i(\zeta_i)_{p+1} = -\frac{1}{\mu}\int_x^b a_i\beta_i\left(\sum_{\lambda=1}^{n} c_{\lambda i}(\zeta_\lambda)_p\right)dx \quad (i < s),$$

$$\beta_s(\zeta_s)_{p+1} = 1 + \frac{1}{\mu}\int_a^x \beta_s\left(\sum_{\lambda=1}^{n} c_{\lambda s}(\zeta_\lambda)_p\right)dx,$$

$$a_i\beta_i(\zeta_i)_{p+1} = \frac{1}{\mu}\int_a^x a_i\beta_i\left(\sum_{\lambda=1}^{n} c_{\lambda i}(\zeta_\lambda)_p\right)dx \quad (i > s);$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

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* See Cotton, *loc. cit.* (1).

† This is essentially Birkhoff's device, *loc. cit.*

Now $|a_i|$ or $\exp\left\{\mathbf{R}\left(\mu\int_a^x(\omega_s-\omega_i)\,dx\right)\right\}$ is an increasing function of x as before when $i>s$, and a decreasing function of x when $i<s$. These constructional equations therefore provide the inequalities

$$|\beta_i(\zeta_i)_{p+1}| \leqslant \frac{1}{|\mu|}\int_x^b\left(\sum_{\lambda=1}^n c_{\lambda i}^{\bullet}\,|(\zeta_\lambda)_p|\right)dx \quad (i<s),$$

$$|\beta_i(\zeta_i)_{p+1}| \leqslant \frac{1}{|\mu|}\int_a^x\left(\sum_{\lambda=1}^n c_{\lambda i}^{\bullet}\,|(\zeta_\lambda)_p|\right)dx \quad (i>s).$$

The ordinary arguments must be slightly modified to establish the existence of limit functions for the sequences $(\zeta_i)_p$. As before we construct sequences of dominant functions $(Z_i)_p$ by the equations

$$(Z_s)_0 = 1,$$

$$(Z_i)_0 = 0 \quad (i \neq s);$$

$$\dots \qquad \dots \qquad \dots$$

$$|\beta_i|(Z_i)_{p+1} = \frac{1}{|\mu|}\int_x^b\left(\sum_{\lambda=1}^n c_{\lambda i}^{\bullet}(Z_\lambda)_p\right)dx \quad (i<s),$$

$$|\beta_s|(Z_s)_{p+1} = 1+\frac{1}{|\mu|}\int_a^x\left(\sum_{\lambda=1}^n c_{\lambda s}^{\bullet}(Z_\lambda)_p\right)dx,$$

$$|\beta\ (Z_s)_{p+1} = \frac{1}{|\mu|}\int_a^x\left(\sum_{\lambda=1}^n c_{\lambda i}^{\bullet}(Z_\lambda)_p\right)dx \quad (i>s);$$

$$\dots \qquad \dots \qquad \dots \qquad \dots \qquad \dots \qquad \dots$$

It is clear that $|(\zeta_i)_p| \leqslant (Z_i)_p$, and it can be shown that limit functions of the sequences $(Z_i)_p$ exist as $p\to\infty$, provided $|\mu|$ is sufficiently large.* We may suppose that the necessary condition is satisfied when μ is in S.

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* If we write $(U_i)_{p+1} = (Z_i)_{p+1}-(Z_i)_p$, and suppose $c_{\lambda i}^{\bullet} \leqslant M$, a constant, then

$$|\beta_i|(U_i)_{p+1} \leqslant \frac{M}{|\mu|}\int_a^b\left(\sum_{\lambda=1}^n (U_\lambda)_p\right)dx \quad (i=1,2,\dots,n),$$

from which, by induction, $\sum_{i=1}^n (U_i)_{p+1} \leqslant \left(\frac{nN(b-a)}{|\mu|}\right)^{p+1}$,

where N is the maximum value of $M/|\beta_i|$. The convergence of the sequences follows at once. The sequences only converge for sufficiently large values of μ because two different upper limits a and b occur in the defining integrals instead of the usual one. This is of course equivalent to the fact that the analogous integral equations are of Fredholm's, not Volterra's type.

The set of limit functions forms a solution of the system of equations

$$(26)\qquad \begin{cases} \dfrac{d}{dx}(|\beta_i|Z_i) = -\dfrac{1}{|\mu|}\sum\limits_{\lambda=1}^{n} c^{*}_{\lambda i} Z_\lambda & (i < s), \\ \dfrac{d}{dx}(|\beta_i|Z_i) = \dfrac{1}{|\mu|}\sum\limits_{\lambda=1}^{n} c^{*}_{\lambda i} Z_\lambda & (i \geqq s), \end{cases}$$

with the terminal conditions (25), which can be shown by the arguments of the next section to be the unique solution with these terminal conditions. Equation (26) provides the most satisfactory method of determining upper limits for the limit functions of $(Z_i)_p$.

It is easily shown that these upper limits m_{si} can be chosen so as to be independent of μ. Finally, the existence of these limit functions of $(Z_i)_p$ implies the existence of the limit functions γ_{si}, of $(\zeta_i)_p$, which are a solution of the required equations with the terminal conditions (25), and satisfy the inequalities (18).

The n solutions g_{ri} of equations (11) constructed in this manner with the help of (17) form the required standard set.

7. *The independence and uniqueness of the functions* g_{ri}.—These n solutions g_{ri} are independent, for their discriminant Δ has the value unity at $x = 0$, and therefore never vanishes. We have, in fact,*

$$(27)\qquad \Delta(a) = \begin{vmatrix} 1 & g_{21}(a) & g_{31}(a) & \dots & g_{n1}(a) \\ 0 & 1 & g_{32}(a) & \dots & g_{n2}(a) \\ 0 & 0 & 1 & \dots & g_{n3}(a) \\ \dots & \dots & \dots & & \dots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix} = 1.$$

It is necessary for the next argument to show that the s-th solution g_{si} which satisfies the terminal conditions (25) is unique—this fact does not follow at once from standard theorems. We have only to show that there is no solution of equation (11) other than zero, satisfying the terminal conditions

$$v_i(b) = 0 \quad (i < s),$$

$$v_i(a) = 0 \quad (i \geqq s).$$

Since $\Delta \neq 0$, any solution can be put in the form

$$v_i = A_1 g_{1i} + A_2 g_{2i} + \dots + A_n g_{ni} \quad (i = 1, 2, \dots, n),$$

* Owing to the form of (17), $g_{ri}(a) = \gamma_{ri}(a)$ for all values of r and i.

where the A's are constants. The terminal conditions give first of all

$$v_1(b) = A_1 g_{11}(b) + A_2 g_{21}(b) + \ldots + A_n g_{n1}(b) = 0.$$

But $g_{r1}(b) = 0$ when $r > 1$; also, since $\Delta(b) \neq 0$, $g_{11}(b) \neq 0$. Therefore $A_1 = 0$, and similarly the other conditions ($i < s$) give in order

$$A_2 = A_3 = \ldots = A_{s-1} = 0.$$

For the remaining $n-s+1$ equations we start at the other end, and by similar arguments show that the remaining constants

$$A_n = A_{n-1} = \ldots = A_s = 0.$$

The solution g_{si} is therefore unique.

8. *The standard set of solutions G_{ri}.*—The functions G_{ri}, when multiplied by Δ, are the co-factors of g_{ri} in the determinant Δ itself. The terminal values of these functions can now be specified. On referring to (27) we see that

$$\left\{\begin{aligned} G_{ii}(a) &= 1, \\ G_{ri}(a) &= 0 \quad (r > i). \end{aligned}\right. \tag{28}$$

If further we substitute $x = b$ in Δ, we find that, since

$$g_{ri}(b) = 0 \quad (r > i),$$

$$G_{ri}(b) = 0 \quad (r < i). \tag{29}$$

The functions G_{ri} therefore satisfy the adjoint differential equations (12) and the terminal conditions (28) and (29). They are necessarily unique. But the terminal conditions (28) and (29) are precisely the terminal conditions (25) of the functions g_{ri} with a and b interchanged when $r \neq i$, and the adjoint equations (12) are precisely analogous to (11) with the signs of all the functions ω_i changed. It follows that the precise method of solution, by which the functions g_{ri} and their dominant functions m_{ri} were obtained, may be applied at once to the equations (12) to obtain the functions G_{ri} and their dominant functions M_{ri}. It is only necessary to replace $\int_a^x$ by $\int_b^x$, or *vice versa*, wherever it occurs in the equations of formation, except in the principal term containing 1 where $\int_a^x$ stands unaltered.

9. *The error term in an approximate solution or asymptotic expansion.*—Suppose now that by any means whatever we construct an approximate

solution or asymptotic expansion of a solution of (in the first instance) the equations (11) or (14) of the form

$$v_i = q_i \quad (i = 1, 2, \ldots, n). \tag{30}$$

If we make the substitution $v_i = q_i + \zeta_i$, the functions ζ_i must be a solution of equations of the form

$$\zeta_i' = (\mu\omega_i + b_{ii})\zeta_i + \frac{1}{\mu}\sum_{\lambda=1}^{n} c_{\lambda i}\zeta_\lambda + E_i e^{\mu\Omega} \quad (i = 1, 2, \ldots, n), \tag{30$'$}$$

of which a particular solution is given by the formula (15), with $e^{\mu\Omega}$ inserted in each integrand. If we select in a suitable manner the range of integration* (a, x) or (b, x) for each term we can obtain in every case, from (16)–(19) inequalities of the form

$$|\zeta_i| \leqslant \left\{ \sum_{r, j=1}^{n} m_{ri} \int |E_j| M_{rj}\, dx \right\} e^{\mathbf{R}(\mu\Omega)}. \tag{31}$$

The inequalities (31) give an upper limit as required for $|\zeta_i|$ which represents the error in the approximate solution. More precisely the reasoning shows that there is *a solution* of the original equations represented by the approximation (30) with an error for which an upper limit is assigned by (31).

Now consider the problem of determining approximate solutions or asymptotic expansions for systems of equations in more general untransformed forms such as (1), (4), or (9); for definiteness consider the nonhomogeneous form of (1), or

$$y_i' = \sum_{\lambda=1}^{n} a_{\lambda i} y_\lambda + f_i e^{\mu\Omega} \quad (i = 1, 2, \ldots, n). \tag{32}$$

It may be convenient to start by transforming this equation to the form (14), and then to construct asymptotic expansions, in which case the preceding arguments apply unaltered. In general, however, it may be more convenient to construct the asymptotic expansions for the original or for some otherwise modified form of (32). If then we denote a definite number of terms of the expansions by q_i as before, and a solution of (32)

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* The range of integration must be (a, x) if

$$\mathbf{R}\{\mu(\Omega' - \omega_r)\} \geqslant 0,$$

and (x, b) in the contrary case. We then reapply the arguments of § 6 and obtain the relation (31). It is necessary to assume, if Ω' is not one of the ω_r, that $\mathbf{R}\{\mu(\Omega' - \omega_r)\}$ does not change sign when μ is in S and x in (a, b).

by $q_i+\zeta_i$, we shall in general find that ζ_i satisfies equations of the form

$$(33) \qquad \zeta_i' = \sum_{\lambda=1}^{n} a_{\lambda i}\zeta_\lambda + E_i,$$

where E_i is a function of x and μ satisfying the relation

$$E_i = O\,|\,\mu^{-n}e^{\mu\Omega}\,|\,.$$

In order to establish the asymptotic nature of the solutions obtained, it is then only necessary to suppose that these ζ-equations are transformed and the preceding arguments applied. It is to be remembered that Schlesinger's transformation of the equations (1) or (32) to the forms (4) or (9) does not involve μ at all. Consequently the order of the error terms in μ is unaltered in the transformed equations. Further, it is easily verified that the extra transformation step from (4) or (9) to (11) or (30)′, though depending on μ, cannot increase the order of the error terms.

From the point of view of theory this method is unexceptionable, and the standard set of solutions which we have constructed for the transformed equations enables the asymptotic nature of any such expansion to be established at once. From the practical point of view, there is the disadvantage that the transformation of the ζ-equations must still be actually carried out. This or some equivalent labour seems however to be unavoidable by any method.

The asymptotic nature of all the expansions obtained by Horn, Schlesinger, and Birkhoff, may be at once established by the arguments suggested, whatever the precise form of construction.

10. *Extensions.*—The foregoing discussion provides as it stands a slight extension of previous results, but the extension of Birkhoff's results which is from equation (7) to equations (1) is really only trivial. The form of the foregoing discussion enables us however to extend the results in certain cases to complex values of the independent variable.

We suppose, in the first place, that all the functions of x that occur in the equations are holomorphic within and on the boundary of a region X of the x-plane, and that a is a point on the boundary of X. We suppose further that the roots of the discriminant (3) are all distinct within and on the boundary of X, so that the functions ω_i are holomorphic in this region. It follows that $\int_a^x \omega_r\,dx$ taken along any path γ in X from a to x is one-valued and a holomorphic function of x in X. The arguments of § 5 hold unaltered. The fundamental point in the argument is the deriva-

tion of the inequality (22) from the equations of formation (21). All the integrals concerned are independent of the path γ in X. We must be able to find one particular path γ from a to x inside X along which we can apply the arguments of § 6. It is essential for the argument that on γ

$$\mathbf{R}\left\{\mu\int_a^x(\omega_1-\omega_i)\,dx\right\} \geqslant \mathbf{R}\left\{\mu\int_a^\xi(\omega_1-\omega_i)\,dx\right\} \quad (i=2,3,\ldots,n),$$

where ξ is any point on γ between a and x. We require therefore along γ

$$\mathbf{R}\left\{\mu\int_\xi^x(\omega_1-\omega_i)\,dx\right\} \geqslant 0,$$

which is equivalent to

$$\mathbf{R}\left(\mu\omega_1\frac{dx}{dt}\right) \geqslant \mathbf{R}\left(\mu\omega_i\frac{dx}{dt}\right) \quad (i=2,3,\ldots,n)$$

on the curve γ, where t is a parameter defining γ which increases steadily as the representative point goes from a to x. If these conditions hold we can construct the solution γ_{1i} with the required properties. For the construction of the remaining solutions of the set we require the point b on the boundary of X and corresponding inequalities along curves from b to x in X. It is not difficult to see that the complete conditions required to replace (8) may be stated as follows.

For the construction of the standard approximating set of solutions of equations (11) *in a region X of the x-plane when μ is in S, it may be assumed that there exist points a and b on the boundary of X such that through any point x of X a curve γ can be drawn from a to b inside X, and that at every point of γ*

$$(34) \qquad \mathbf{R}\left(\mu\omega_1\frac{dx}{dt}\right) \geqslant \mathbf{R}\left(\mu\omega_2\frac{dx}{dt}\right) \geqslant \ldots \geqslant \mathbf{R}\left(\mu\omega_n\frac{dx}{dt}\right),$$

where t is a parameter defining γ.

It is of course assumed in addition that the coefficients are all holomorphic and the ω's all distinct in X, and that the series for the coefficients all converge when x is in X and μ in S.

Under these conditions the argument proceeds unaltered to the end of § 8; to obtain the inequalities (31) for a solution of the equations (30)′, we have only to suppose further that Ω is holomorphic in X, and that $\mathbf{R}\left(\mu\Omega'\frac{dx}{dt}\right)$ must fit into some permanent position in the scheme (34) for all values of x in X and μ in S, that is to say, for all the curves γ we

must have

$$\mathbf{R}\left(\mu\omega_r \frac{dx}{dt}\right) > \mathbf{R}\left(\mu\Omega' \frac{dx}{dt}\right) > \mathbf{R}\left(\mu\omega_{r+1} \frac{dx}{dt}\right) \tag{35}$$

for one and the same value of r. Under conditions (34) and (35) the whole argument extends to complex values of x. We may state these results in the following

THEOREM I.—*If the conditions* (34) *and* (35) *are satisfied when x is in X and μ in S, the asymptotic expansions of the solutions of equations* (1) *or* (9), *as established for real x can be established for all values of x in X.*

III. *The asymptotic forms of particular integrals of the equations* (9).

11. *Forced oscillations without resonance.*—We propose to determine a particular integral of the system of equations

$$\begin{cases} z_1' = \mu\omega_1 z_1 + \sum\limits_{\lambda=1}^{n} b_{\lambda 1} z_\lambda + f e^{\mu\Omega}, \\ z_i' = \mu\omega_i z_i + \sum\limits_{\lambda=1}^{n} b_{\lambda i} z_\lambda \quad (i = 2, 3, \ldots, n), \end{cases} \tag{36}$$

confining ourselves to real values of x; the function f has the form specified in (10), and may be put in the form

$$f = f_0 + \frac{f_1}{\mu} + \ldots + \frac{f_p}{\mu^p} + \frac{F}{\mu^{p+1}}, \tag{37}$$

where $$|F| < E,$$

and E is a function of x independent of μ when μ is in S and x in (a, b). The function Ω also depends only on x. Since integrals of the equations are additive, we may, for simplicity of exposition, consider only the case in which the "disturbing function" f_i is zero in all the equations except one, which may with loss of generality be taken as above to be the first equation. We attempt to find a particular integral of the form

$$z_i = e^{\mu\Omega}\left({}_0u_i + \frac{{}_1u_i}{\mu} + \frac{{}_2u_i}{\mu^2} + \ldots\right) \quad (i = 1, 2, \ldots, n), \tag{38}$$

where the coefficients are functions of x independent of μ. If we substi-

tute formally for z_i from equations (38) in equations (36), we obtain

$$
\begin{aligned}
{}_0u'_i+\frac{{}_1u'_i}{\mu}+\frac{{}_2u'_i}{\mu^2}+\ldots+\mu(\Omega'-\omega_i)\left({}_0u_i+\frac{{}_1u_i}{\mu}+\frac{{}_2u_i}{\mu^2}+\ldots\right)& \\
=\sum_{\lambda=1}^{n}\left({}_1b_{\lambda i}+\frac{{}_2b_{\lambda i}}{\mu}+\frac{{}_3b_{\lambda i}}{\mu^2}+\ldots\right)&\left({}_0u_\lambda+\frac{{}_1u_\lambda}{\mu}+\frac{{}_2u_\lambda}{\mu^2}+\ldots\right) \\
+\left[f_0+\frac{f_1}{\mu}+\frac{f_2}{\mu^2}+\ldots\right]\quad (i&=1,2,\ldots,n),
\end{aligned}
$$

the terms [ ] occurring only when $i=1$. If we equate powers of μ in each equation, we obtain in succession if $\Omega'-\omega_i$ never vanishes for any i or any x in (a, b),

$$
(39)\qquad \begin{cases}
\qquad {}_0u_i=0 \quad (i=1,2,\ldots,n); \\
(\Omega'-\omega_1)\,{}_1u_1=f_0, \\
\qquad {}_1u_i=0 \quad (i=2,3,\ldots,n); \\
(\Omega'-\omega_1)\,{}_2u_1=-{}_1u'_1+{}_1b_{11}({}_1u_1)+f_1, \\
(\Omega'-\omega_i)\,{}_2u_i={}_1b_{1i}({}_1u_1) \quad (i=2,3,\ldots,n); \\
(\Omega'-\omega_1)\,{}_3u_1=-{}_2u'_1+\sum_{\lambda=1}^{n}{}_1b_{\lambda 1}({}_2u_\lambda)+{}_2b_{11}({}_1u_1)+f_2, \\
(\Omega'-\omega_i)\,{}_3u_i=-{}_2u'_i+\sum_{\lambda=1}^{n}{}_1b_{\lambda i}({}_2u_\lambda)+{}_2b_{1i}({}_1u_1) \quad (i=2,3,\ldots,n); \\
\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots
\end{cases}
$$

and so on. The law of formation of the successive coefficients by this method is sufficiently obvious; it is clear that each coefficient is continuous and determined uniquely in terms of the preceding ones: the first effective coefficient ${}_1u_1$ is fixed by the value of f_0.

If we construct in this way the first $p+1$ terms of the solution, we can write

$$
(40)\qquad z_i=e^{\mu\Omega}\left(\frac{{}_1u_i}{\mu}+\frac{{}_2u_i}{\mu^2}+\ldots+\frac{{}_{p+1}u_i}{\mu^{p+1}}\right)+\zeta_i \quad (i=1,2,\ldots,n).
$$

The coefficients ${}_pu_i$ are known functions of x determined by the laws of formation (39). If we form the equations of type (33) satisfied by ζ_i it follows from the method of formation of the coefficients in (40) that the error terms E_i satisfy inequalities of the form

$$
(41)\qquad |E_i|\leqslant\frac{H(x)\,e^{\mathbf{R}(\mu\Omega)}}{|\mu|^{p+1}},
$$

where $H(x)$ is independent of μ when μ is in S and x in (a, b). It follows at once by the arguments of § 9 that there exists a particular integral of the equations (36) of the form (40), where the error terms ζ_i are such that $\zeta_i = O\{|\mu|^{-p-1} e^{\mathbf{R}(\mu\Omega)}\}$ as $|\mu| \to \infty$, provided that no one of $\mathbf{R}\{\mu(\Omega'-\omega_i\}$ ever changes sign in the region considered. We have therefore proved the following

THEOREM II.—*If μ is in S and x in (a, b), $\Omega'-\omega_i \neq 0$, and $\mathbf{R}\{\mu(\Omega'-\omega_i)\}$ never changes sign for any value of i or x, there exists a particular integral of the equations* (36) *whose asymptotic expansion as $|\mu| \to \infty$ in S takes the form*

$$z_i = e^{\mu\Omega}\left\{\frac{{}_1u_i}{\mu}+\frac{{}_2u_i}{\mu^2}+\ldots+\frac{{}_pu_i}{\mu^p}+O\left(\frac{1}{|\mu|^{p+1}}\right)\right\} \quad (i = 1, 2, \ldots, n), \tag{42}$$

where the coefficients ${}_pu_i$ are determined successively by the laws of formation (39).

12. *Forced oscillations with resonance.*—In this case the function Ω' is permanently equal to one of the ω_i. The solutions take slightly different forms according as to whether the Ω' of the disturbing function exciting resonance agrees with the ω_i of the equation in which it occurs or with the ω_i of some other equation. These two cases may be referred to as *direct resonance* and *cross resonance* respectively.

As a typical case of direct resonance we may consider the equations

$$\begin{cases} z_1' = \mu\omega_1 z_1 + \sum\limits_{\lambda=1}^{n} b_{\lambda 1} z_\lambda + f e^{\mu\int_a^x \omega_1\,dx}, \\ z_i' = \mu\omega_i z_i + \sum\limits_{\lambda=1}^{n} b_{\lambda i} z_\lambda \quad (i = 2, 3, \ldots, n). \end{cases} \tag{43}$$

We assume a solution of the usual form

$$z_i = e^{\mu\int_a^x \omega_1\,dx}\left({}_0u_i+\frac{{}_1u_i}{\mu}+\frac{{}_2u_i}{\mu^2}+\ldots\right) \quad (i = 1, 2, \ldots, n). \tag{44}$$

Substitute formally for z_i in (43) and remove the exponential factor. We obtain

$${}_0u_1'+\frac{{}_1u_1'}{\mu}+\frac{{}_2u_1'}{\mu^2}+\ldots = \sum_{\lambda=1}^{n}\left({}_1b_{\lambda 1}+\frac{{}_2b_{\lambda 1}}{\mu}+\frac{{}_3b_{\lambda 1}}{\mu^2}+\ldots\right)\left({}_0u_\lambda+\frac{{}_1u_\lambda}{\mu}+\frac{{}_2u_\lambda}{\mu^2}+\ldots\right)$$
$$+f_0+\frac{f_1}{\mu}+\frac{f_2}{\mu^2}+\ldots,$$

$$\mu(\omega_1-\omega_i)\left({}_0u_i+\frac{{}_1u_i}{\mu}+\frac{{}_2u_i}{\mu^2}+\ldots\right)+{}_0u_i'+\frac{{}_1u_i'}{\mu}+\frac{{}_2u_i'}{\mu^2}+\ldots$$
$$=\sum_{\lambda=1}^{n}\left({}_1b_{\lambda i}+\frac{{}_2b_{\lambda i}}{\mu}+\frac{{}_3b_{\lambda i}}{\mu^2}+\ldots\right)\left({}_0u_\lambda+\frac{{}_1u_\lambda}{\mu}+\frac{{}_2u_\lambda}{\mu^2}+\ldots\right)\quad (i=2, 3, \ldots, n),$$

in which we equate coefficients of powers of μ. We obtain the laws of formation

$$(45)\qquad \begin{cases} {}_0u_i = 0 \quad (i=2, 3, \ldots, n); \\ {}_0u_i' - {}_1b_{11}({}_0u_1) = f_0, \end{cases}$$

which latter integrates and gives

$$(45)'\qquad \begin{cases} {}_0u_1 = e^{\int_a^x {}_1b_{11}\,dx}\int_a^x f_0 e^{-\int_a^x {}_1b_{11}\,dx}\,dx; \\ (\omega_1-\omega_i)\,{}_1u_i = {}_1b_{1i}({}_0u_1) \quad (i=2, 3, \ldots, n), \\ {}_1u_1'-{}_1b_{11}({}_1u_1) = f_1+{}_2b_{11}({}_0u_1)+\sum_{\lambda=2}^{n}{}_1b_{\lambda 1}({}_1u_\lambda), \end{cases}$$

which latter integrates and gives ${}_1u_1$ in the same form as ${}_0u_1$;

$$(45)''\qquad \begin{cases} (\omega_1-\omega_i)\,{}_2u_i = -{}_1u_i'+{}_2b_{1i}({}_0u_1)+\sum_{\lambda=1}^{n}{}_1b_{\lambda i}({}_1u_\lambda) \quad (i=2, 3, \ldots, n), \\ {}_2u_1'-{}_1b_{11}({}_2u_1) = f_2+{}_3b_{11}({}_0u_1)+\sum_{\lambda=1}^{n}{}_2b_{\lambda 1}({}_1u_\lambda)+\sum_{\lambda=2}^{n}{}_1b_{\lambda 1}({}_2u_\lambda), \end{cases}$$

which integrates as before, and so on. At each stage the necessary coefficient of the form ${}_pu_1$ is determined as the solution of a simple differential equation of the form $y'-{}_1b_{11}y = Y$, where Y is a known function of x. To make the law precise we may specify (as above) that at each stage we take that solution which vanishes* for $x = a$. The remaining coefficients of the form ${}_pu_i$ $(i > 1)$ are determined by simple linear (algebraic) equations as in the former case without resonance.

As a typical case of cross resonance we may consider the equations

$$(46)\qquad \begin{cases} z_i' = \mu\omega_i z_i+\sum_{\lambda=1}^{n} b_{\lambda i}z_\lambda \quad (i \neq 2), \\ z_2' = \mu\omega_2 z_2+\sum_{\lambda=1}^{n} b_{\lambda 2}z_\lambda+fe^{\mu\int_a^x \omega_1\,dx}. \end{cases}$$

* Or, has any other assigned value.

In this case the laws of formation of the coefficients take the form

$$(47)\quad \begin{cases} {}_0u_i = 0 \quad (i = 1, 2, \dots, n); \\ (\omega_1 - \omega_2)\, {}_1u_2 = f_0, \\ {}_1u_i = 0 \quad (i = 3, 4, \dots, n), \\ {}_1u_1' - {}_1b_{11}({}_1u_1) = {}_1b_{21}({}_1u_2), \end{cases}$$

leading to

$$(47)'\quad \begin{cases} {}_1u_1 = e^{\int_a^x {}_1b_{11}\,dx} \int_a^x {}_1b_{21}\,({}_1u_2)\, e^{-\int_a^x {}_1b_{11}\,dx}\, dx\,; \\ (\omega_1 - \omega_2)\, {}_2u_2 = f_1 - {}_1u_2' + \sum\limits_{\lambda=1,\,2} {}_1b_{\lambda 2}({}_1u_\lambda), \\ (\omega_1 - \omega_i)\, {}_2u_i = \sum\limits_{\lambda=1,\,2} {}_1b_{\lambda i}({}_1u_\lambda) \quad (i = 3, 4, \dots, n), \\ {}_2u_1' - {}_1b_{11}({}_2u_i) = \sum\limits_{\lambda=2}^{n} {}_1b_{\lambda 1}({}_2u_\lambda) + \sum\limits_{\lambda=1,\,2} {}_2b_{\lambda 1}({}_1u_\lambda), \end{cases}$$

which integrates as before, and so on. In both cases, if we apply the arguments of § 9, we see that there is actually a particular integral of the equations (43) or (46) of the form (44), and so obtain the following

Theorem III.—*If* μ *is in* S *and* x *in* (a, b) *there exists a particular integral of the equations* (43) *or* (46) *whose asymptotic expansion as* $|\mu| \to \infty$ *in* S *takes the form*

$$(48)\qquad z_i = e^{\mu \int_a^x \omega_1\,dx} \left({}_0u_i + \frac{{}_1u_i}{\mu} + \frac{{}_2u_i}{\mu^2} + \dots + \frac{{}_\nu u_i}{u^\nu} + O\frac{1}{|\mu|^{\nu+1}} \right)$$

$$(i = 1, 2, \dots, n),$$

where the coefficients ${}_pu_i$ *are determined successively by the laws of formation* (45) *or* (47) *respectively.*

It is of interest to observe the difference between the solutions in the cases in which there is and is not resonance. When there is no resonance the principal term is

$$z_1 = \frac{1}{\mu}\, \frac{f_0}{\Omega' - \omega_1}\, e^{\mu\Omega},$$

all other terms being $O(|\mu|^{-2})$. When there is (direct) resonance, the principal term is

$$z_1 = e^{\int_a^x (\mu\omega_1 + {}_1b_{11})\,dx} \int_a^x f_0\, e^{-\int_a^x {}_1b_{11}\,dx}\, dx,$$

all other terms being $O(|\mu|^{-1})$. If we consider the case in which μ is a pure imaginary and all the functions of x are real, we see that in the former case z_1 remains bounded by the value of $f_0/(\Omega'-\omega_1)$ irrespective of the length of the interval (a, x); while, in the latter case, this is not necessarily so, for if ${}_1b_{11} = 0$, and f_0 is constant for example, then z_1 increases with x like $f_0(x-a)$.*

Under conditions (34) and (35), we can extend the domain of validity of the asymptotic expansions of our particular integrals to the region X in the complex x-plane.

* Compare the behaviour of the particular integrals of the equation $y'' + n^2y = A \cos pt$ when p is not and is equal to n.

TIDAL OSCILLATIONS IN GULFS AND RECTANGULAR BASINS

By G. I. Taylor.

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Introduction and Summary of Results.

In some recent work* on the dissipation of energy in the tides of the Irish Sea, use was made of the observed fact that the tidal streams in the South Channel of the Irish Sea move backwards and forwards in a straight line. The rotation of the earth causes the rise and fall of the tide to be four times as great on the Holyhead side of the Channel as it is on the Irish side, but apparently it does not give rise to any appreciable elliptic motion of the water particles as one might have expected.

In the work referred to above, it is shown that the tidal observations taken on both sides of the Channel and the observations of tidal streams taken at various points across it, are explicable if they are due to two tidal waves of the "Kelvin" type moving in opposite directions, up and down the Channel.

The Kelvin type of tidal wave is one which can be propagated in a rotating channel, or in a channel on the surface of the earth, if the dimensions of the system are such that it does not cover more than a small range of latitude. The motion of the particles of water is confined to one dimension, that of the length of the channel, the deflecting force, due to the rotation being counterbalanced by a horizontal pressure gradient which is due to the fact that the amplitude is greater on one side of the channel than on the other. In the case of a channel in the Northern Hemisphere this "deflecting force" necessitates a pressure gradient acting from right to left at the crest of the wave where the water particles are moving forward along the channel in the same direction as the wave. This pressure gradient is provided by an increase in the amplitude of the wave on the right-hand side of the channel.

In the case of the southern entrance of the Irish Sea observations

* "Tidal Friction in the Irish Sea," *Phil. Trans.*, (A), Vol. 220, p. 1.

point to the existence of two Kelvin waves, one inward and the other outward bound. It appears, therefore, that the Irish Sea acts as a kind of reflector which reflects back Kelvin waves of the same type as those entering. On the other hand, the reflection is evidently not of the ordinary type. The entering wave is greater on the shore which lies on the right of an observer facing inwards, while the outgoing wave is greater to his left. As Poincaré has pointed out* the Kelvin wave cannot be regularly reflected because, when two such waves, moving in opposite directions, are superposed, it is impossible to find any line across the channel such that there is no motion across it. It is impossible, therefore, to place a solid boundary across any part of the channel without affecting the motion.

From the theoretical point of view, considerable interest attaches to finding the mechanism by means of which the reflection of Kelvin waves can be brought about. From the practical standpoint the question is also important because the tides at the open ends of a large majority of gulfs, deep bays and partially enclosed seas appear to be similar in character to those at the entrance to the Irish Sea.

The outstanding feature of the Kelvin type of wave is the absence of any motion across the channel. In the case of a very narrow channel, the particles of water are constrained to move parallel to its walls. The Kelvin type of wave is, therefore, to be anticipated in this case. On the other hand, however wide the channel may be, it is possible for it to contain two Kelvin waves moving in opposite directions; and the question naturally arises whether a Kelvin wave is always reflected as a Kelvin wave, even when the channel is quite wide. In other words, will the parallel coasts of a deep gulf like the North Sea, for instance, force the tidal current to move parallel to the two parallel coasts at some distance from the closed end? If so, how far from the end does the effect of the end stretch? If not, what type of tide may be expected?

In the work which follows, these questions are investigated. The reflection of tidal waves at the rectangular end of a channel is expressed mathematically by means of an infinite series of complex terms.

The physical meaning of the expression is not at once obvious, but one result springs immediately from the form of the result, namely, that in a given channel, rotating at a given speed, a Kelvin wave is reflected completely at the closed end, provided its period is greater than a certain quantity. Kelvin waves of period less than this, however, cannot be reflected.

In the case of the principal semi-diurnal tidal waves on the earth, the

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* *Leçons de Mécanique Céleste*, t. 3 (Théorie des Marées), p. 124.

period and angular velocity are given. The result stated above is therefore equivalent to the statement that in channels of given depth and closed at one end, a semi-diurnal Kelvin type of wave can be reflected provided the channel is narrower than a certain width, but that if it is wider than this critical width, perfect reflection cannot take place.

As explained above, the physical significance of the result is obscured in a cloud of symbols. A particular case has therefore been worked out numerically.

The case chosen is that of a channel situated in latitude 53 whose width is 250 nautical miles and depth 40 fathoms. This corresponds, roughly, to the case of the North Sea, which is in fact nearly rectangular, though the water is shallower than 40 fathoms at the Southern end.

The results are exhibited by means of the two diagrams shown in Figs. 1 and 2. The first diagram (Fig. 1) represents the height of the tidal wave by means of cotidal lines which are drawn through the points where it is high water at any specified time. These lines are drawn for every hour (or rather for every $\frac{1}{12}$ part of a period), the successive times of high water being marked by figures round the edge of the diagram.

The amounts of rise and fall of tide in different parts of the basin are shown in Fig. 1 by means of dotted lines.

In the second diagram (Fig. 2) a series of ellipses with varying axes and orientations are drawn. The radius vectors of these represent the magnitude and direction of the velocity of the tidal stream at different states of the tide. They also represent, on a different scale of course, the actual paths of the particles of water.

An inspection of these two diagrams at once reveals the nature and mechanism of the reflection.

In the lower part of the basin at a distance greater than about 250 miles from the closed end, the cotidal lines and the motion of the particles correspond very nearly to two equal Kelvin waves moving up and down the channel. The tidal streams are very nearly parallel to the sides of the channel and the cotidal lines move in along the right hand shore (*i.e.* the left-hand side of the figure). The tidal wave then sweeps round the end wall of the basin at a rate rather greater than the velocity of the Kelvin wave, and moves back along the opposite shore to that along which it approached the end. In turning at right angles in order to cross the end of the channel, the wave produces a bigger rise and fall of tide at the two corners than anywhere else in the field. On the scale chosen the range of tide at the corners is represented by the number 1·95, whereas the greatest range in the distant parts of the channel, far from the influence of the end, is represented by 1·61.

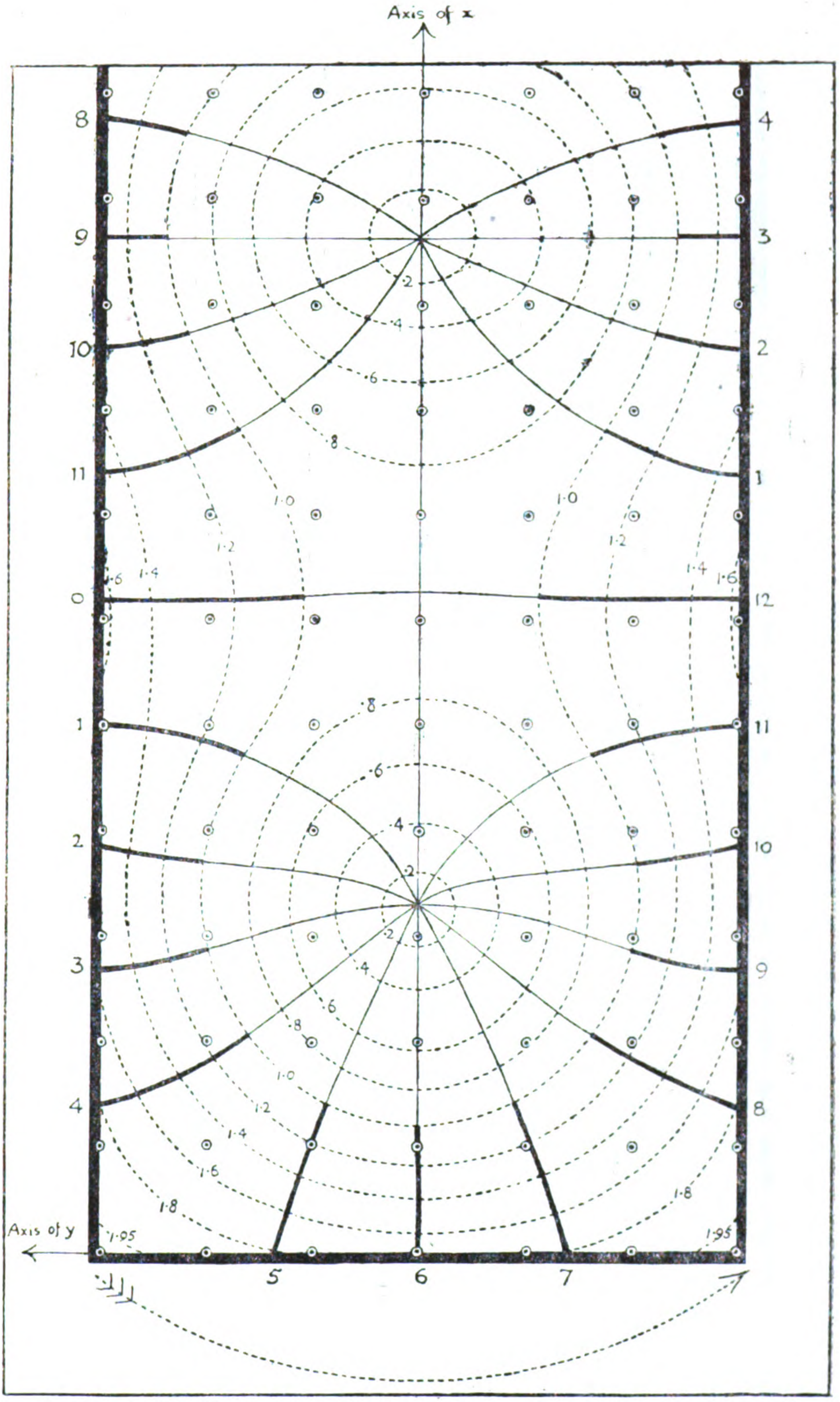

FIG. 1.—*Cotidal lines in basin where a Kelvin wave is being reflected.*—Full lines are cotidal lines. Figures outside the edge of the basin show time of high water on corresponding cotidal line. Dotted lines are lines of equal tidal range. Figures inside basin show amount of tidal range. Small circles with central dot show positions at which tidal motion was calculated. Curved arrow shows direction of rotation of system.

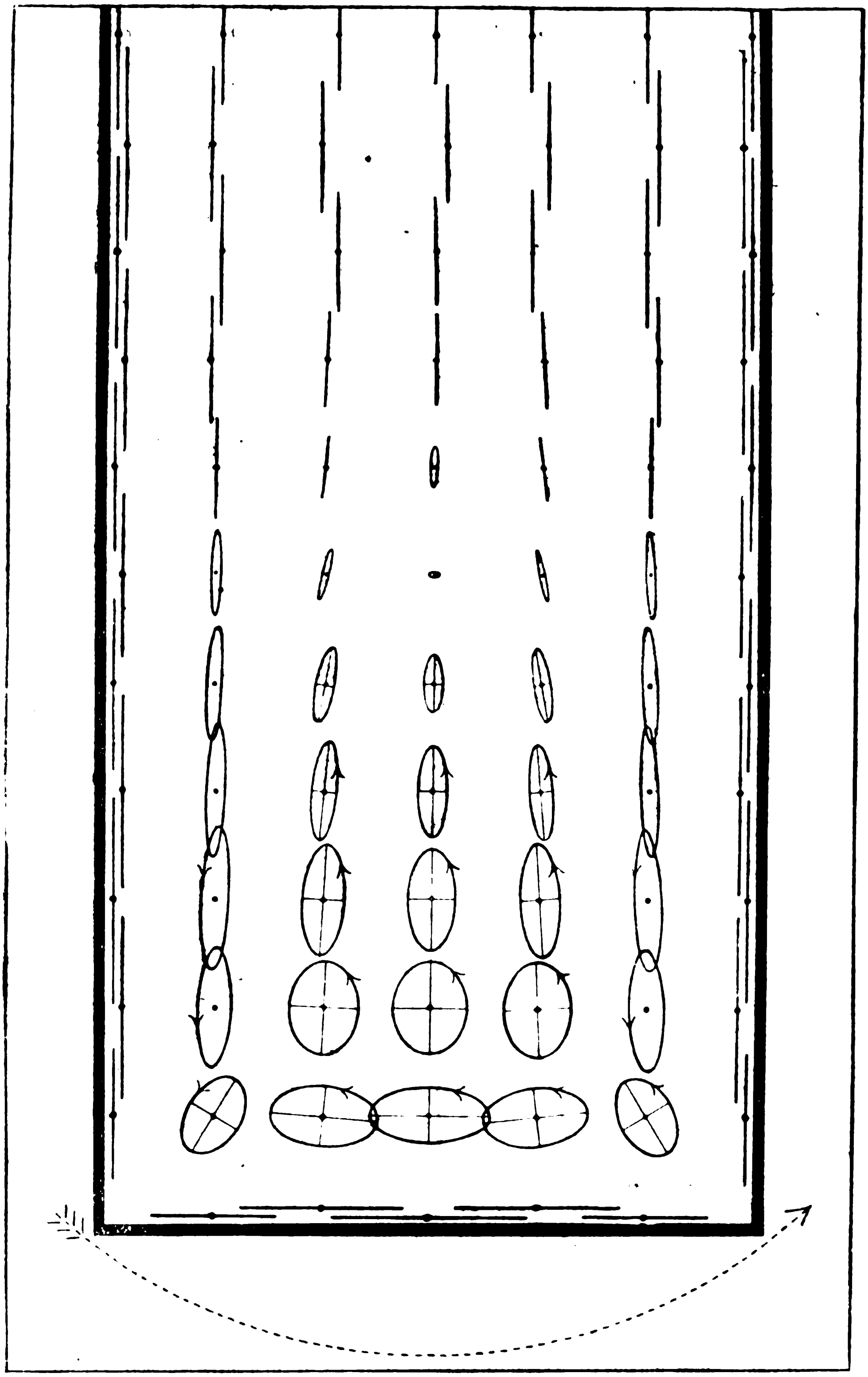

Fig. 2.—Tidal ellipses, showing motion of water in region where tidal wave is being reflected.

In order to show up more conspicuously the nature of the motion, the cotidal lines have been drawn in Fig. 1 with heavy lines in the region where the rise and fall is, on the scale chosen, greater than 1, that is to say in the parts where the range of tide is greater than half the maximum range at the corners. The way in which the strongly marked parts of the cotidal lines move down the left side of the diagram, cross the end and move up the right side, is conspicuous.

In the distant parts of the channel the tidal streams are parallel to the shores at all states of the tide. At distances from the end less than a distance about equal to the width, however, the particles of water move in ellipses, except, of course, close in shore, where they continue to move parallel to the shore. The direction in which the particles move round all the larger ellipses is the same as that of the rotation of the earth; but at some distance from the closed end the phase of the up-and-down-channel component of velocity changes, while that of the cross-channel component does not, so the direction of rotation of the particles in the ellipses must be reversed. The reversed elliptic paths are, however, small compared with the direct ones.

The maximum tidal currents occur at certain points close to the parallel shores, the greatest being at a distance from the end about equal to half the breadth of the channel; but the cross-channel current at the mid-point of the end is very nearly as great as this maximum.

The currents near the central part of the basin are considerably smaller than those close to the shore. At a distance of 80 miles from the end (equal to about one-third of the breadth of the channel), the cross-channel current has died down to considerably less than half its value at the end, and the paths of particles are nearly circular. Near the South end of the North Sea where the depths are all less than 40 fathoms the tidal streams might be expected to die down to half value in a much smaller distance than 80 miles, which is one-third the breadth of the North Sea.

Perhaps the most remarkable feature of the motion is the magnitude of the cross-channel currents a short distance from the closed end.

Case of Comparatively Narrow Channel.

These investigations were begun partly in order to find out why there is no appreciable elliptic motion in the waters of the South Channel to the Irish Sea, though the rotation of the earth produces such a large effect on the rise and fall of tide there. In order to elucidate this point the case of reflection of tidal waves in a narrow channel is worked out.

A formula is obtained for the maximum cross-channel current, and it is shown that in the case of the Irish Sea this could not be more than half a knot at the most. It is pointed out, moreover, that this occurs only at the end of the channel where the up-and-down-channel currents vanish. At the point where the up-and-down-channel current is a maximum, the cross-channel current is extremely small, being only a fraction equal to 5×10^{-8} of the main current. It appears, therefore, that the phenomena which occur at the entrance to the Irish Sea are what tidal theory would lead us to expect.

Tidal Oscillations in a Rectangular Basin.

The method developed for dealing with the reflection of tidal waves at the end of a closed channel can be applied to solve the problem of the tidal oscillations in a rectangular basin of uniform depth. This problem is of much less interest from the hydrographic point of view than the problem which has been solved, but it derives a certain interest from the fact that so few cases of tidal motion in limited basins have been solved. The only cases which appear to have been solved so far, are those of the circular* and nearly circular† basins, and the infinite channel.‡

Lord Rayleigh§ has solved the case of tidal motion in a rectangular basin when the period of rotation is large compared with the period of oscillation. This limitation introduces considerable simplification into the analysis, but unfortunately it also reduces its usefulness. In the first place fundamental changes in the types of oscillation which are possible are obscured. In the second place the periods of the tides with which we have to deal in nature are of the same order as the period of rotation of the earth. Lord Rayleigh's conclusions cannot, therefore, be applied to tides in the sea, though, as he remarks, they might apply to the tides in a comparatively small enclosed sheet of water, especially if it were situated near, but not on, the equator.

In 1909 Lord Rayleigh again returned to the subject‖ and published a method of approximating to the small change in period which results from a small rotation of the system. He found that the period is increased by the rotation, a result opposite to that applicable to a circular basin, but in agreement with the conclusions reached in the present paper.

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* Lamb, *Hydrodynamics*, 4th ed., p. 311.

† Proudman, *Proc. London Math. Soc.*, Ser. 2, Vol. 12 (1913), p. 453.

‡ Poincaré, *Leçons de Mécanique Céleste*, t. 3 (*Théorie des Marées*), p. 125; and Proudman, *l.c.* p. 419.

§ *Phil. Mag.*, Ser. 6, Vol. 5 (1903), p. 297.

‖ *Proc. Roy. Soc.*. (A), Vol. 82 (1909), pp. 448-464.

The physical reason suggested by the present analysis for this increase in period is that the oscillations consist of tidal waves which move round the basin. If the system were not rotating a wave travelling from one end of the basin to the other would be reflected from the end directly it got there. In the case of a rotating basin the period is increased because the tidal wave has to cross the end of the channel before it can be reflected back along the opposite side.

It is shown that two types of oscillation exist, one in which the elevation of the surface is symmetrical about the centre, and one in which it is anti-symmetrical, that is it is of the same magnitude but opposite in sign, at diametrically opposite points. These probably consist of systems in which odd and even numbers of tidal waves follow one another round the basin.

Reasons are given for believing that the number of possible periods is very much smaller than in the case of a non-rotating rectangular basin, that they form in fact a singly infinite series, while those of the non-rotating basin form a doubly infinite series. The physical reason for this difference appears to be connected with the way in which tidal waves are reflected by being deflected along the rectangular end of the basin. The same tidal waves therefore flow along the sides and the ends of the basin, so that the lengths of the waves travelling parallel to the sides must be closely related with the lengths of the waves travelling parallel to the ends.

In the case of a non-rotating system the wave lengths of the oscillations parallel to the sides and the ends are quite independent of one another.

Numerical Verification.

The method is subjected to a numerical test by calculating the slowest period of oscillation in the case of a basin whose length is twice its breadth, when the period of rotation of the system is equal to the slowest period of free oscillation of the basin in the absence of rotation.

It is found that in this case the slowest period is increased by the rotation in the ratio $1 : 1{\cdot}14$.

In order to test the conclusion reached in the course of the work, that all possible solutions can be obtained from a consideration of a pair of Kelvin waves moving parallel to one of the sides of the basin, the slowest period was calculated by two methods: (1) the original pair of Kelvin waves were taken as being parallel to the long sides, and (2) parallel to the short sides of the basin.

The period obtained by these two methods was exactly the same. It appears therefore that the two methods are equally available for representing the same set of oscillations.

A comparison is made between the oscillations of a rectangular basin and those of a circular basin.

REFLECTION OF TIDAL WAVES FROM THE CLOSED END OF A ROTATING CHANNEL WHICH IS INFINITE IN ONE DIRECTION.

Before entering on the details of the solution it is useful, for reference purposes, to give a list of the symbols which will be used.

x : the axis of x is the line mid-way between the sides of the channel.

y : the axis of y is perpendicular to the axis of x, and parallel to the end of the channel.

u and v : components of velocity parallel to x and y respectively.

ζ : height of the tide above the mean level.

n, angular velocity of rotation : in the case of a channel on the earth's surface this may be taken to be $\omega \sin \lambda$, where λ is the latitude, and ω the angular velocity of the Earth about its axis.

t : time.

$\sigma = 2\pi \div$ (period of tidal oscillation), so that u, v, and ζ may be taken as proportional to $e^{i\sigma t}$.

h : depth of water, supposed uniform.

g : acceleration due to gravity.

$c = \sqrt{(gh)}$: the velocity of a long wave in a non-rotating channel.

$a = 2n/c$.

$i = \sqrt{(-1)}$.

m : any positive integer.

$k^2 = (\sigma^2 - 4n^2)/c^2$.

$s_m = \sqrt{(k^2 - m^2)}$ when $m^2 < k^2$, and $s_m = \sqrt{(m^2 - k^2)}$ when $m^2 > k^2$.

$r_m = 2n\sigma / m s_m c^2$.

A_m, B_m, C_m, D_m : constants determined in the course of the analysis.

$\beta_1, \beta_3, \beta_5, \ldots, \gamma_2, \gamma_4, \gamma_6, \ldots$ are undetermined multipliers.

$z = \tan(\sigma x_1/c)$.

Assuming that u, v, and ζ are proportional to $e^{i\sigma t}$ the equations of motion and continuity of tides in a rotating basin of uniform depth* reduce to†

$$i\sigma u - 2nv = -g\frac{\partial\zeta}{\partial x}, \tag{1a}$$

$$i\sigma v + 2nu = -g\frac{\partial\zeta}{\partial y}, \tag{1b}$$

$$\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}+\frac{\sigma^2-4n^2}{c}\zeta = 0, \tag{1c}$$

or

$$(\nabla^2+k^2)\zeta = 0, \tag{1d}$$

where

$$k^2 = (\sigma^2-4n^2)/c^2.$$

It is evident that u and v also satisfy

$$(\nabla^2+k^2)\,u,\, v = 0.$$

It will be noticed that these equations are unaltered if ζ, x, y and c are all reduced in a constant ratio. No loss of generality will result therefore in taking the breadth of the channel to be equal to π. This introduces considerable simplification into the analysis. The dimensions of c then become the same as those of σ, namely t^{-1}, and the sides of the channel are the lines $y = \pm\frac{1}{2}\pi$.

The boundary conditions which have to be satisfied are that v shall vanish at $y = \pm\frac{1}{2}\pi$, and that u shall vanish at the closed end of the channel, which will be taken as the line $x = x_1$.

The principle on which the solution which follows is based is to add together a number of special solutions of equations (1) which all satisfy the first of the boundary conditions, namely, $v = 0$ at $y = \pm\frac{1}{2}\pi$, but which do not satisfy the second boundary condition at $x = x_1$. By making an appropriate combination of such special solutions, however, it is found possible to satisfy this last condition also.

If the channel were not rotating the motion would be found by superposing two equal wave trains moving up and down the channel respectively. At certain points along the channel there are planes perpendicular to its length across which there is no motion. At any one of these planes a barrier could be erected without affecting the motion on either side of it.

* In order that the depth may be uniform, the bottom must be of such a shape that it is parallel to the free surface of the rotating liquid.

† See Lamb, *Hydrodynamics*, Chap. VIII.

The motion can, therefore, be confined to one side of the barrier, and it consists of a wave train which moves up the channel and is reflected at the end.

A similar method can be adopted in the case of a rotating channel. Taking the incident wave as the Kelvin wave represented by*

$$u_1 = e^{(-2ny - i\sigma x)/c + i\sigma t},$$

$$v = 0,$$

the reflected wave may be taken as

$$u_2 = -e^{(2ny + i\sigma x)/c + i\sigma t},$$

$$v = 0.$$

On superposing these two waves it will be found that there is no value of x for which $u_1 + u_2 = 0$ for all values of y and t; though there are an infinite number of *points* at which this is the case, namely, those for which $y = 0$ and x is a multiple of $\pi c/\sigma$. It is impossible, therefore, to place a fixed barrier across the channel without altering the motion.

It will now be shown that there is a tidal motion satisfying the boundary condition $v = 0$ at $y = \pm \frac{1}{2}\pi$, which can be superposed on the two Kelvin waves so as to make $u = 0$ for all values of y at a certain value, x_1, of x. A barrier can therefore be erected across the channel at $x = x_1$ without affecting the motion on either side of it. The total motion therefore represents an incident and reflected wave train. It will appear in the course of the work that the superposed motion due to the barrier, vanishes at points far distant from the barrier, provided that the period is greater than a certain value.

On multiplying $u_1 + u_2$ by a constant quantity $\frac{1}{2}Si$, and dropping the factor $e^{i\sigma t}$, the motion due to the incident and reflected wave train may be represented by

$$u = \tfrac{1}{2}Si(u_1 + u_2) = S\{\cosh ay \sin(\sigma x/c) - i \sinh ay \cos(\sigma x/c)\}, \qquad (2)$$

$$v = 0,$$

where $$a = 2n/c.$$

The problem is to find another type of tidal motion satisfying $v = 0$ at $y = \pm \frac{1}{2}\pi$ and $u = \frac{1}{2}Si(u_1 + u_2)$ at $x = x_1$.

It may be seen from (1) that v satisfies the equation

$$(\nabla^2 + k^2)\, v = 0.$$

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* See Lamb, *loc. cit.*, Chap. VIII.

A particular solution of this equation is

$$v = e^{-s_m x + imy}, \tag{3}$$

provided

$$s_m^2 = m^2 - k^2. \tag{4}$$

Here m may have any value, and s_m may be regarded as being determined by (4), but, in order that v may vanish at $y = \pm \frac{1}{2}\pi$, m must have integral values.

Solutions of this type assume four different forms according as m is odd or even, and according as k^2 is greater or less than m^2.

Assume therefore for v the following forms, which vanish at $y = \pm \frac{1}{2}\pi$,

$$m^2 < k^2, \quad s_m^2 = k^2 - m^2, \quad \begin{cases} m \text{ even}, & v = D_m \cos s_m x \sin my, & (5a) \\ m \text{ odd}, & v = iC_m \sin s_m x \cos my, & (5b) \end{cases}$$

$$m^2 > k^2, \quad s_m^2 = m^2 - k^2, \quad \begin{cases} m \text{ even}, & v = D_m e^{-s_m x} \sin my, & (5c) \\ m \text{ odd}, & v = iC_m e^{-s_m x} \cos my, & (5d) \end{cases}$$

where now s_m is real in all cases.

For u assume the forms

$$m^2 < k^2, \quad u = A_m \sin s_m x \cos my + iB_m \cos s_m x \sin my, \tag{6a}$$

$$m^2 > k^2, \quad u = A_m e^{-s_m x} \cos my + iB_m e^{-s_m x} \sin my. \tag{6b}$$

The constants A_m and B_m must be chosen so that equations (1a) and (1b) are satisfied. Eliminating ζ between (1a) and (1b) it will be found that

$$i\sigma \frac{\partial u}{\partial y} - 2n \frac{\partial u}{\partial x} = i\sigma \frac{\partial v}{\partial x} + 2n \frac{\partial v}{\partial y}. \tag{7}$$

Substituting from (5) and (6) in (7), coefficients of $\cos my$ and $i \sin my$ on the two sides of the equation may be equated. The two resulting equations determine A_m and B_m in terms of C_m or D_m; or, if A_m and B_m be regarded as being given, the equations determine C_m or D_m in terms of them, and also establish a relationship which must exist between A_m and B_m in order that v may be of the assumed form.

If r_m be written for $2n\sigma/(ms_m c^2)$ these relationships are

$$m^2 < k^2, \quad \begin{cases} m \text{ even}, & A_m/B_m = -1/r_m, & (8a) \\ m \text{ odd}, & A_m/B_m = r_m, & (8b) \end{cases}$$

$$m^2 > k^2, \quad \begin{cases} m \text{ even}, & A_m/B_m = -1/r_m, & (8c) \\ m \text{ odd}, & A_m/B_m = -r_m. & (8d) \end{cases}$$

And C_m and D_m are then given by

$$m^2 < k^2, \quad \begin{cases} m \text{ even}, & D_m = \dfrac{m}{s_m} A_m - \dfrac{2n}{\sigma} B_m, & (9a) \\[2ex] m \text{ odd}, & C_m = \dfrac{2n}{\sigma} A_m + \dfrac{m}{s_m} B_m, & (9b) \end{cases}$$

$$m^2 > k^2, \quad \begin{cases} m \text{ even}, & D_m = \dfrac{m}{s_m} A_m - \dfrac{2n}{\sigma} B_m, & (9c) \\[2ex] m \text{ odd}, & C_m = \dfrac{2n}{\sigma} A_m - \dfrac{m}{s_m} B_m. & (9d) \end{cases}$$

It now remains to be seen whether it is possible to choose a series of values for A_m and B_m so that the value of u due to the sum of all the terms is equal to $\frac{1}{2}Si(u_1+u_2)$ for all values of y at some value x_1, of x.

The value of u for any value of x is expressed as a Fourier series in $\cos my$ and $\sin my$. It is necessary therefore to express $\frac{1}{2}Si(u_1+u_2)$ by means of a similar Fourier series.

To do this first write down the Trigonometrical Series expressing $\cosh ay$ and $\sinh ay$ between the limits $y = \pm\frac{1}{2}\pi$ in terms of cosines of even and sines of odd multiples of y respectively.

These are

$$\cosh ay = \frac{4a}{\pi} \sinh \frac{a\pi}{2} \left(\frac{1}{2a^2} - \frac{\cos 2y}{a^2+2^2} + \frac{\cos 4y}{a^2+4^2} - \ldots + (-1)^{\frac{1}{2}m} \frac{\cos my}{a^2+m^2} \ldots\right), \tag{10a}$$

$$\sinh ay = \frac{4a}{\pi} \cosh \frac{a\pi}{2} \left(\frac{\sin y}{a^2+1^2} - \frac{\sin 3y}{a^2+3^2} + \ldots + (-1)^{\frac{1}{2}(m-1)} \frac{\sin my}{a^2+m^2} \ldots\right). \tag{10b}$$

Hence the value of u due to the original pair of Kelvin waves may be expressed in the form

$$\begin{aligned} \tfrac{1}{2}Si(u_1+u_2) = {} & \frac{4aS}{\pi} \sin \frac{a\pi}{2} \sin \frac{\sigma x}{c} \left[\frac{1}{2a^2} + \sum_{m \text{ even}} (-1)^{\frac{1}{2}m} \frac{\cos my}{a^2+m^2}\right] \\ & - \frac{4aSi}{\pi} \cosh \frac{a\pi}{2} \cos \frac{\sigma x}{c} \left[\sum_{m \text{ odd}} (-1)^{\frac{1}{2}(m-1)} \frac{\sin my}{a^2+m^2}\right]. \end{aligned} \tag{10c}$$

For convenience the amplitude of the incident and reflected waves will be chosen so that $4aS/\pi = 1$.

It will be noticed that in this expression (10c) only cosines of even

multiples, and sines of odd multiples of y occur; whereas both sines and cosines of my necessarily occur in every term of (6a) and (6b). It is not possible therefore to equate coefficients of $\cos my$ and $\sin my$ in (6) and (10) directly.

Though no cosines of odd multiples of my occur in (10a), yet it is possible to construct an infinite number of trigonometrical series which do contain them and yet represent the function $\cosh ay$ between the limits $\pm\frac{1}{2}\pi$. To understand how this can be done it is only necessary to remember that the cosine of an odd multiple of y can be expanded in a trigonometrical series of even multiples of y which is valid between $\pm\frac{1}{2}\pi$. If therefore any multiple of a cosine of an odd multiple of y be added to the series (10a), and at the same time the same multiple of this trigononometrical expansion for this cosine be subtracted, a new series will be obtained for $\cosh ay$, which is valid between the limits $\pm\frac{1}{2}\pi$ and contains a cosine of an odd multiple of y as well as the cosines of the even multiples of y.

The same argument applies to the case of the expansion of $\sinh ay$ in terms of sines of odd multiples of y.

The trigonometrical series referred to for the sines of even multiples of y and the cosines of odd multiples of y are

$$s \text{ odd}, \quad (-1)^{\frac{1}{2}(s-1)}\frac{\pi}{4s}\cos sy = \frac{1}{2s^2}+\frac{\cos 2y}{2^2-s^2}-\frac{\cos 4y}{4^2-s^2}+\frac{\cos 6y}{6^2-s^2}-\dots, \tag{11a}$$

$$\text{and } s \text{ even}, \quad (-1)^{\frac{1}{2}s}\frac{\pi}{4s}\sin sy = \frac{\sin y}{1-s^2}-\frac{\sin 3y}{3^2-s^2}+\frac{\sin 5y}{5^2-s^2}-\dots. \tag{11b}$$

Multiples $\beta_1, \beta_3, \dots, \beta_s, \dots$ of $\cos sy$ (s odd), and $\gamma_2, \gamma_4, \dots, \gamma_s, \dots$ of $\sin sy$ (s even), may now be added to the series for $\frac{1}{2}Si\,(u_1+u_2)$, and the corresponding series subtracted. In this way it will be found that

$$\begin{aligned}\tfrac{1}{2}Si(u_1+u_2) &= \sinh\frac{a\pi}{2}\sin\frac{\sigma x}{c}\left[\frac{1}{2a^2}-\frac{\cos 2y}{a^2+2^2}+\frac{\cos 4y}{a^2+4^2}-\dots\right.\\ &\quad\left.+\sum_{s\text{ odd}}\beta_s\left\{(-1)^{\frac{1}{2}(s-1)}\frac{\pi\cos sy}{4s}-\left(\frac{1}{2s^2}+\frac{\cos 2y}{2^2-s^2}-\frac{\cos 4y}{4^2-s^2}+\dots\right)\right\}\right]\\ &\quad -i\cosh\frac{a\pi}{2}\cos\frac{\sigma x}{c}\left[\frac{\sin y}{a^2+1^2}-\frac{\sin 3y}{a^2+3^2}+\frac{\sin 5y}{a^2+5^2}-\dots\right.\\ &\quad\left.+\sum_{s\text{ even}}\gamma_s\left\{(-1)^{\frac{1}{2}s}\frac{\pi\sin sy}{4s}-\left(\frac{\sin y}{1^2-s^2}-\frac{\sin 3y}{3^2-s^2}+\frac{\sin 5y}{5^2-s^2}-\dots\right)\right\}\right],\end{aligned} \tag{12}$$

where β_s and γ_s are as yet undetermined.

The condition that, at the section $x = x_1$, of the channel, it shall be possible to choose values of A_m and B_m so that the motion given by a series of terms of the forms (6a) and (6b) may represent the same value of u as (12), is obtained from equations (8). The condition is that the ratio of the coefficient of $\cos my$ in (12) to the coefficient of $i \sin my$ shall be equal to

$$\left.\begin{array}{lll} -(1/r_m)\tan s_m x_1 & \text{when} & m^2 < k^2 \text{ and } m \textit{ even} \\ r_m \tan s_m x_1 & \text{when} & m^2 < k^2 \text{ and } m \textit{ odd} \\ -(1/r_m) & \text{when} & m^2 > k^2 \text{ and } m \textit{ even} \\ -r_m & \text{when} & m^2 > k^2 \text{ and } m \textit{ odd} \end{array}\right\}. \tag{13}$$

Taking all integral values of m, equations are obtained which determine $\beta_1, \beta_3, \beta_5, \ldots$ and $\gamma_2, \gamma_4, \gamma_6, \ldots$. The following are specimens :—

$$m \textit{ odd} \quad \left\{\begin{array}{l} \dfrac{1}{a^2+m^2} - \dfrac{\gamma_2}{m^2-2^2} - \dfrac{\gamma_4}{m^2-4^2} - \dfrac{\gamma_6}{m^2-6^2} - \ldots \\ \qquad = \dfrac{-\beta_m \pi}{4mr_m} \tanh\dfrac{a\pi}{2} \tan\dfrac{\sigma x_1}{c} \cot s_m x_1, \quad (m^2 < k^2), \quad (14a) \\ \qquad = \dfrac{\beta_m \pi}{4mr_m} \tanh\dfrac{a\pi}{2} \tan\dfrac{\sigma x_1}{c}, \qquad (m^2 > k^2), \quad (14b) \end{array}\right.$$

$$m \textit{ even} \quad \left\{\begin{array}{l} \dfrac{1}{a^2+m^2} + \dfrac{\beta_1}{m^2-1^2} + \dfrac{\beta_3}{m^2-3^2} + \dfrac{\beta_5}{m^2-5^2} + \ldots \\ \qquad = \dfrac{\gamma_m \pi}{4mr_m} \coth\dfrac{a\pi}{2} \cot\dfrac{\sigma x_1}{c} \tan s_m x_1, \quad (m^2 < k^2), \quad (14c) \\ \qquad = \dfrac{\gamma_m \pi}{4mr_m} \coth\dfrac{a\pi}{2} \cot\dfrac{\sigma x_1}{c}, \qquad (m^2 > k^2). \quad (14d) \end{array}\right.$$

From these equations it is possible to determine the β's and γ's for any given value of x_1, but there is still one more condition which must be satisfied, namely, that the terms in (12) which do not involve y must vanish. This extra condition determines the value of x_1. It is

$$\frac{1}{a^2} - \frac{\beta_1}{1^2} - \frac{\beta_3}{3^2} - \frac{\beta_5}{5^2} - \ldots = 0. \tag{15}$$

In order to determine x_1, the β's and γ's must be eliminated between equations (14) and (15). The equation for x_1 then involves an infinite determinant. It is

$$\begin{vmatrix} \frac{1}{a^2} & \frac{-1}{1^2} & 0 & \frac{-1}{3^2} & 0 & \frac{-1}{5^2} & \cdots \\ \frac{-1}{a^2+1^2} & L_1 z & \frac{1}{1^2-2^2} & 0 & \frac{1}{1^2-4^2} & 0 & \cdots \\ \frac{1}{a^2+2^2} & \frac{1}{2^2-1^2} & \frac{-M_2}{z} & \frac{1}{2^2-3^2} & 0 & \frac{1}{2^2-5^2} & \cdots \\ \frac{-1}{a^2+3^2} & 0 & \frac{1}{3^2-2^2} & L_3 z & \frac{1}{3^2-4^2} & 0 & \cdots \\ \frac{1}{a^2+4^2} & \frac{1}{4^2-1^2} & 0 & \frac{1}{4^2-3^2} & \frac{-M_4}{z} & \frac{1}{4^2-5^2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0, \quad (16)$$

where z has been written for $\tan(\sigma x_1/c)$ and

$$\left.\begin{aligned} L_m &= -(\pi/4mr_m)\tanh(\tfrac{1}{2}a\pi)\cot s_m x_1, && (m \text{ odd}, \ m^2 < k^2) \\ &= \ \ (\pi/4mr_m)\tanh(\tfrac{1}{2}a\pi), && (m \text{ odd}, \ m^2 > k^2) \\ M_m &= \ \ (\pi/4mr_m)\coth(\tfrac{1}{2}a\pi)\tan s_m x_1, && (m \text{ even}, \ m^2 < k^2) \\ &= \ \ (\pi/4mr_m)\coth(\tfrac{1}{2}a\pi), && (m \text{ even}, \ m^2 > k^2) \end{aligned}\right\}. \quad (17)$$

The reason why L_m and M_m have been used in this form is that, if there are no terms for which $m^2 < k^2$, *i.e.* if $k^2 < 1$, then L_m and M_m do not contain x_1. This case will now be treated separately.

Case when $k^2 < 1$.

In this case (16) is a simple equation, for, multiplying the first, third, and all odd columns by z, it will be found that every term in the second, fourth, and all even rows, contains z. Hence, dividing these by z, all the terms containing z are removed to the first column, where they occur in

alternate rows. The resulting simple equation for z is therefore

$$
z\begin{vmatrix}
\frac{1}{a^2} & \frac{-1}{1^2} & 0 & \frac{-1}{3^2} & 0 & \cdots \\
0 & L_1 & \frac{1}{1^2-2^2} & 0 & \frac{1}{1^2-4^2} & \cdots \\
\frac{1}{a^2+2^2} & \frac{1}{2^2-1^2} & -M_2 & \frac{1}{2^2-3^2} & 0 & \cdots \\
0 & 0 & \frac{1}{3^2-2^2} & L_3 & \frac{1}{3^2-4^2} & \cdots \\
\frac{1}{a^2+4^2} & \frac{1}{4^2-1^2} & 0 & \frac{1}{4^2-3^2} & -M_4 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots &
\end{vmatrix}
$$

$$
= -\begin{vmatrix}
0 & \frac{-1}{1^2} & 0 & \frac{-1}{3^2} & 0 & \cdots \\
\frac{-1}{a^2+1^2} & L_1 & \frac{1}{1^2-2^2} & 0 & \frac{1}{1^2-4^2} & \cdots \\
0 & \frac{1}{2^2-1^2} & -M_2 & \frac{1}{2^2-3^2} & 0 & \cdots \\
\frac{-1}{a^2+3^2} & 0 & \frac{1}{3^2-2^2} & L_3 & \frac{1}{3^2-4^2} & \cdots \\
0 & \frac{1}{4^2-1^2} & 0 & \frac{1}{4^2-3^2} & -M_4 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots &
\end{vmatrix}. \qquad (18)
$$

Hence z, or $\tan(\sigma x_1/c)$, can be found.

Case when $k^2 > 1$.

In this case the equation (16) does not reduce to a simple equation for $\tan(\sigma x_1/c)$, because some of the L's and M's contain $\tan s_m x_1$, or $\cot s_m x_1$, It is therefore necessary to approximate numerically to the solution of (16) by assuming various values for x_1, and finding when the determinant (16) changes sign.

Determination of the β's and γ's.

When x_1 has been found, its value may be substituted in equations

(14). The resulting equations for the β's and γ's have now numerical coefficients. They can therefore be solved numerically.

Reflection of Kelvin Waves.

It is evident from the form of the result that when the incident and reflected Kelvin waves are given, the whole motion which is necessary to produce the effect of reflection is determined by the equations.

In the case when $k^2 < 1$, the effect of the end of the channel which is represented by the terms

$$u = \Sigma\,(A_m e^{-s_m x} \cos my + iB_m e^{-s_m x} \sin my),$$

and the corresponding expressions for v, decrease indefinitely at great distances from the closed end. Kelvin waves are therefore reflected perfectly.

In the case when $k^2 > 1$, one at least of the terms in the expressions (5) and (6) contains sines and cosines of a multiple of x. These are finite for infinite values of x. Hence, if $k^2 > 1$, perfect reflection of Kelvin waves cannot take place. It appears, in fact, that the channel is too wide to force the reflected tidal wave back into the condition in which the particles of water move only parallel to the walls.

The terms, mentioned above, which occur when $k^2 > 1$, and contain sines and cosines of multiples of x, represent a pair of waves of a type to which Poincaré* refers in his book *Théorie des Marées*. They were also discovered independently by Proudman.†

Numerical Solutions.

The solution which has been given of the motion in a region where tidal waves are being reflected from the end of a channel must contain the physical explanation of the phenomenon; but the form of the result is so complicated that it would be difficult to discuss the tidal regime in a general way. A particular case has therefore been worked out in detail. The case chosen is that for which $k = 0{\cdot}5$, $a = 0{\cdot}7$. This corresponds with the tidal motion in a channel 250 miles wide and 40 fathoms deep situated in lat. 53° N. when a tidal wave of period 12 hours is reflected at the closed end. These figures have been chosen because they correspond, roughly, to the dimensions of the North Sea; but it is also worth noticing

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* *Leçons de Mécanique Céleste*, t. 3 (*Théorie des Marées*), Chap. VI, p. 126.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 12 (1913), p. 469.

that the case for which k lies in the middle of the range, 0–1, in which perfect reflection takes place, may be expected to present typical features.

Case when $k = 0{\cdot}5$, $a = 0{\cdot}7$.

In working a numerical example it is found convenient to transfer the origin to the mid-point of the end of the channel. The equation for the Kelvin wave system then becomes

$$\tfrac{1}{2}Si(u_1+u_2) = S\{\cosh ay \sin\sigma(x+x_1)/c - i\sinh ay\cos\sigma(x+x_1)/c\}, \quad (19)$$

$$v = 0.$$

The values of A_m and B_m are calculated for 10 terms up to A_{10} and B_{10}.

It should be noticed that all the quantities which occur in the equations, such as σ/c, s_m, r_m, L_m, M_m are functions of a and k only. These are first determined and are inserted in the determinants of equation (18). Their values are given in columns 2, 3, 4 and 5 of Table I.

Taking first only 2 rows and columns for each determinant in (18), a value $z = {\cdot}363$ is obtained. Taking successively, 3, 4 and 5 rows and columns in each determinant, the values ·383, ·385 and ·385 are obtained. It appears, therefore, that this method of approximating to the value of z is very rapid.

The value of σ/c is $\sqrt{(a^2+k^2)} = {\cdot}860$. Taking the value of $\tan\sigma x_1/c$ to be ·385 this gives $\sigma x_1/c = 21^\circ 3'$ or ·367 in circular measure. Hence

$$x_1 = {\cdot}427.$$

This value, ·385, for z, or $\tan\sigma x_1/c$, is next inserted in the equations (14). The first equation, namely,

$$\frac{\beta_1\pi}{4r_1}\tanh\frac{a\pi}{2}\tan\frac{\sigma x_1}{c} = \frac{1}{a^2+1} - \frac{\gamma_2}{1^2-2^2} - \frac{\gamma_4}{1^2-4^2} + \dots$$

becomes

$${\cdot}348\beta_1 - {\cdot}333\gamma_2 - {\cdot}0667\gamma_4 - {\cdot}0286\gamma_6 - {\cdot}0159\gamma_8 - {\cdot}0101\gamma_{10} - {\cdot}6711 = 0.$$

As a first approximation β_1 is taken as

$$\frac{{\cdot}6711}{{\cdot}348} = 1{\cdot}93.$$

The second equation of (14) is then written down. It is:

$${\cdot}223 + {\cdot}333\beta_1 - 8{\cdot}19\beta_3 - {\cdot}048\beta_5 - \dots = 0$$

As a first approximation γ_2 is taken as

$$(\cdot 333\beta_1 + \cdot 223)/8\cdot 19 = \cdot 106.$$

First approximations are thus found for all the β's and γ's. These values are then inserted in (14) and a new value is found for β_1, namely, 2·03. This is then used to obtain a better approximation for γ_2, and so on. It is found that two applications of this method are sufficient.

The results are given in columns 6 and 7 of the following table:—

TABLE I.

Table giving numerical data involved in calculating tides for the case when
$k = 0\cdot 5, \quad a = 0\cdot 7.$

Columns 1	2	3	4	5	6	7	8	9	10	11	12	13
m	s_m	r_m	L_m	M_m	β_m	γ_m	A_m	B_m	C_m	D_m	*odd*, $-B_m s_m - mC_m$; *even*, $-B_m s_m$	$s_m A_m$; $s_m A_m - mD_m$
1	·866	·695	·905	—	2·03	—	+ ·765	−1·100	+1·892	—	−·946	+·663
2	1·937	·155	—	3·15	—	·1080	− ·427	+ ·066	—	−·495	−·128	+·161
3	2·96	·0680	3·09	—	·0714	—	− ·0090	+ ·132	− ·138	—	+·023	−·027
4	3·97	·0380	—	6·44	—	·0124	+ ·1000	− ·004	—	+·103	+·016	−·015
5	4·98	·0242	5·19	—	·0165	—	+ ·0012	− ·050	+ ·051	—	−·006	+·006
6	5·98	·0167	—	9·72	—	·0035	− ·0425	+ ·001	—	−·043	−·003	+·003
7	6·99	·0123	7·30	—	·0061	—	− ·0003	+ ·027	− ·027	—		
8	7·99	·0094	—	12·98	—	·0015	+ ·0245	− ·0002	—	+·024		
9	9·00	·0074	9·40	—	·0030	—	+ ·0001	− ·017	+ ·017	—		
10	10·00	·0060	—	16·25	—	·0007	− ·0147	+ ·0001	—	−·015		

The values of A_m and B_m can be found from (12). Remembering that the origin has now been transferred to the end of the channel it will be found that for odd values of m,

$$A_m = (-1)^{\frac{1}{2}(m-1)} (\beta_m \pi/4m) \sinh(\tfrac{1}{2}a\pi) \sin(\sigma x_1/c);$$

and for even values of m,

$$B_m = -(-1)^{\frac{1}{2}m} (\gamma_m \pi/4m) \cosh(\tfrac{1}{2}a\pi) \cos(\sigma x_1/c).$$

To determine B_m when m is odd, and A_m when m is even, the relations (8) may be used. Their values are shown in columns 8 and 9 of Table I.

C_m and D_m are found from the formulæ (9). They are given in columns 10 and 11 of Table I.

The tidal range ζ is found from the formula

$$\frac{\sigma\zeta}{h} = i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right).$$

Finally, the whole motion is represented by

$$u = 1{\cdot}122\,\{\cosh {\cdot}7y \sin {\cdot}860(x+{\cdot}427)-i \sinh {\cdot}7y \cos {\cdot}860(x+{\cdot}427)\}$$

$$-\sum_{m=1}^{m=10}(A_m e^{-s_m x}\cos my+iB_m e^{-s_m x}\sin my, \quad (20a)$$

$$v = -\sum_{m\text{ even}} D_m e^{-s_m x}\sin my - i\sum_{m\text{ odd}} C_m e^{-s_m x}\cos my, \quad (20b)$$

$$\sigma\zeta/h = {\cdot}965\,\{-\sinh {\cdot}7y \sin {\cdot}860(x+{\cdot}427)+i \cosh {\cdot}7y \cos {\cdot}860(x+{\cdot}427)\}$$

$$+\sum_{m\text{ odd}}\{(-B_m s_m - mC_m)\,e^{-s_m x}\sin my + is_m A_m e^{-s_m x}\cos my\}$$

$$+\sum_{m\text{ even}}\{-B_m s_m e^{-s_m x}\sin my + i\,(s_m A_m - mD_m)\,e^{-s_m x}\cos my\}. \quad (20c)$$

Verification of the Solution.

In order to verify the accuracy of this solution, it should be noticed that the value of u at $x=0$ should be 0 for all values of y. Taking first the case where $y=0$, the value of u is $\cdot403-\Sigma A_m$. Adding column 8 of the table it is found that $\Sigma A_m = \cdot398$. Hence u very nearly vanishes at the mid-point of the end of the channel.

At the corner $x=0$, $y=\frac{1}{2}\pi$, the value of u due to the incident and reflected Kelvin waves, is found to be $0{\cdot}67-1{\cdot}40i$.

The part due to the terms which are inserted to make u vanish at $x=0$ is

$$(A_2-A_4+A_6-A_8+A_{10})-i(B_1-B_3+B_5-B_7+B_9) = -0{\cdot}61+1{\cdot}33i.$$

Hence it will be seen that both at $y=0$ and at $y=\frac{1}{2}\pi$ the two motions very nearly neutralise one another. The solution is better at $y=0$ than it is at $y=\frac{1}{2}\pi$; but this is to be expected because at $y=0$ the terms are alternately positive and negative numbers; while at $y=\frac{1}{2}\pi$, both the real and imaginary parts are composed entirely of terms of one sign, so that if account were taken of the higher terms the agreement would be better.

Representation of the Results.

In order to represent in an intelligible manner the tidal motion represented, mathematically, by (20), the values of u, v and $\sigma\zeta/h$ have been calculated from the formulæ (20) for the 63 points

$$x = 0,\ \pi/6,\ \pi/3,\ \pi/2,\ 2\pi/3,\ 5\pi/6,\ \pi,\ 4\pi/3,\ 5\pi/3;$$

$$y = 0,\ \pm\pi/6,\ \pm\pi/3,\ \pm\pi/2.$$

Rise and fall of tide.

Taking the value of $\sigma\zeta/h$ at any point to be $P+iQ$, the phase θ, of the tide is given by $\tan\theta = Q/P$. The cotidal lines are lines of constant θ, *i.e.* lines at all points of which it is high water simultaneously. These have been drawn for values of θ differing by $\pi/6$ which corresponds to 1 hour's difference in the state of the tide. They are shown on the full lines in Fig. 1.

The range of tide is proportional to $|\sigma\zeta/h|$ which is equal to $P\sec\theta$. Lines of equal range of tide are shown as dotted lines in Fig. 1. The diagram shows the motion of the tidal wave down one side of the channel; and the way in which it sweeps round the end to return along the opposite side. Further remarks about it will be found in the introduction.

Tidal currents.

The tidal currents are represented by the ellipses in Fig. 2. Each ellipse is centred at the point to which it applies. The velocity and direction of the tidal stream is represented by a vector from the point in question. The ellipses also represent, on another scale, the actual paths of water during the tidal motion.

The method adopted for finding the magnitude and direction of the axes was the following:—

Let
$$u = (A+iB)\,e^{i\sigma t},$$
$$v = (C+iD)\,e^{i\sigma t},$$

where A, B, C, D are real numbers.

The components of velocity at any time are found by taking the real parts of these.

Let P be the point whose coordinates are (A, B) on a system of rectangular axes $O\xi$, $O\eta$ (Fig. 3). Let Q be the point whose coordinates are $(-D, C)$ on the same system. Take the mid-point R of PQ. Join OR.

Then the major axis of the ellipse is $OR+QR$, the minor axis is $OR-QR$, and the inclination of the major axis to the axis of x, *i.e.* to the walls of the channel (see Fig. 2), is half the angle QRO.

This may be proved by dropping perpendiculars PM, QN on $O\xi$, $O\eta$ respectively. ON and OM are then the components of velocity at time $t = 0$. If PM and QN meet in L, the resultant velocity is represented by OL. As t increases the points P, Q and R revolve uniformly round O. Since QLP is a right angle, L describes a circle of radius QR round R. Since R describes a circle of radius OR round O, the maximum value of OL occurs when O, R and L are in the same straight line and its length is then $OR+QR$. Similarly the minimum length of OL is $OL-QR$. These then are the major and minor axes of the ellipse.

The direction in which the particles of water move is determined by the relative magnitudes of OR and QR. If $OR > QR$ the rotation is positive, *i.e.* from Ox towards Oy, which is also the direction of rotation. If $OR < QR$ the rotation of the particles of water is opposite to the direction of rotation of the system.

The phase of the motion at time $t = 0$ can also be found from the diagram (Fig. 3).

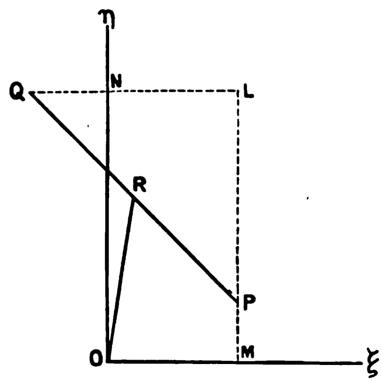

FIG. 3.—Construction used in finding positions and magnitudes of axes in tidal ellipses.

The results are shown in Fig. 2. They are commented on in the introduction. The maximum tidal current in the Kelvin wave system is $(1{\cdot}122) \cosh \frac{1}{2}\alpha\pi = 1{\cdot}87$. The maximum cross-channel current occurs at the mid-point of the closed end; its value is

$$v = C_1+C_3+C_5+C_7+C_9 = 1{\cdot}80.$$

It will be seen that in this example the maximum tidal current across the

end is very nearly as great as the maximum tidal current in the Kelvin wave system.

Case of narrow channel.

For this case a and k are small. Hence approximately

$$s_m = m, \quad \sigma/c = k, \quad r_m = ak/m^2, \quad \tanh \tfrac{1}{2}a\pi = \sinh \tfrac{1}{2}a\pi = \tfrac{1}{2}a\pi,$$

$$L_m = m\pi^2/8k, \quad M_m = m/2a^2k,$$

$$\sigma x_1/c = 8ka^2/\pi^2, \quad \beta_m = m^{-3}a^{-2}.$$

The terms containing the γ's are small compared with the terms containing the β's,

$$A_m = (-1)^{\frac{1}{2}(m-1)}\, akm^{-4},$$

$$C_m = -B_m = (-1)^{\frac{1}{2}(m-1)}\, m^{-2}.$$

Hence maximum cross-channel velocity at mid-point of the end of the channel is*

$$1^{-2} - 3^{-2} + 5^{-2} - 7^{-2} + \ldots = \cdot 916.$$

The maximum velocity along the channel is $S = \pi/4a$. Hence ratio of maximum cross-channel velocity to maximum velocity along the channel is $\cdot 916\,(4a/\pi) = 1\cdot 16a$ or roughly a.

In the case of the south channel to the Irish Sea, which is 50 miles wide from Holyhead to Ireland, and 40 fathoms deep, $a = 0\cdot 14$. The maximum up and down channel currents occur between Holyhead and Ireland, and they are about 3 knots. The maximum cross-channel current which can be expected anywhere in the system is therefore less than half a knot, and it should be remembered that this does not occur anywhere near the region where the maximum currents occur. In fact the ratio of the cross-channel current to the down-channel current, in the region of maximum currents, is only $e^{-\pi/2k}\,(4a/\pi)$, which in the case of the Irish Sea, where $a = 0\cdot 14$, $k = 0\cdot 1$, would be only 5×10^{-8} knots.

TIDAL OSCILLATIONS IN A RECTANGULAR BASIN.

If instead of the forms ($5c$) and ($5d$) we had assumed in the case where $m^2 > k^2$ that v contains terms involving $e^{s_m x}$ as well as $e^{-s_m x}$, and

* See Bromwich, *Infinite Series*, p. 479.

corresponding extra terms containing $\cos s_m x$ and $\sin s_m x$ in the case where $m^2 < k^2$, then the extra constants introduced would have enabled us to make u vanish for two different values of x_1. In this case a solution would have been obtained for the vibrations of a rectangular sheet of liquid. It will be shown later, however, that in this case, as might be expected, there is an extra condition which determines the periods of the oscillations.

The analysis is very much simplified by taking the origin of coordinates at the centre of the rectangle. In that case it will be found that the oscillations are of two classes: (1) those in which the tidal wave is symmetrical about the centre, and (2) those in which it is anti-symmetrical (*i.e.* those in which the range of tide is the same at opposite points, but the phase is opposite). The slowest mode belongs to the latter class.

To determine the anti-symmetrical types consider the pair of Kelvin waves given by

$$u = S\{\cosh ay \cos(\sigma x/c) + i \sinh ay \sin(\sigma x/c)\}, \tag{21}$$

$$v = 0.$$

Suppose, as before, that the width of the basin is π. Let its length be $2l$.

To the original Kelvin waves add the oscillation represented by

$$m^2 < k^2 \begin{cases} m \text{ odd}, & v = iC_m \cos s_m x \cos my, & (22a) \\ m \text{ even}, & v = D_m \sin s_m x \sin my, & (22b) \end{cases}$$

$$m^2 > k^2 \begin{cases} m \text{ odd}, & v = iC_m \cosh s_m x \cos my, & (22c) \\ m \text{ even}, & v = D_m \sinh s_m x \sin my, & (22d) \end{cases}$$

$$m^2 < k^2, \quad u = A_m \cos s_m x \cos my + iB_m \sin s_m x \sin my, \tag{23a}$$

$$m^2 > k^2, \quad u = A_m \cosh s_m x \cos my + iB_m \sinh s_m x \sin my. \tag{23b}$$

Substituting in the equation

$$i\sigma \frac{\partial u}{\partial y} - 2n \frac{\partial u}{\partial x} = i\sigma \frac{\partial v}{\partial x} + 2n \frac{\partial v}{\partial y},$$

it will be found that

$$m^2 < k^2 \begin{cases} m \textit{ odd}, & A_m/B_m = -r_m, \quad C_m = (2n/\sigma)A_m - (m/s_m)B_m, & (24a) \\ m \textit{ even}, & A_m/B_m = 1/r_m, \quad D_m = -(m/s_m)A_m - (2n/\sigma)B_m, & (24b) \end{cases}$$

$$m^2 > k^2 \begin{cases} m \textit{ odd}, & A_m/B_m = r_m, \quad C_m = (2n/\sigma)A_m + (m/s_m)B_m, & (24c) \\ m \textit{ even}, & A_m/B_m = 1/r_m, \quad D_m = -(m/s_m)A_m - (2n/\sigma)B_m, & (24d) \end{cases}$$

where $$r_m = (2n\sigma)/(ms_m c^2),$$

as before.

Expanding (21) as a Fourier series in y at the end, $x = l$, of the rectangle,

$$u = \frac{4aS}{\pi}\sinh\frac{a\pi}{2}\cos\frac{\sigma l}{c}\left(\frac{1}{2a^2} - \frac{\cos 2y}{a^2+2^2} + \frac{\cos 4y}{a^2+4^2} - \frac{\cos 6y}{a^2+6^2} + \dots\right)$$

$$+\frac{4aSi}{\pi}\cosh\frac{a\pi}{2}\sin\frac{\sigma l}{c}\left(\frac{\sin y}{a^2+1^2} - \frac{\sin 3y}{a^2+3^2} + \frac{\sin 5y}{a^2+5^2} - \dots\right). \quad (25)$$

Take $S = \pi/4a$ as before, add the undetermined multiples $\beta_1, \beta_3, \dots, \gamma_2, \gamma_4, \dots$ of the series for $\cos sy$ and $\sin sy$.

Write down the equations corresponding with (14). They are

$$m \textit{ odd} \begin{cases} \dfrac{-1}{a^2+m^2} + \dfrac{\gamma_2}{m^2-2^2} + \dfrac{\gamma_4}{m^2-4^2} + \dots \\ = \dfrac{\beta_m \pi}{4mr_m}\tanh\tfrac{1}{2}a\pi\cot\sigma l/c\tan s_m l = \lambda_m\beta_m, \quad m^2 < k^2, & (26a) \\ = -\dfrac{\beta_m \pi}{4mr_m}\tanh\tfrac{1}{2}a\pi\cot\sigma l/c\tanh s_m l = -\lambda_m\beta_m, \quad m^2 > k^2, & (26b) \end{cases}$$

$$m \textit{ even} \begin{cases} \dfrac{1}{a^2+m^2} + \dfrac{\beta_1}{m^2-1^2} + \dfrac{\beta_3}{m^2-3^2} + \dfrac{\beta_5}{m^2-5^2} + \dots \\ = \dfrac{\gamma_m \pi}{4mr_m}\coth\tfrac{1}{2}a\pi\tan\sigma l/c\cot s_m l = -\mu_m\gamma_m, \quad m^2 < k^2, & (26c) \\ = \dfrac{\gamma_m \pi}{4mr_m}\coth\tfrac{1}{2}a\pi\tan\sigma l/c\coth s_m l = -\mu_m\gamma_m, \quad m^2 > k^2, & (26d) \end{cases}$$

where $-\lambda_m$ and $-\mu_m$ are written for the coefficients of β_m and γ_m on the right-hand sides of these equations. To these must be added the equation which is necessary in order that there be no constant term left over.

This is the same as before, namely,

$$\frac{1}{a^2} - \frac{\beta_1}{1^2} - \frac{\beta_3}{3^2} - \frac{\beta_5}{5^2} - \ldots = 0. \tag{27}$$

Eliminating the β's and γ's, the following equation is obtained

$$\begin{vmatrix} \frac{1}{a^2} & \frac{-1}{1^2} & 0 & \frac{-1}{3^2} & 0 & \ldots \\ \frac{-1}{a^2+1^2} & \lambda_1 & \frac{1}{1^2-2^2} & 0 & \frac{1}{1^2-4^2} & \ldots \\ \frac{1}{a^2+2^2} & \frac{1}{2^2-1^2} & \mu_2 & \frac{1}{2^2-3^2} & 0 & \ldots \\ \frac{-1}{a^2+3^2} & 0 & \frac{1}{3^2-2^2} & \lambda_3 & \frac{1}{3^2-4^2} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \end{vmatrix} \tag{28}$$

If l is fixed there is now only one unknown left in the equation, namely, σ. Equation (28) is, therefore, a period equation, and its roots determine a set of free periods.

It will be noticed that when the motion is determined in this way so that u is zero at $x = l$, it is also zero at $x = -l$.

The solution therefore represents one set of the oscillations of a sheet of liquid confined between the lines $x = \pm l$, $y = \pm \frac{1}{2}\pi$. Since

$$\zeta = \left(\frac{ih}{\sigma}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right),$$

it will be seen that the surface of the sea is at any instant anti-symmetrical.

To determine the symmetrical oscillations, take as the original pair of Kelvin waves

$$u = S\left\{\cosh ay \sin \sigma x/c - i \sinh ay \cos \sigma x/c\right\}, \quad v = 0. \tag{29}$$

The period equation is the same as (28) except that the λ's and μ's assume slightly different forms. To distinguish the two types of oscillation the anti-symmetrical type will be called Type A, while the symmetrical type will be called Type B.

The values of the λ's and μ's in the period equation are given in the following Table II.

TABLE II.

Showing values of λ_m and μ_m in the period equation (28).

TYPE A.—*Anti-symmetrical oscillations.*

$$m \text{ odd} \begin{cases} m^2 < k^2, & \lambda_m = -(\pi/4mr_m)\tanh \tfrac{1}{2}a\pi \cot \sigma l/c \tan s_m l, \\ m^2 > k^2, & \lambda_m = (\pi/4mr_m)\tanh \tfrac{1}{2}a\pi \cot \sigma l/c \tanh s_m l, \end{cases}$$

$$m \text{ even} \begin{cases} m^2 < k^2, & \mu_m = -(\pi/4mr_m)\coth \tfrac{1}{2}a\pi \tan \sigma l/c \cot s_m l, \\ m^2 > k^2, & \mu_m = -(\pi/4mr_m)\coth \tfrac{1}{2}a\pi \tan \sigma l/c \coth s_m l. \end{cases}$$

TYPE B.—*Symmetrical oscillations.*

$$m \text{ odd} \begin{cases} m^2 < k^2, & \lambda_m = (\pi/4mr_m)\tanh \tfrac{1}{2}a\pi \tan \sigma l/c \tan s_m l, \\ m^2 > k^2, & \lambda_m = -(\pi/4mr_m)\tanh \tfrac{1}{2}a\pi \tan \sigma l/c \tanh s_m l, \end{cases}$$

$$m \text{ even} \begin{cases} m^2 < k^2, & \mu_m = (\pi/4mr_m)\coth \tfrac{1}{2}a\pi \cot \sigma l/c \cot s_m l, \\ m^2 > k^2, & \mu_m = (\pi/4mr_m)\coth \tfrac{1}{2}a\pi \cot \sigma l/c \coth s_m l. \end{cases}$$

It will be noticed that when an oscillation of Type B has been determined in this way so that $u = 0$ at $x = l$, then $u = 0$ also at $x = -l$, and therefore oscillations of Type B also take place in a closed basin of breadth π and length $2l$.

Method gives all possible oscillations.

We have now seen how two types of oscillation may be found. The question whether there are any other possible types of oscillation still remains.

First it should be noticed that it is impossible to obtain any other types beside A and B from any other combination of the original Kelvin waves, for any other combination may be represented by the expression

$$u = S_1(\cosh ay \cos \sigma x/c + i \sinh ay \sin \sigma x/c)$$
$$+ S_2(\cosh ay \sin \sigma x/c - i \sinh ay \cos \sigma x/c), \quad v = 0.$$

If the β's and γ's and σ be determined so that $u = 0$ at $x = l$, it will be found that u is not equal to 0 at $x = -l$ unless either S_1 or S_2 is equal

to 0. Hence Types A and B are the only possible types of oscillation which can be obtained by the method described above.

Next let us enquire whether it is possible for the sheet of liquid to oscillate in any other way.

Consider the oscillation of period $2\pi/\sigma$ in a liquid enclosed between two walls at $y = \pm\frac{1}{2}\pi$. The value of v at any section may be expressed by a unique series of the form

$$v = f_1 \cos y + f_3 \cos 3y + f_5 \cos 5y + \dots$$
$$+ g_2 \sin 2y + g_4 \sin 4y + g_6 \sin 6y + \dots,$$

where $f_1, f_3, \dots, g_2, g_4, \dots$ are functions of x.

From equation (3) it will be seen that $f_1, f_3, \dots, g_2, g_4, \dots$ may be expressed in the form

$$f_m = E_m e^{s_m x} + F_m e^{-s_m x},$$

where E_m and F_m are numbers which may be complex.

Corresponding with each term in the series for v at a given section $x = x_1$, say, there are two terms in a series for u, which are obtained from equations similar to (24).

Besides these there are the Kelvin wave systems corresponding with $v = 0$.

It appears, therefore, that all possible oscillations of a sheet of liquid contained between $y = \pm\frac{1}{2}\pi$ can be expressed as the sum of terms of these types. The ways in which the Kelvin terms can be combined with the others so as to make $u = 0$ at $x = \pm l$ have already been discussed.

If it were possible to select a combination of terms which did not include the Kelvin terms and yet made $u = 0$ at $x = \pm l$, it would be possible to choose a value of σ so that solutions to the equations (26) could be found when the constant terms $1/(a^2+m^2)$ and $1/a^2$ have been removed.

It is evident that this is not possible in general. It seems probable, therefore, that the oscillations of Types A and B are the only possible types of oscillation in a rectangular sheet of rotating liquid.

Numerical verification.

In order to test these conclusions I have calculated, in two ways, the slowest mode of oscillation of a rectangular basin whose length is twice its breadth.

(I) Taking the original pair of Kelvin waves as being parallel to the longer side, and

(II) Taking them as being parallel to the shorter side.

The periods obtained by these two methods were exactly the same although nearly all the quantities concerned were different, and the manner in which the infinite determinants converged was quite different in the two cases. This is good evidence that the oscillations in the two cases were exactly the same.

Let the breadth of the basin be B and its length $2B$. Then in working out the period by method (I), in which the axis of x is parallel to the longest side of the rectangle, the breadth is reduced to π so that

$$c = (\pi/B)\sqrt{(gh)}.$$

In adopting the method (II), in which the axis of x is parallel to the shortest side of the rectangle, c', which will be used to denote the value of c in this case, is equal to $(\pi/2B)\sqrt{(gh)}$.

Hence $$c' = \tfrac{1}{2}c.$$

The amount of rotation is so chosen that $2n/c = 1$ so that, in (I), $a = 1$; and in (II), $a' = 2$, where $a' = 2n/c'$. In this case the period of rotation is the same as that of the slowest oscillation of the basin when not rotating.

Since in (I) the length in the direction of the axis is twice the breadth, $l = \pi$. In (II) where the side parallel to the axis of x is only half the side parallel to the y axis, l', the value of l, in this case, is $\pi/4$.

I. Putting $a = 1$, $l = \pi$ in the period equation (28), the smallest root was determined as follows :—

A particular value was chosen for σ/c, the values of the determinants obtained by taking the first 1, 2, 3 and 4 rows and columns of (28) were then calculated. These are represented by Δ_1, Δ_2, Δ_3, Δ_4.

The calculations were repeated for a series of values of σ/c. The results are given in Table $3a$. They are exhibited graphically in Fig. 4. It will be seen that the curves Δ_2, Δ_3, and Δ_4 all pass through practically the same point on the axis. This gives an idea of the rapidity with which the roots of Δ_2, Δ_3, Δ_4, &c., converge to a fixed value. The value of the root of Δ_4, determined graphically, is

$$\sigma/c = 0{\cdot}429. \tag{32}$$

TABLE 3.

Numerical values of quantities used in finding the period of the slowest oscillation in a basin whose length is twice its breadth when the period of rotation is equal to the slowest period of the same basin when not rotating.

TABLE (3a).

$a = 1, \quad l = \pi$			
σc	Δ_2	Δ_3	Δ_4
·30	+1·90	−16·4	−89·6
·35	+ ·938	− 9·84	−32·1
·40	+ ·294	− 4·08	− 7·42
·42	+ ·097	− 1·51	− 1·99
·45	− ·160	+ ·475	+ 3·87
·47	− ·295	+13·05	+ 6·35

TABLE (3b).

$a = 2, \quad l = \pi/4$				
σ/c'	Δ_2	Δ_3	Δ_4	Δ_5
·80	·130	−·0605	−·137	+·214
·85	·08	−·010	−·0184	+·0250
·90	·042	+·0266	+·0472	−·0814

II. Putting $a = 2$, $l = \pi/4$ in (28) the values of Δ_2, Δ_3, Δ_4, and Δ_5 are calculated for values of σ/c equal to 0·80, 0·85, and 0·90. Their values are given in Table (3*b*).

On drawing the graphs it is again found that Δ_3, Δ_4, and Δ_5 all cross the axis very nearly at the same point.

The root of Δ_5 is found graphically to be

$$\sigma/c_1 = 0·859.$$

Remembering that $c' = \frac{1}{2}c$, it will be seen that method (II) gives

$$\sigma/c = \tfrac{1}{2}\sigma/c' = \tfrac{1}{2}(0·859) = 0·4295. \tag{33}$$

Comparing this with the value 0·429 obtained by method (I) it is evident that the two methods are giving the same oscillation.

It is worth noticing that the slowest period, $2\pi/\sigma$, of the same basin in the absence of rotation is given by $\sigma/c = ·50$. The period is therefore increased in the ratio ·50 : ·429 = 1·14 by a rotation whose period is equal to that of the longest free period of the basin when not rotating.

Character of the Oscillations.

Reasoning by analogy with the motion determined in the first part of this paper, it seems probable that the oscillations consist of a series of tidal

waves following one another round the basin in the direction of rotation. In this case oscillations of Type A would consist of an odd number of waves, while oscillations of Type B would consist of an even number. If this were true then the roots of the period equation for oscillations of Type B should always fall between two roots of the period equation for Type A, and *vice versa*. It might be possible to prove this, but I do not feel competent to do so. On the other hand, it should not be difficult to determine the roots in a particular numerical case and so verify this suggestion.

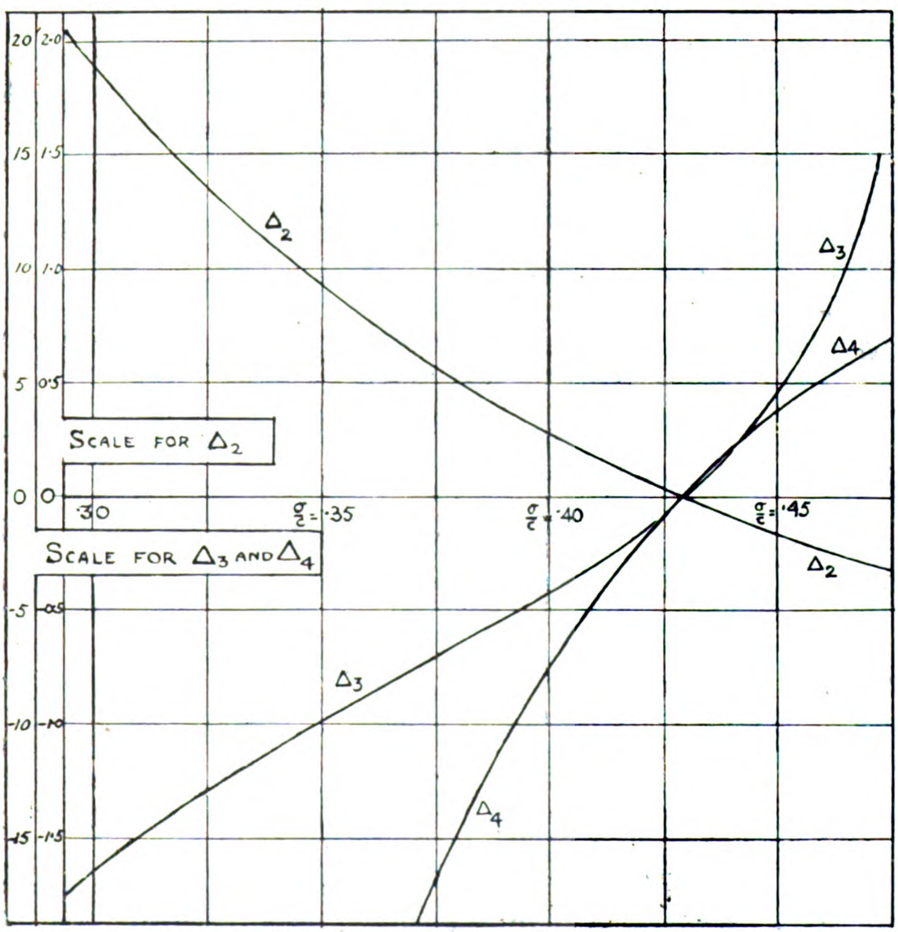

FIG. 4.—Graphical solution of period equation. Curves from figures in Table (3*a*).

Comparison with Oscillations in a Circular Basin.

The oscillations of liquid in a rotating circular basin have been worked

out by Lamb.* Lamb comes to the conclusion that there are two types of oscillation, one in which the tidal waves flow round the basin in the direction of the rotation of the basin, and another in which the tidal waves move in the opposite direction. It is, therefore, of interest to find out what difference there is, if any, between positive and negative roots of (28).

It will be seen from Table II that a change of sign in σ/c, unaccompanied by a change in its absolute magnitude, leaves the values of the λ's and μ's in (28) unaltered. Similarly a change in sign in a does not affect them either. For every positive root of (28) there is therefore an equal negative root.

At first sight one might be disposed to think that the existence of roots of both signs means, as it does in the case of the circular basin, that it is possible for a wave to proceed round the tank in either direction. This is not true, however. The negative and positive roots of (28) both represent the same oscillation.

To prove this suppose that a positive root of (28) has been found, and that the values of A_m, B_m, &c. have been found.

Now consider the motion when the sign, but not the magnitude, of σ/c is changed. From (26) it will be seen that the β's and γ's are unaffected. All the A's are reversed, but the B's remain unaffected. The term

$$A_m \sin s_m x \cos my + iB_m \cos s_m x \sin my$$

in the expression for u now becomes

$$-A_m \sin s_m x \cos my + iB_m \cos s_m x \sin my.$$

When multiplied by $e^{-i\sigma t}$ the real part is

$$-A_m \sin s_m x \cos my \cos \sigma t + B_m \cos s_m x \sin my \sin \sigma t.$$

The real part of the corresponding oscillation in the case of a positive root of (28) is

$$A_m \sin s_m x \cos my \cos \sigma t - B_m \cos s_m x \sin my \sin \sigma t,$$

which is the same as in the case of the negative root except for a change in sign.

It is evident, therefore, that the direction in which the waves move round the basin is the same in the two cases, and it seems clear that this must be the direction of rotation. It seems probable, therefore, that the type of oscillation discovered by Lamb in the case of a circular basin in

* *Hydrodynamics*, p. 311, 1916.

which the wave moves in a direction opposite to that of its rotation is peculiar to the circular basin and is not a general feature of tidal oscillations. It is worth noticing that no oscillation of this type appears to have been observed in the ocean.

Comparison with Non-Rotating System.

The oscillations which this work indicates are quite different from the oscillations which a non-rotating system can execute. In the case of a non-rotating rectangular basin of uniform depth there is a doubly infinite system of periods, corresponding with giving all integral values to m and n in the equation

$$\sigma^2/c = \pi^2(m^2/a^2+n^2/b^2),$$

where a and b are the lengths of the sides of the rectangle.* In the present system there is only a singly infinite series of periods.

The physical reason for this appears to be connected with the way in which tidal waves are reflected from a wall which is perpendicular to their direction of motion. It is shown in the first part of the paper that Kelvin waves are reflected at the end of a channel by a process which involves their being deflected so that they move for a time parallel to the end. It appears, therefore, that the wave length of the waves proceeding along one side of a channel cannot be independent of the wave length of tidal waves moving parallel to the other.

In conclusion I wish to express my thanks to Prof. E. T. Whittaker for his valuable advice on questions involved in the numerical work described in this paper.

* See Lamb's *Hydrodynamics*, Chap. VIII.

ON THE PARTIAL DERIVATES OF A FUNCTION OF MANY VARIABLES

By GRACE CHISHOLM YOUNG.

[Read December 9th, 1920.]

1. Let $$f(x, y) = f(x, y_1, y_2, \ldots, y_{n-1}, \ldots)$$
denote a function of any number* of variables, finite everywhere, and measurable for each fixed $y \equiv (y_1, y_2, \ldots, y_{n-1}, \ldots)$.

We shall denote *the partial derivates* (*upper right-hand, upper left-hand, lower right-hand, lower left-hand*) *with respect to* x, by

$$f^{+,0}(x, y), \quad f^{-,0}(x, y), \quad f_{+,0}(x, y), \quad f_{-,0}(x, y).$$

These are the upper and lower limits of the *incrementary ratio with respect to* x,

$$R_f(x, x+h; y) = \{f(x+h, y) - f(x, y)\}/h,$$

respectively for $h > 0$ (*right*), *and* $h < 0$ (*left*).

We then have the following theorems:—

THEOREM 1.—*The points, if any, at which the upper partial derivate on one side with respect to* x *is less than the lower partial derivate on the other side, form a set of content zero whose section by every line* $y =$ *constant is a countable set, i.e.*

$$f_{-,0}(x, y) \leqslant f^{+,0}(x, y), \quad f_{+,0} \leqslant f^{-,0}(x, y). \quad [P.\,p.]\dagger$$

This is an immediate consequence of a theorem given by myself in the *Acta Math.*‡ on the derivates of a function of a single variable. We do

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* Not necessarily finite or even countably infinite.

† $P.\,p.$ means *presque partout*, that is "except at a set of content zero."

‡ "*Except at a countable set of points, the lower derivate on either side is less than or equal to the upper derivate on the other side, i.e.* $f_-(x) \leqslant f^+(x)$, and also $f_+(x) \leqslant f^-(x)$." "A Note on Derivates and Differential Coefficients," *Acta Math.*, Vol. 37, p. 144.

not even here need to assume the function $f(x, y)$ to be measurable for each fixed y.

THEOREM 2.—*The points, if any, at which the upper partial derivate with respect to x on one side has the value* $+\infty$, *while the lower partial derivate on the other side has a value other than* $-\infty$, *i.e.* $f^{+,0}(x, y) = +\infty$, $f_{-,0}(x, y) \neq -\infty$, *or* $f^{-,0} = +\infty$, $f_{+,0}(x, y) \neq -\infty$, *form a set of content zero, whose section by every line* $y =$ *constant is a set of linear content zero.*

This is an immediate consequence of Theorem 1 of my former communication to the Society on the subject of derivates.*

THEOREM 3.—*The points, if any, at which* $f(x, y)$ *has a partial differential coefficient* $\dfrac{\partial f(x, y)}{\partial x}$ *which is infinite with determinate sign, or at which it has a forward or backward partial differential coefficient which is infinite with determinate sign, that is* $f^{+,0}(x, y) = f_{+,0}(x, y) = +\infty$ *or* $-\infty$, *or* $f^{-,0}(x, y) = f_{-,0}(x, y) = +\infty$ *or* $-\infty$, *form a set of content zero, whose section by every line* $y =$ *constant is a set of linear content zero.*

This follows immediately from Theorems 1 and 2 above.

THEOREM 4*a*.—*The points, if any, at which one of the upper (lower) partial derivates with respect to x, being finite, is not equal to the lower (upper) partial derivate on the other side, that is*

$$+\infty > f^{+,0}(x, y) \neq f_{-,0}(x, y); \quad +\infty > f^{-,0}(x, y) \neq f_{+,0}(x, y);$$

$$-\infty < f_{-,0}(x, y) \neq f^{+,0}(x, y); \quad -\infty < f_{+,0}(x, y) \neq f^{-,0}(x, y),$$

form a set of content zero, whose section by every line $y =$ *constant is a set of linear content zero.*

THEOREM 4*b*.—*The points, if any, at which one of the upper partial derivates with respect to x and one of the lower partial derivates are finite and different from one another, form a set of content zero, whose section by every line* $y =$ *constant is a set of values of x of linear content zero.*

The second of these theorems follows at once from Theorem 3 of the

* "On the Derivates of a Function," *Proc. London Math. Soc.*, Ser. 2, Vol. 15 (1916), p. 368.

communication last quoted. As however I am able to prove a somewhat more extended result by dividing the proof into two parts, I proceed to do so now.* The original theorem is now divided into two, in the first of which the assumption of finitude is made only for a single derivate, while in the second, the enunciation of which is that of the original theorem, the finitude of two derivates is hypothecated.

We proceed then to prove the first of these theorems from which Theorem 4*a* above at once follows:—

THEOREM.—*If $f(x)$ is a finite measurable function, the points, if any, at which one of the upper (lower) derivates, being finite, is not equal to the lower (upper) derivate on the other side, form a set of content zero.*

Let the given derivate be $f^-(x)$. Except at a countable set, we have

$$f_+(x) \leqslant f^-(x),$$

and, since $f^-(x)$ is not equal to $+\infty$ at the points we are going to consider, $f_+(x)$ is not $-\infty$ except possibly at a set of content zero, while $f_+(x)$ can only be $+\infty$ at a set of content zero; these results have already been quoted. Thus, suppressing a possible set of content zero, $f_+(x)$ *is finite, and* $\leqslant f^-(x)$, *wherever* $f^-(x)$ *is finite.*

We proceed to show that

$$f^-(x) = f_+(x)$$

at every point of the set S of points at which both these derivates are finite, and at which $f_+(x) \leqslant f^-(x)$, excepting only a sub-set of content zero.

Assume, if possible, that the set S has positive content. Let S_r denote the sub-set of S at whose points

$$-r < f_+(x) \leqslant f^-(x) < r,$$

r being any positive integer. Then each set S_r contains its predecessor S_{r-1}, and the outer limiting set is S itself. Therefore for some value of r the content of S_r must be positive. Let r denote the least integer for which this is true. Now write

$$f(x) = g(x) + rx, \quad A = 2r.$$

* At the same time I am enabled to add, to the list of corrections already given, the argument by which the different possible cases may be deduced from that explicitly treated. I have also given in full the justification for the statement that the end-points of the intervals r_x and t_i belong to the set E.

Then S_r is the set of points at which

$$0 < g_+(x) \leqslant g^-(x) < A. \tag{α}$$

Since $g(x)$, like $f(x)$, possesses the C-property,* and S_r is of positive content, we can remove a sub-set of sufficiently small content, leaving over a complementary sub-set S'_r of positive content, with respect to which $f(x)$ is continuous. By Lusin's Lemma† we can then find a perfect sub-set S''_r of S'_r, which is throughout‡ of positive content.

Finally, since at each point of S''_r we have (α), we can§ choose a fundamental interval (a, b), so as to contain a part G of S''_r, and such that, for points x, $x-h$, and $x+h$ in it, we have

$$\left.\begin{array}{l} \dfrac{g(x)-g(x-h)}{h} \leqslant A \\ \qquad\qquad 0 \leqslant \dfrac{g(x+h)-g(x)}{h} \end{array}\right\} \tag{β}$$

(h being > 0), provided x belongs to the set G. We may clearly assume a and b to be points of G. Let G_k denote the sub-set of G at which

$$g^-(x) - g_+(x) > k. \tag{1}$$

Then, if k assumes in succession the values A, $\frac{1}{2}A$, $\frac{1}{4}A$, ..., each set G_k is contained in the following, and the outer limiting set is G. Hence again, for one of these values of k—which we may take to be the greatest such—G_k must have positive content.

Now this set G_k is itself the sum of the finite number of sets $H_{k,y}$ at which, besides (1), we have

$$\tfrac{1}{2}(y-1)\,k < g_+(x) \leqslant \tfrac{1}{2}yk, \tag{2}$$

y denoting any positive integer up to that for which $(y-1)\,k = 2A$. Hence at least one of these sets must have positive content: let y have the least of the values for which this is true. Then there is a perfect sub-set of $H_{k,y}$ which is throughout of positive content; let us call this E.

Let e be any chosen small positive quantity, satisfying the inequality

$$e < \tfrac{1}{2}kE/[(y+1)\,k+2A]. \tag{3}$$

* See the London Mathematical Society paper above quoted, § 2.

† *Ibid.*, § 3.

‡ That is of positive content in every interval containing a point of the set.

§ By the Lemmas of § 4, *loc. cit.*

Now divide the interval (a, b) into a finite number of compartments, namely,

(i) *Black intervals of* E, so chosen that the sum of the remaining black intervals is less than e; and

(ii) *Complementary compartments*, whose sum is accordingly $\geqslant E$, and $< E+e$.

We shall take each of these compartments (ii) separately.

To each point x of the compartment considered, which is a left-hand end-point, or internal point, of a black interval of the set E, we adjoin the part r_x of that black interval on the right of the point x. For each interval r_x then, the *right-hand* end-point belongs to the set E.

At each of the remaining points of the compartment, since it is a point of E, (1) and (2) hold. Therefore

$$g_+(x) < \tfrac{1}{2}yk, \tag{4}$$

$$\tfrac{1}{2}(y+1)k < g^-(x). \tag{5}$$

Therefore we can find intervals $(x, x+h_1)$ and $(x-h_2, x)$ on the right and on the left of x, inside our compartment, and such that

$$g(x+h_1)-g(x) < \tfrac{1}{2}ykh_1, \tag{6}$$

$$\tfrac{1}{2}(y+1)kh_2 < g(x)-g(x-h_2). \tag{7}$$

By Young's First Lemma* we can choose out a finite number of the intervals $(x, x+h_1)$ and r_x, nowhere overlapping, and such that the sum of the complementary intervals, say $t_1, t_2, \ldots, t_m$ filling up the compartment, is less than e/n.

We may assume that both end-points of any one of these intervals r_x *and* t_i *is a point of the set* E. For, if the right-hand end-point of any interval t_i is not a point of the set E, we merely have to suppress the part of this t_i internal to the black interval of E containing the right-hand end-point of this t_i; this is equivalent to replacing one of the chosen r_x's by another r_x. Having done this, every chosen t_i, like every r_x, will have a point of E for its *right-hand* end-point.

If now any r_x or t_i chosen has for left-hand end-point a point x not belonging to E, we only have to add to our chosen r_x's the whole of the

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* "On the Derivates of a Function," *loc. cit.*, p. 368. If preferred, Lebesgue's Lemma, of which I have recently given a proof without Cantor's numbers (*Bull. d. l. Soc. Math. de France*).

black interval of E containing that end-point, suppressing at the same time the r_x, or that part of the t_i in question, which is internal to the newly chosen r_x, as well as any parts of other of the chosen intervals which may encroach on the black interval introduced among them.

Let us do the same in each of the compartments (ii). Then the sum of all the chosen intervals r_x is, by our choice of the compartments (i), less than e, and so is the sum of the intervals t_i, since there are n compartments (ii), and in each the sum of these intervals t_i is $< e/x$.

Now let P_1, p_1 and P denote respectively the sum of the increments of $g(x)$ over the chosen intervals $(x, x+h_1)$, over all the chosen intervals r_x and the intervals t_i, and over the compartments (i).

Then, since all these intervals together form a finite number of abutting intervals reaching from a to b,

$$g(b)-g(a) = P_1+p_1+P.$$

But, by (6),
$$P_1 < \tfrac{1}{2}yk(E+e),$$

since the content of the chosen intervals $(x, x+h_1)$ is not greater than that of the compartments (ii) in which they lie. Also, by (β),

$$0 \leqslant p_1 < 2Ae,$$

since both end-points of each r_x or t_i belong to the set E, and the sum of these intervals is $< 2e$. Hence

$$g(b)-g(a) < \tfrac{1}{2}yk(E+e)+2Ae+P. \tag{8}$$

Similarly, working with the intervals $(x-h_2, x)$ instead of $(x, x+h)$, and with intervals l_x on the left, instead of r_x on the right, and denoting the sum of the increments of $g(x)$ over the chosen intervals $(x-h_2, x)$ by P_2, and over the chosen intervals l_x and the intervals t_i by p_2, we have

$$g(b)-g(a) = P_2+p_2+P.$$

But, by (7),
$$\tfrac{1}{2}(y+1)\,k(E-e) < P_2,$$

since these intervals contain all the points of E, except a sub-set of content $< e$.

Hence since, as we saw for p_1,

$$0 \leqslant p_2,$$

we get
$$\tfrac{1}{2}(y+1)\,k(E-e)+P < g(b)-g(a). \tag{9}$$

Combining (8) and (9),

$$\tfrac{1}{2}(y+1)\,k(E-e) < \tfrac{1}{2}yk(E+e)+2Ae,$$

whence, *a fortiori*, $\frac{1}{2}kE < e[(y+1)k+2A]$,

which is in contradiction with (3).

Thus our assumption is untenable.

This proves the theorem.

2. The second theorem is as follows:—

If $f(x)$ is a finite measurable function, the points, if any, at which one of the upper derivates and one of the lower derivates are finite and different from one another, form a set of content zero.

By the preceding theorem this is true if the derivates are one on one side and one on the other. It only remains to discuss the case when they are both on the same side, say $f^+(x)$ and $f_+(x)$. Then, by the preceding theorem,

$$\left.\begin{aligned} f^+(x) &= f_-(x), \\ f_+(x) &= f^-(x), \end{aligned}\right\} \quad [P.\,p.]$$

and therefore, since $f^+(x) \geqslant f_+(x)$ and $f_-(x) \leqslant f^-(x)$,

$$f^+(x) = f_+(x), \quad [P.\,p.]$$

which proves the theorem.

3. We have hitherto assumed, except in Theorem 1, that $f(x, y)$ was a finite function. Using, however, the more general results of my former communication, we see that not only Theorem 1, but also Theorem 2 and Theorems $4a$ and $4b$, remain true when $f(x)$ is infinite at certain points, while Theorem 2 takes the following form:—

THEOREM 2 *bis*.—*The points at which $f(x, y)$ has an infinite partial forward or backward differential coefficient, with determinate sign, consist of the infinities of $f(x, y)$ and possibly an additional set of plane content zero, whose section by y = constant is a set of zero linear content.*

The points at which $f(x, y)$ has an infinite partial differential coefficient with determinate sign, form a set of plane content zero, whose section by y = constant is a set of zero linear content.

THE PRODUCT OF TWO HYPERGEOMETRIC FUNCTIONS

By G. N. WATSON.

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IT is possible to establish a relation which connects the product of two hypergeometric functions

$$F(a, \beta;\ \gamma;\ z) \times F(a, \beta;\ \gamma;\ Z)$$

with the hypergeometric function of two variables of Appell's fourth type

$$F_4[a, \beta;\ \gamma,\ a+\beta-\gamma+1;\ zZ,\ (1-z)(1-Z)].$$

The reader will remember that the definition* of Appell's function is

$$F_4[a, \beta;\ \gamma, \gamma';\ \xi, \eta] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(\beta)_{m+n}}{(\gamma)_m (\gamma')_n\, m!\, n!} \xi^m \eta^n,$$

where a symbol of the form $(a)_m$ denotes

$$a(a+1)(a+2) \dots (a+m-1).$$

In the special case in which $z = Z$, the existence of the relation has been indicated to a certain extent by Appell† himself, for he has shown that $\{F(a, \beta;\ \gamma;\ Z)\}^2$ and $F_4[a, \beta;\ \gamma,\ a+\beta-\gamma+1;\ Z^2, (1-Z)^2]$ are solutions of the same linear differential equation of the third order.

The more general case in which z and Z are unequal, which is the subject of this paper, would appear to give the best theorem concerning the expression of functions of the fourth type in terms of hypergeometric functions, just as Appell's theorem‡ that

$$F_1(a;\ \beta, \gamma-\beta;\ \gamma;\ X, Y) = (1-Y)^{-a} F\left(a, \beta;\ \gamma;\ \frac{X-Y}{1-Y}\right)$$

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* *Comptes Rendus*, t. 90 (1880), pp. 296, 731.

† *Journal de Math.*, Sér. 3, t. 10 (1884), pp. 418–421.

‡ *Journal de Math.*, Sér. 3, t. 8 (1882), p. 175; see also Barnes, *Proc. London Math. Soc.* Ser. 2, Vol. 6 (1908), p. 169.

is, in all probability, the best theorem concerning functions of the first type.

In this paper I propose to establish the general relation with the aid of contour integrals of Barnes' types, after considering the special case of the relation in which a is a negative integer, so that the hypergeometric series reduce to polynomials. The fact that the relation is of a somewhat abstruse character is indicated by the impracticability of proving it in a simple manner in the special case without making use of infinite series.

The importance of the relation arises from its existence, and not from the methods used in proving it, for the proof requires only a certain amount of analytical ingenuity. I may state that the method by which I discovered the relation was a consideration of various types of normal solutions of the wave-equation in four dimensions which have been the subject of a paper by Bateman.*

2. When a is a negative integer $-n$, the relation to be proved assumes the simple form†

$$F(-n,\beta+n;\gamma;z)\times F(-n,\beta+n;\gamma;Z)$$
$$=(-)^n\frac{(\beta-\gamma+1)_n}{(\gamma)_n}F_4[-n,\beta+n;\gamma,\beta-\gamma+1;zZ,(1-z)(1-Z)].$$

To prove the relation, we transform the expression on the right in the following manner, using Vandermonde's theorem in the fifth and sixth lines of the analysis:

$$(1-Z)^{\beta-\gamma}F_4[-n,\beta+n;\gamma,\beta-\gamma+1;zZ,(1-z)(1-Z)]$$
$$=\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(\beta+n)_{r+s}}{(\gamma)_r(\beta-\gamma+1)_s r!\,s!}z^rZ^r(1-z)^s(1-Z)^{\beta-\gamma+s}$$
$$=\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(\beta+n)_{r+s}}{(\gamma)_r r!}\sum_{k=0}^{s}\frac{(-)^k z^{r+k}}{k!\,(s-k)!}\sum_{p=0}^{\infty}\frac{(-)^p Z^{r+p}}{p!\,(\beta-\gamma+1)_{s-p}}$$
$$=\sum_{r=0}^{n}\sum_{s=0}^{n-r}\sum_{l=r}^{r+s}\sum_{q=r}^{\infty}\frac{(-n)_{r+s}(\beta+n)_{r+s}(-)^{l+q}z^lZ^q}{(\gamma)_r(\beta-\gamma+1)_{r+s-q}r!\,(l-r)!\,(r+s-l)!\,(q-r)!}$$
$$=\sum_{l=0}^{n}\sum_{q=0}^{\infty}\sum_{r=0}^{q}\sum_{s=l-r}^{n-r}\frac{(-n)_{r+s}(\beta+n)_{r+s}(-)^{l+q}z^lZ^q}{(\gamma)_r(\beta-\gamma+1)_{r+s-q}r!\,(l-r)!\,(r+s-l)!\,(q-r)!}$$

* *Proc. London Math. Soc.*, Ser. 2, Vol. 3 (1905), pp. 111–123.

† It is convenient to take $\beta+n$ as the second element in order to retain the usual notation for Jacobi's polynomials.

$$= \sum_{l=0}^{n} \sum_{q=0}^{\infty} \sum_{r=0}^{q} \frac{(-n)_l\,(\beta+n)_l\,(\gamma+l+q)_{n-l}\,(-)^{n+q}\,z^l Z^q}{(\gamma)_r\,(\beta-\gamma+1)_{n-q}\,r!\,(l-r)!\,(q-r)!}$$

$$= \sum_{l=0}^{n} \sum_{q=0}^{\infty} \frac{(-n)_l\,(\beta+n)_l}{(\gamma)_l\,l!}\,z^l\,\frac{(\gamma+q)_n\,(\gamma-\beta)_{q-n}}{q!}\,Z^q$$

$$= (-)^n \frac{(\gamma)_n}{(\beta-\gamma+1)_n} F(-n,\,\beta+n\,;\,\gamma\,;\,z) \times F(\gamma-\beta-n,\,\gamma+n;\,\gamma;\,Z)$$

$$= (-)^n \frac{(\gamma)_n}{(\beta-\gamma+1)_n} (1-Z)^{\beta-\gamma} F(-n,\,\beta+n\,;\,\gamma;\,z) \times F(-n,\,\beta+n;\,\gamma;\,Z),$$

by a well-known transformation of hypergeometric functions; and this establishes the stated relation.

3. In order to establish one form of the general relation let us consider the expression

$$(1-Z)^{a+\beta-\gamma} \left(\frac{1}{2\pi i}\right)^2 \int_{-\infty i}^{\infty i} \int_{-\infty i}^{\infty i} \Gamma(a+s+t)\,\Gamma(\beta+s+t)\,\Gamma(1-\gamma-s) \times \Gamma(\gamma-a-\beta-t)\,\Gamma(-s)\,\Gamma(-t)\,\{zZ\}^s\,\{(1-z)(1-Z)\}^t\,ds\,dt,$$

which is an absolutely convergent double integral, provided that

$$|\arg\{zZ\}| < 2\pi, \quad |\arg\{(1-z)(1-Z)\}| < 2\pi,$$

and it is supposed* that the contours have loops, if necessary, to ensure that the points $-a$, $-a-1$, $-a-2$, ..., $-\beta$, $-\beta-1$, $-\beta-2$, ... lie on the left of the s-contour, and the other poles of the integrand lie on the right of the contours.

We now define $-z$ and $-Z$ by the equations

$$-z = ze^{\pm\pi i}, \quad -Z = Ze^{\pm\pi i},$$

where $\quad |\arg(-z)| < \pi, \quad |\arg(-Z)| < \pi,$

and then $\quad |\arg(zZ)| = |\arg(-z)+\arg(-Z)| < 2\pi.$

* It simplifies the argument if it is first supposed that a, β, γ, $\gamma-a-\beta$ have positive real parts, and then at the end of the reasoning to use the theory of analytic continuation to remove these restrictions.

In the double integral we now make use of the formulæ

$$\Gamma(-t)(1-z)^t = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(-\phi)\,\Gamma(\phi-t)(-z)^{\phi}\,d\phi,$$

$$\Gamma(\gamma-a-\beta-t)(1-Z)^{a+\beta-\gamma+t}$$
$$= \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(-\psi)\,\Gamma(\psi-a-\beta+\gamma-t)(-Z)^{\psi}\,d\psi,$$

whence it follows that the double integral is equal to

$$\left(\frac{1}{2\pi i}\right)^4 \int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i} \Gamma(a+s+t)\,\Gamma(\beta+s+t)\,\Gamma(1-\gamma-s)\,\Gamma(-s)$$
$$\times\Gamma(-\phi)\,\Gamma(-\psi)\,\Gamma(\phi-t)\,\Gamma(\psi-a-\beta+\gamma-t)(-z)^{s+\phi}\,(-Z)^{s+\psi}\,d\phi\,d\psi\,ds\,dt$$
$$= \left(\frac{1}{2\pi i}\right)^4 \int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i} \Gamma(a+s+t)\,\Gamma(\beta+s+t)\,\Gamma(1-\gamma-s)\,\Gamma(-s)$$
$$\times\Gamma(s-\phi)\Gamma(s-\psi)\,\Gamma(\phi-s-t)\Gamma(\psi-a-\beta+\gamma-s-t)(-z)^{\phi}(-Z)^{\psi}$$
$$d\phi\,d\psi\,ds\,dt$$
$$= \left(\frac{1}{2\pi i}\right)^3 \int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i} \Gamma(1-\gamma-s)\,\Gamma(-s)\,\Gamma(s-\phi)\,\Gamma(s-\psi)$$
$$\times\frac{\Gamma(a+\phi)\,\Gamma(\beta+\phi)\,\Gamma(\psi+\gamma-a)\,\Gamma(\psi+\gamma-\beta)}{\Gamma(\phi+\psi+\gamma)}(-z)^{\phi}\,(-Z)^{\psi}\,ds\,d\phi\,d\psi$$
$$= \left(\frac{1}{2\pi i}\right)^2 \int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i} \Gamma(-\phi)\,\Gamma(-\psi)\,\Gamma(1-\gamma-\phi)\,\Gamma(1-\gamma-\psi)$$
$$\times\Gamma(a+\phi)\Gamma(\beta+\phi)\,\Gamma(\psi+\gamma-a)\,\Gamma(\psi+\gamma-\beta)$$
$$\times\frac{\sin(\phi+\psi+\gamma)\pi}{\pi}(-z)^{\phi}(-Z)^{\psi}\,d\phi\,d\psi.$$

In each of the last two lines, Barnes' lemma,* that

$$\frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(a_1+w)\,\Gamma(a_2+w)\,\Gamma(\beta_1-w)\,\Gamma(\beta_2-w)\,dw$$
$$= \frac{\Gamma(a_1+\beta_1)\,\Gamma(a_2+\beta_2)\,\Gamma(a_2+\beta_1)\,\Gamma(a_2+\beta_2)}{\Gamma(a_1+a_2+\beta_1+\beta_2)},$$

has been used.

We now evaluate the initial and final integrals by calculating the residues at the poles on the right of the contours, and after dividing by

* *Proc. London Math. Soc.*, Ser. 2, Vol. 6 (1908), pp. 154, 155.

$(1-Z)^{\alpha+\beta-\gamma}$, we find that

$$\Gamma(\alpha)\,\Gamma(\beta)\,(1-\gamma)\,\Gamma(\gamma-\alpha-\beta)\,F_4[\alpha, \beta;\ \gamma, \alpha+\beta-\gamma+1;\ zZ, (1-z)(1-Z)]$$
$$+(zZ)^{1-\gamma}\,\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)\,\Gamma(\gamma-1)\,\Gamma(\gamma-\alpha-\beta)$$
$$\times F_4[\alpha-\gamma+1, \beta-\gamma+1;\ 2-\gamma, \alpha+\beta-\gamma+1;\ zZ, (1-z)(1-Z)]$$
$$+\{(1-z)(1-Z)\}^{\gamma-\alpha-\beta}\,\Gamma(\gamma-\beta)\,\Gamma(\gamma-\alpha)\,\Gamma(1-\gamma)\,\Gamma(\alpha+\beta-\gamma)$$
$$\times F_4[\gamma-\beta, \gamma-\alpha;\ \gamma, \gamma-\alpha-\beta+1;\ zZ, (1-z)(1-Z)]$$
$$+(zZ)^{1-\gamma}\{(1-z)(1-Z)\}^{\gamma-\alpha-\beta}\,\Gamma(1-\beta)\,\Gamma(1-\alpha)\,\Gamma(\gamma-1)\,\Gamma(\alpha+\beta-\gamma)$$
$$\times F_4[1-\beta, 1-\alpha;\ 2-\gamma, \gamma-\alpha-\beta+1;\ zZ, (1-z)(1-Z)]$$
$$=\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\gamma)}\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)\,\Gamma(1-\gamma)\,F(\alpha, \beta;\ \gamma;\ z)$$
$$\times(1-Z)^{\gamma-\alpha-\beta}\,F(\gamma-\beta, \gamma-\alpha;\ \gamma;\ Z)$$
$$+(zZ)^{1-\gamma}\,\frac{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)}{\Gamma(2-\gamma)}\,\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(\gamma-1)$$
$$\times F(\alpha-\gamma+1, \beta-\gamma+1;\ 2-\gamma;\ z)\,(1-Z)^{\gamma-\alpha-\beta}\,F(1-\beta, 1-\alpha;\ 2-\gamma;\ Z)$$
$$=\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\gamma)}\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)\,\Gamma(1-\gamma)\,F(\alpha, \beta;\ \gamma;\ z)\,F(\alpha, \beta;\ \gamma;\ Z)$$
$$+(zZ)^{1-\gamma}\,\frac{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)}{\Gamma(2-\gamma)}\,\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(\gamma-1)$$
$$\times F(\alpha-\gamma+1, \beta-\gamma+1;\ 2-\gamma;\ z)\,F(\alpha-\gamma+1, \beta-\gamma+1;\ 2-\gamma;\ Z),$$

and this is an equation of the specified type.

4. If we had dealt in a similar manner with the integral

$$(1-Z)^{\alpha+\beta-\gamma}\left(\frac{1}{2\pi i}\right)^2\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\frac{\Gamma(\alpha+s+t)\,\Gamma(\beta+s+t)}{\Gamma(\gamma+s)}\,\Gamma(-s)\,z^s\,(-Z)^s$$
$$\times\Gamma(-t)\,\Gamma(\gamma-\alpha-\beta-t)\,\{(1-z)(1-Z)\}^t\,ds\,dt,$$

which is convergent when

$$|\arg z+\arg(-Z)|<\pi, \quad |\arg(1-z)+\arg(1-Z)|<2\pi,$$

we should have found it equal to

$$\left(\frac{1}{2\pi i}\right)^3\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\frac{\Gamma(-s)\,e^{\mp s\pi i}}{\Gamma(\gamma+s)}\,\Gamma(s-\phi)\,\Gamma(s-\psi)$$
$$\times\frac{\Gamma(\alpha+\phi)\,\Gamma(\beta+\phi)\,\Gamma(\psi+\gamma-\alpha)\,\Gamma(\psi+\gamma-\beta)}{\Gamma(\phi+\psi+\gamma)}\,(-z)^\phi\,(-Z)^\psi\,ds\,d\phi\,d\psi,$$

and, when $R(\gamma)$ is positive, this is equal to

$$\left(\frac{1}{2\pi i}\right)^2 \int_{-\infty i}^{\infty i} \int_{-\infty i}^{\infty i} \frac{\Gamma(-\phi)\,\Gamma(-\psi)}{\Gamma(\gamma+\phi)\,\Gamma(\gamma+\psi)}\,\Gamma(a+\phi)\,\Gamma(\beta+\phi)$$
$$\times\Gamma(\psi+\gamma-a)\,\Gamma(\psi+\gamma-\beta)(-z)^{\phi}(-Z)^{\psi}\,d\phi\,d\psi.$$

If we calculate the residues of the initial and final integrals at the poles on the right of the contours, and then divide by $(1-Z)^{a+\beta-\gamma}$, we find that

$$\frac{\Gamma(a)\,\Gamma(\beta)}{\Gamma(\gamma)}\Gamma(\gamma-a-\beta)\,F_4[a,\beta;\gamma,a+\beta-\gamma+1;zZ,(1-z)(1-Z)]$$
$$+\{(1-z)(1-Z)\}^{\gamma-a-\beta}\frac{\Gamma(\gamma-a)\,\Gamma(\gamma-\beta)}{\Gamma(\gamma)}\Gamma(a+\beta-\gamma)$$
$$\times F_4[\gamma-\beta,\gamma-a;\gamma,\gamma-a-\beta+1;zZ,(1-z)(1-Z)]$$
$$=\frac{\Gamma(a)\,\Gamma(\beta)\,\Gamma(\gamma-a)\,\Gamma(\gamma-\beta)}{\{\Gamma(\gamma)\}^2}F(a,\beta;\gamma;z)\,F(a,\beta;\gamma;Z),$$

and the restriction that $R(\gamma)>0$ may now be removed by the theory of analytic continuation.

Since

$$F_4(a,\beta;\gamma,\gamma';\xi,\eta)=\sum_{m=0}^{\infty}\frac{(a)_m\,(\beta)_m}{(\gamma)_m\,m!}\xi^m F(a+m,\beta+m;\gamma';\eta),$$

the last result may be written in the form

$$\frac{\Gamma(a)\,\Gamma(\beta)\,\Gamma(\gamma-a)\,\Gamma(\gamma-\beta)}{\{\Gamma(\gamma)\}^2}F(a,\beta;\gamma;z)\,F(a,\beta;\gamma;Z)$$
$$=\Gamma(\gamma-a-\beta)\sum_{m=0}^{\infty}\frac{\Gamma(a+m)\,\Gamma(\beta+m)}{\Gamma(\gamma+m)\,m!}(zZ)^m F[a+m,\beta+m,a+\beta-\gamma+1;(1-z)(1-Z)]$$
$$+\Gamma(a+\beta-\gamma)\sum_{m=0}^{\infty}\frac{\Gamma(\gamma-\beta+m)\,\Gamma(\gamma-a+m)}{\Gamma(\gamma+m)\,m!}(zZ)^m\{(1-z)(1-Z)\}^{\gamma-a-\beta}$$
$$\times F[\gamma-\beta+m,\gamma-a+m,\gamma-a-\beta+1;(1-z)(1-Z)].$$

If we combine corresponding terms of the series on the right, we find that they are expressible in terms of

$$F(a+m,\beta+m,\gamma+2m,z+Z-zZ),$$

so that we finally get

$$\frac{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}{\{\Gamma(\gamma)\}^2} F(\alpha, \beta;\ \gamma;\ z)\, F(\alpha, \beta;\ \gamma;\ Z)$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma(\alpha+m)\,\Gamma(\beta+m)\,\Gamma(\gamma-\alpha+m)\,\Gamma(\gamma-\beta+m)}{\Gamma(\gamma+m)\,\Gamma(\gamma+2m)\, m!} (zZ)^m$$

$$\times F(\alpha+m,\ \beta+m;\ \gamma+2m;\ z+Z-zZ).$$

We can therefore express the product as a double series, thus

$$F(\alpha, \beta;\ \gamma;\ z) \times F(\alpha, \beta;\ \gamma;\ Z)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}\,(\beta)_{m+n}\,(\gamma-\alpha)_m\,(\gamma-\beta)_m}{(\gamma)_m\,(\gamma)_{2m+n}\, m!\, n!} (zZ)^m (z+Z-zZ)^n.$$

The series on the right is not one of Appell's functions as it stands, but, as we have seen, it is expressible in terms of two functions of Appell's fourth type.

DIFFUSION BY CONTINUOUS MOVEMENTS

By G. I. Taylor.

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Introduction.

It has been shown by the author,* and others, that turbulent motion is capable of diffusing heat and other diffusible properties through the interior of a fluid in much the same way that molecular agitation gives rise to molecular diffusion. In the case of molecular diffusion the relationship between the rate of diffusion and the molecular constants is known; a large part of the Kinetic Theory of Gases is devoted to this question. On the other hand, nothing appears to be known regarding the relationship between the constants which might be used to determine any particular type of turbulent motion and its "diffusing power."

The propositions set down in the following pages are the result of efforts to solve this problem.

In order to simplify matters as much as possible the transmission of heat in one direction only, that of the axis of x, will be considered. We shall take the case of an incompressible fluid whose temperature θ, at time $t = 0$, depends only on x, and increases or decreases uniformly with x. Initially therefore $\partial\theta/\partial x$ is constant and equal to β, say.

Now suppose that the fluid is moving in turbulent motion, so that the distribution of temperature is continually altering. Suppose that the turbulent motion could be defined by means of the Lagrangian equations of fluid motion, so that the coordinates (x, y, z) of a particle are given in terms of its initial coordinates (a, b, c) at the time $t = 0$, and of t.

Since the temperature of any particle is supposed to remain constant during the motion, the temperature at the point (x, y, z) at time t, which will be represented by the symbol $\theta\,(x, y, z)$ is $\theta\,(a, 0)$, which represents the temperature at $x = a$ at time $t = 0$.

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* "Eddy Motion in the Atmosphere," *Phil. Trans.*, 1915, p. 1.

Since the rate of increase in temperature with x is constant when $t = 0$,

$$\theta(a, 0) = \theta(x, 0) - (x-a)\beta.$$

The average rate at which heat is being conveyed across unit area of a plane perpendicular to the axis of x is evidently equal to $-\rho\sigma\beta$ multiplied by the average value of $u(x-a)$ over a large area of a plane perpendicular to the axis of x. In these expressions u represents the velocity of a particle of fluid in the direction of the axis of x, ρ is the density, and σ the specific heat, so that $\rho\sigma$ is the heat capacity of unit volume of the fluid.

No doubt the average value of $u(x-a)$, which must be obtained from considerations of the particular nature of the turbulent motion in question, depends on the mean motion of the fluid; but if experimental data exist, as in fact they do, which enable its value to be calculated, it is of interest to enquire what types of turbulent motion are capable of producing the observed distribution of temperature.

In order to simplify matters still further it will be assumed that the turbulent motion is uniformly distributed throughout space. The mean value of $u(x-a)$ will then be the same for every layer and will be equal to the mean value throughout space. This quantity will be expressed by the symbol $[u(x-a)]$.

Owing to the fact that the fluid is incompressible $[u(x-a)]$ could be calculated either by taking a rectangular element $\delta x\,\delta y\,\delta z$, at time t, finding the corresponding value of $u(x-a)$ and integrating throughout space; or by taking an element $\delta a\,\delta b\,\delta c$ at time $t = 0$, finding the corresponding value of $u(x-a)$ at time t, and integrating. The second method will be adopted.

Fixing our attention on a particle of fluid, it will be noticed that

$$u = \frac{\partial x}{\partial t} \quad \text{and} \quad x-a = \int_0^t u\,dt.$$

Hence, writing X for $x-a$,

$$[u(x-a)] = \left[X\frac{dX}{dt}\right] = \tfrac{1}{2}\left[\frac{dX^2}{dt}\right] = \tfrac{1}{2}\frac{d}{dt}[X^2].$$

In this ideally simplified system therefore the rate at which heat is transferred in the direction of the axis of x is determined by the rate of increase of the mean value of the square of the distance, parallel to the axis of x, which is moved through by a particle of fluid in time t.

If a physicist were to try to define the characteristic features of any particular case of turbulent motion, with a view to discussing statistically

its effect as a conductor of heat, he would probably first fix his attention on the mean energy of the motion. That is to say, he would determine $[u^2]$.

He would then perhaps notice that it is not sufficient to determine $[u^2]$. With a given value of $[u^2]$ it is possible for the turbulent motion to be associated with a small or a large transfer of heat, according to whether a particle frequently, or infrequently, reverses its direction of motion. It would therefore be necessary to define some characteristic of the motion which differentiates between the cases in which the changes in the velocity of a particle are rapid, and those in which they are slow. A suitable characteristic to choose would be $\left[\left(\frac{du}{dt}\right)^2\right]$.

Further investigation would show that it is necessary also to define

$$\left[\left(\frac{d^2u}{dt^2}\right)^2\right], \ldots, \left[\left(\frac{d^nu}{dt^n}\right)^2\right], \ldots .$$

The relationship between

$$\tfrac{1}{2}\frac{d}{dt}[X^2] \text{ and } [u^2], \quad \left[\left(\frac{du}{dt}\right)^2\right], \ldots, \left[\left(\frac{d^nu}{dt^n}\right)^2\right], \ldots$$

is discussed in the following pages. The problem is in some respects similar to that known as "The drunkard's walk," or to Karl Pearson's* problem of the random migration of insects, when the motion is limited to one dimension; but in the course of the investigation some curious propositions have come to light concerning the mean values of continuously varying quantities which may perhaps be of interest to mathematicians, as well as to physicists.

In the course of the work no discussion of the convergency of the series used is attempted. The work must therefore be regarded as incomplete. The author feels that such questions might be examined with advantage by a pure mathematician, and it is in the hope of interesting one of them that he wishes to offer this paper to the London Mathematical Society.

Discontinuous Motion.

Before proceeding to consider the continuous version of the problem of random migration in one dimension, the discontinuous case will be

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* Drapers' Company *Memoirs*.

extended slightly, so as to make it bear some resemblance to the continuous case.

Suppose that a point starts moving with uniform velocity v along a line, and that after a time τ it suddenly makes a fresh start and either continues moving forward with velocity v or reverses its direction and moves back over the same path with the same velocity v. Suppose that this process is repeated n times and that we consider the mean values of the quantity concerned for a very large number of such paths.

Let x_r be the distance moved over in the r-th interval. Then x_r is numerically equal to $v\tau$, but its sign may be either positive or negative and each occurs an equal number of times in considering the average. If X_n is the standard deviation or "root mean square" of the distance moved by the point from the original position after time $n\tau$, then

$$X_n^2 = [(x_1+x_2+x_3+\ldots+x_n)^2],$$

where the square bracket indicates that the mean value is taken for all the paths.

$$\text{Hence} \qquad X_n^2 = nd^2+2[x_1x_2+x_1x_3+\ldots+x_rx_s+\ldots], \tag{1}$$

$$\text{where} \qquad d = v\tau.$$

If there is no correlation between any two x's,

$$[x_rx_s] = 0.$$

$$\text{Hence} \qquad X_n^2 = nd^2, \quad \text{or} \quad X_n = d\sqrt{n} = v\sqrt{(\tau T_n)},$$

where T_n is the total time during which the migration has been taking place. It will be seen therefore that X_n is proportional to $\sqrt{T_n}$.

Actually in a turbulent fluid or in any continuous motion there is necessarily a correlation between the movement in any one short interval of time and the next. This correlation will evidently increase as the interval of time diminishes, till, when the time is short compared with the time during which a finite change in velocity takes place, the coefficient of correlation tends to the limiting value unity.

This idea will now be introduced into equation (1).

To begin with let us make the arbitrary assumption that x_r is correlated with x_{r+1} by a correlation coefficient c. Suppose also that the partial correlations of x_r with $x_{r+2}, x_{r+3}, \ldots$ are all zero. The correlation coefficient between x_r and x_{r+2} is then c^2. Between x_r and x_{r+s} it is c^s.

The value of $2[x_1x_2+x_1x_3+\ldots+x_rx_s+\ldots]$ is then

$$2d^2\{nc+(n-1)c^2+(n-2)c^3+\ldots+c^n\}.$$

The series in the { } bracket is easily summed. Substituting its value in (1) it will be found that

$$X_n^2 = d^2 \left\{ n + \frac{2nc}{1-c} - \frac{2c^2(1-c^n)}{(1-c)^2} \right\},$$

or, putting $n = T_n/\tau$, and $d = v\tau$,

$$X_n^2 = v^2 \left\{ \left(\frac{1+c}{1-c}\tau\right) T_n - \frac{2c^2(1-c^n)\tau^2}{(1-c)^2} \right\}. \tag{2}$$

By reducing τ indefinitely we can evidently make the case approximate to some sort of continuous migration, but in order that X_n, v and T_n may be finite and tend to a definite limit as τ is decreased, it is necessary that $\left(\frac{1+c}{1-c}\right)\tau$ and $\frac{2c^2(1-c^n)\tau^2}{(1-c)^2}$ must also tend to a definite limit. That is to say, $1-c$ must be proportional to τ.

Let $\frac{\tau}{1-c}$ tend to the limit A when τ and $1-c$ tend to zero.

Then X_n^2 tends to the limiting value

$$v^2 \{2AT_n - 2A^2(1-e^{-T_n/A})\},$$

or, dropping the suffixes which are no longer necessary,

$$\sqrt{[X^2]} = v\sqrt{\{2AT - 2A^2(1-e^{-T/A})\}}, \tag{3}$$

where X is the distance traversed by a particle during a flight extending over an interval of time T, and the "root mean square" is taken for a large number of such flights.

When T is small this reduces to $\sqrt{[X^2]} = vT$, which is exactly what we should expect when the time is so short that the correlation coefficient c^n, between the first and last small element of migration has not fallen appreciably away from unity.

When T is large $\sqrt{[X^2]} = v\sqrt{(2AT)}$, so that the amount of "diffusion" is proportional to the square root of the time. The constant A evidently measures the rate at which the correlation coefficient between the direction of an infinitesimal path in the migration and that of an infinitesimal path at a time T, say, later, falls off with increasing values of T.

We have now seen how it is possible by introducing the idea of a correlation between the directions of the successive jumps in a random migration, to keep the standard deviation of the distance of migration constant, no matter how small the infinitesimal paths of the migration may be.

The migration is still a discontinuous one however. It suffers also

from the disadvantage of depending on a special assumption, namely, that there is a definite correlation between the direction of motion in one infinitesimal element of path, and that in its immediate neighbours, but that there is no partial correlation between the directions of motion in paths which are not neighbours. This means that there is a special law of correlation between the directions of the paths at finite intervals of time. The correlation coefficient between the direction of an infinitesimal path and that of the path which occurs at a time $T = n\tau$ later, is evidently c^n. This may be written

$$\{1-(1-c)\}^n = (1-\tau/A)^n = (1-\tau/A)^{T/\tau}.$$

When τ is small this tends to the limit $e^{-T/A}$. (4)

We are therefore limiting ourselves to the particular type of motion in which the direction of an infinitesimal path is correlated to that at time T later by the correlation coefficient* $e^{-T/A}$.

Diffusion by continuous Movements.

The work just described, though not particularly useful for our present purpose, is useful in that it gives rise to ideas about how problems of migration or diffusion by continuous movements may be treated. In what follows these ideas are worked out and the conditions of motion which determine the laws of diffusion are found.

Before proceeding to discuss diffusion, however, it will be necessary to prove a few statistical properties of continuously varying quantities.

Suppose that we wish to express the characteristic properties of the variations of some quantity which varies continuously, but which appears to have no very definite law of variation. Suppose, for instance, it is desired to define the characteristic features of a barograph record. There are no obvious periods, nor is there any definite constant amplitude of variation in barometric pressure, yet there are certain properties of the curve which can be defined. If we take the standard deviation of pressure from its mean value during a year, it will be found to be practically constant from year to year. If p represents the deviation from the mean pressure, this standard deviation is $\sqrt{[p^2]}$, where the square bracket now indicates that the mean value of p^2 has been taken over a long period of

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* Incidentally it will be noticed that the correlation between the direction of motion at one instant and that at time *t earlier* is also $e^{-T/A}$. It is obvious that we cannot consider the value of the expression $e^{-T/A}$ when T is negative.

time. One property of the curve which we can define, therefore, is the constancy of $\sqrt{[p^2]}$ during successive long periods.

The statistical properties of the barograph curve are by no means completely determined by this. It is possible to imagine an infinite variety of barograph curves with a given standard deviation of p. They might, for instance, have a large number of peaks in the curve during a given interval of time or a small number. In the former case the standard deviation of dp/dt might be expected to be larger than in the latter. We can, therefore, define the curve still further by specifying the standard deviations of dp/dt.

It appears that, from a given barograph curve, it is theoretically possible to find the standard deviations of p, dp/dt, d^2p/dt^2, ..., d^np/dt^n, ... Let us assume that all these are constant from year to year.

Now suppose that we begin by specifying certain arbitrary standard deviations for p, dp/dt, &c., and that we try to construct a possible barograph curve from them. We are at once brought up against a difficulty. Suppose that we have specified a large number for the standard deviation of dp/dt, *i.e.* $\sqrt{[(dp/dt)^2]}$ and small numbers for $\sqrt{[p^2]}$ and $\sqrt{[(d^2p/dt^2)^2]}$. It is evident that if we begin constructing the curve with a large value of dp/dt at a point where $p = 0$, the fact that the value of $\sqrt{[(d^2p/dt^2)]}$ is small means that it will be a long time before dp/dt changes sign. Hence it will be a long time before p attains its maximum value, and during that time p must have attained a large value. Hence, if the standard deviation of dp/dt is large and that of d^2p/dt^2 is small, the standard deviation of p must be large. It is evident therefore that there must be some relationships between the standard deviations and the curve of which we have not yet taken account. We shall now see what these are.

Suppose that we observe the values $p_1, p_2, p_3, \ldots, p_n$ of p at a large number of successive times $t_1, t_2, t_3, \ldots, t_n$. Suppose further that we observe the values $p_1+\delta p_1$, $p_2+\delta p_2$, $p_3+\delta p_3$, ..., $p_n+\delta p_n$, at times $t_1+\delta t$, $t_2+\delta t$, $t_3+\delta t$, ..., $t_n+\delta t$, where δt is a small interval of time. Then, if $t_1, t_2, \ldots, t_n$ are taken at random

$$[p^2] = (p_1^2+p_2^2+\ldots+p_n^2)/n,$$

and since we are considering a curve in which $[p^2]$ is constant, $[p^2]$ is also, to the first order, equal to

$$\frac{1}{n}\left\{\left(p_1+\frac{dp_1}{dt}\,\delta t\right)^2+\left(p_2+\frac{dp_2}{dt}\,\delta t\right)^2+\ldots+\left(p_n+\frac{dp_n}{dt}\,\delta t\right)^2\right\}$$

$$=[p^2]+2\left[p\,\frac{dp}{dt}\right]\delta t.$$

It appears, therefore, that we can differentiate the quantities inside square brackets which indicate a mean value.

Hence the condition that $[p^2]$ shall be a constant is

$$\left[p\frac{dp}{dt}\right]=0. \tag{5}$$

There is, therefore, no correlation between p and dp/dt.

Now differentiate (5) once more,

$$\left[p\frac{d^2p}{dt^2}\right]+\left[\left(\frac{dp}{dt}\right)^2\right]=0. \tag{6}$$

Hence by the definition of a correlation coefficient, there is a negative correlation between p and d^2p/dt^2 equal to

$$\nu=-\frac{\left[\left(\frac{dp}{dt}\right)^2\right]}{\sqrt{[p^2]}\sqrt{\left[\left(\frac{d^2p}{dt^2}\right)^2\right]}}. \tag{7}$$

A consequence of the existence of this correlation coefficient ν is evidently that $[(dp/dt)^2]$ cannot be greater than $\sqrt{[p^2]}\sqrt{[(d^2p/dt^2)^2]}$, a statement which agrees with the remarks above.

The way in which the correlation coefficient affects the characteristic features of the p, t curve is easily seen. Suppose it is large, *i.e.* nearly equal to -1; then the curve will look something like curve (a), Fig. 1.

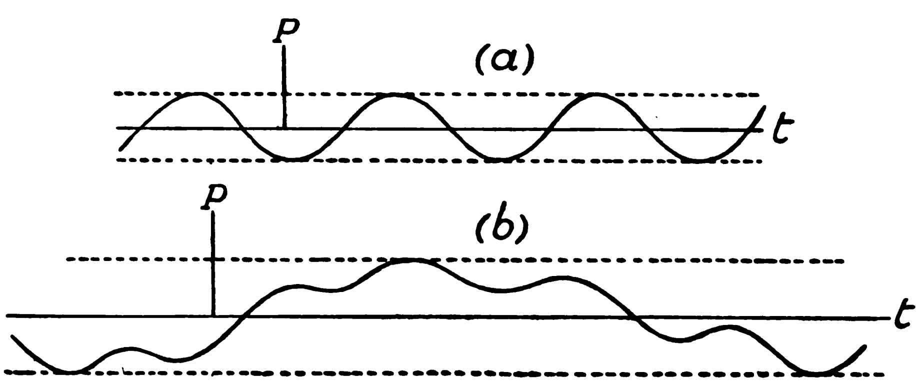

FIG. 1.

Suppose the correlation coefficient between p and d^2p/dt^2 is small, but that the standard deviations of dp/dt and d^2p/dt^2 are the same as in curve (a), Fig. 1, then the slopes and curvatures will be of the same magnitude

as in curve (*a*), but the curvature will not always be concave to the mean line. This is shown in curve (*b*), Fig. 1.

It is evident that the standard deviation of p is greater in (*b*) than it is in (*a*). This is expressed by the formula (7), for if the standard deviations of dp/dt and d^2p/dt^2 are fixed, then the standard deviation of p is, according to (7), inversely proportional to ν.

Since the standard deviation of dp/dt has also been given as constant it can be treated exactly in the same way as the standard deviation of p, thus differentiating $[(dp/dt)^2]$, we have

$$\left[\frac{dp}{dt}\,\frac{d^2p}{dt^2}\right] = 0, \tag{8}$$

and differentiating this again

$$\left[\frac{dp}{dt}\,\frac{d^3p}{dt^3}\right]+\left[\left(\frac{d^2p}{dt^2}\right)^2\right] = 0. \tag{9}$$

But differentiating (6) again

$$\left[p\,\frac{d^3p}{dt^3}\right]+3\left[\frac{dp}{dt}\,\frac{d^2p}{dt^2}\right] = 0.$$

Hence, from (8),

$$\left[p\,\frac{d^3p}{dt^3}\right] = 0. \tag{10}$$

Differentiating (10),

$$\left[p\,\frac{d^4p}{dt^4}\right]+\left[\frac{dp}{dt}\,\frac{d^3p}{dt^3}\right] = 0.$$

Hence, from (9),

$$\left[p\,\frac{d^4p}{dt^4}\right]-\left[\left(\frac{d^2p}{dt^2}\right)^2\right] = 0.$$

Proceeding in this way it can be shown that

$$\left.\begin{aligned} \left[p\,\frac{d^{2n}p}{dt^{2n}}\right] &= (-1)^n\left[\left(\frac{d^np}{dt^n}\right)^2\right], \\ \text{and} \qquad \left[p\,\frac{d^{2n+1}p}{dt^{2n+1}}\right] &= 0. \end{aligned}\right\} \tag{11}$$

The correlations to which p and its differential coefficients must be subject in order that their standard deviations may be constant, have now been established. We can, therefore, now use these standard deviations to define some statistical properties of the curve.

In analysing any actual curve, it may be very difficult and tedious to obtain these standard deviations. There is, however, another method of defining the statistical properties of the curve which is equivalent to that

given above, but which is likely to be much more manageable in practice. This method will now be considered.

Suppose that we take, as before, the values $p_1, p_2, p_3, \ldots, p_n$, of p at a large number of times $t_1, t_2, t_3, \ldots, t_n$, chosen at random. Let us correlate them with the values $p'_1, p'_2, \ldots, p'_n$, of p at times $t_1+\xi$, $t_2+\xi$, $\ldots$, $t_n+\xi$, where ξ is a finite interval of time which may be positive or negative. Let the coefficient of correlation so found be R_ξ. Then R_ξ must evidently be a function of ξ.

If p_t be the value of p at time t, and $p_{t+\xi}$ be the value of p at time $t+\xi$, then by definition

$$[p_t p_{t+\xi}] = R_\xi \sqrt{[p_t^2]} \sqrt{[p_{t+\xi}^2]};$$

but by hypothesis the standard deviation of p does not vary, hence

$$[p_t^2] = [p^2] = [p_{t+\xi}^2],$$

and

$$R_\xi = [p_t p_{t+\xi}] / [p^2]. \tag{12}$$

Now expand $p_{t+\xi}$ in powers of ξ,

$$p_{t+\xi} = p_t + \xi \frac{dp}{dt} + \frac{\xi^2}{2!} \frac{d^2p}{dt^2} + \ldots.$$

Hence

$$[p_t p_{t+\xi}] = [p_t^2] + \xi \left[p \frac{dp}{dt}\right] + \frac{\xi^2}{2!} \left[p \frac{d^2p}{dt^2}\right] + \frac{\xi^3}{3!} \left[p \frac{d^3p}{dt^3}\right]. \tag{13}$$

Substituting for $\left[p \dfrac{d^np}{dt^n}\right]$ from (11), (13) becomes

$$[p_t p_{t+\xi}] = [p^2] + \xi(0) - \frac{\xi^2}{2!} \left[\left(\frac{dp}{dt}\right)^2\right] + \frac{\xi^3}{3!}(0) + \frac{\xi^4}{4!} \left[\left(\frac{d^2p}{dt^2}\right)^2\right] \ldots.$$

Hence, from (12),

$$R_\xi = 1 - \frac{\xi^2}{2!} \frac{\left[\left(\frac{dp}{dt}\right)^2\right]}{[p^2]} + \frac{\xi^4}{4!} \frac{\left[\left(\frac{d^2p}{dt^2}\right)^2\right]}{[p^2]} - \ldots + (-1)^n \frac{\xi^{2n}}{2n!} \frac{\left[\left(\frac{d^np}{dt^n}\right)^2\right]}{[p^2]}. \tag{14}$$

It will be seen that, as might have been expected, R_ξ is an even function of ξ.

As an example of the method let us take the case where it is known that $p = \sin(t+\epsilon)$, where ϵ may take all possible values, all of which are equally probable. In this case

$$[p^2] = \tfrac{1}{2}, \quad [(dp/dt)^2] = \tfrac{1}{2}, \quad \ldots, \quad [(d^np/dt^n)^2] = \tfrac{1}{2}, \quad \ldots,$$

(14) therefore becomes

$$R_\xi = 1 - \frac{\xi^2}{2!}\frac{(\frac{1}{2})}{(\frac{1}{2})} + \frac{\xi^4}{4!}\frac{(\frac{1}{2})}{(\frac{1}{2})} - \dots.$$

This is the series for $\cos\xi$. Hence $R_\xi = \cos\xi$. The correlation between the value of p at any time and its value when t is increased by any odd multiple of $\frac{1}{2}\pi$ is 0. This is obviously true since there is no correlation between $\sin(t+\epsilon)$ and $\sin\{t+\epsilon+(2n+1)(\frac{1}{2}\pi)\}$ as ϵ varies.

The correlations between p and its differential coefficients given in (11) are evidently also true.

Application to Diffusion by continuous Movements.

The theorems which have just been proved will now be used to find out what are the essential properties of the motion of a turbulent fluid which makes it capable of diffusing through the fluid properties such as temperature, smoke content, colouring matter or other properties which adhere to each particle of the fluid during its motion.

Consider a condition in which the turbulence in a fluid is uniformly distributed so that the average conditions of every point in the fluid are the same. Let u be the velocity parallel to a fixed direction, which we will call the axis of x, of the particle on which our attention is fixed. It will now be shown that the statistical properties which were defined above (now in relation to u instead of p) are sufficient to determine the law of diffusion, *i.e.* the law which governs the average distribution of particles initially concentrated at one point, at any subsequent time.

Suppose that the statistical properties of u are known in the form given above, that is to say, suppose that $[u^2]$ and R_ξ are known. R_ξ is now the correlation coefficient between the value of u for a particle at any instant, and the value of u for the same particle after an interval of time ξ.

Let u_t represent the value of u at time t. Consider the value of the definite integral

$$\int_0^t [u_t u_\xi]\, d\xi.$$

By the definition of R_ξ this is equal to

$$[u_t^2]\int_0^t R_{\xi-t}\, d\xi.$$

Hence, since $[u^2]$ does not vary with t, and R_ξ is an even function of ξ,

$$\int_0^t [u_t u_\xi]\, d\xi = [u^2] \int_0^t R_\xi\, d\xi. \tag{15}$$

Evidently one can integrate inside the square bracket just as one can differentiate. Hence

$$\int_0^t [u_t u_\xi]\, d\xi = \left[u_t \int_0^t u_\xi\, d\xi\right] = [u_t X],$$

or, in the notation of the introduction, $[uX]$.

Hence
$$[u^2] \int_0^t R_\xi\, d\xi = [uX] \tag{16}$$

$$= \tfrac{1}{2} \frac{d}{dt} [X^2], \tag{17}$$

and
$$[X^2] = 2[u^2] \int_0^T \int_0^t R_\xi\, d\xi\, dt, \tag{18}$$

where X is the distance traversed by a particle in time T.

Equation (18) is rather remarkable because it reduces the problem of diffusion, in a simplified type of turbulent motion, to the consideration of a single quantity, namely, the correlation coefficient between the velocity of a particle at one instant and that at a time ξ later.

Let us now consider the physical meaning of (18), when T is so small that R_ξ does not differ appreciably from 1 during the interval T. In this case

$$\int_0^T \int_0^t R_\xi\, d\xi\, dt = \tfrac{1}{2} T^2,$$

so that (18) becomes
$$[X^2] = [u^2]\, T^2,$$

or
$$\sqrt{[X^2]} = T\sqrt{[u^2]}. \tag{19}$$

That is to say, the standard deviation of a particle from its initial position is proportional to T when T is small. This is what we should expect provided the time T is so small that the velocity does not alter appreciably while the particle is moving over the path.

Now consider how one would anticipate that R_ξ would vary with ξ in a turbulent fluid. The most natural assumption seems to be that R_ξ would fall to zero for large values of ξ. It might remain positive as in

the curve shown in Fig. 2, or it might become negative or oscillate before

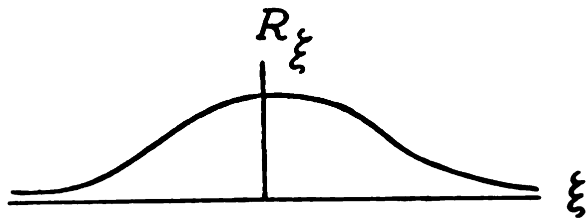

FIG. 2.

falling off to zero. In either case it seems probable that it will be possible to define an interval of time T_1, such that the velocity of the particle at the end of the interval T_1 has no correlation with the velocity at the beginning. In this case suppose that $\lim_{t\to\infty}\int_0^t R_\xi\,d\xi$ is finite and equal to I. Then at time $T\,(> T_1)$ after the beginning of the motion

$$\frac{d}{dt}[X^2] = 2\,[u^2]\,I,$$

so that $[X^2]$ increases at a uniform rate. In the limit when $[X^2]$ is large

$$\sqrt{[X^2]} = \sqrt{(2IT\,[u^2])}, \tag{20}$$

so that the standard deviation of X is proportional to the square root of the time.

This, therefore, is a property which a continuous eddying motion may be expected to have which is exactly analogous to the properties of discontinuous random migration in one dimension.

It will be noticed that when $T > T_1$,

$$[Xu] = [u^2]\,I. \tag{21}$$

Hence $[Xu]$ is constant in spite of the fact that $[X^2]$ continually increases. In order that this may be the case X must always be positively correlated with u, but the correlation coefficient must decrease with increasing $[X^2]$. If ν_{Xu} represents the correlation coefficient between X and u

$$\nu_{Xu} = -\frac{[Xu]}{\sqrt{[X^2]}\sqrt{[u^2]}} = \frac{I\sqrt{[u^2]}}{\sqrt{[X^2]}},$$

and in the limit when $T \to \infty$,

$$\nu_{Xu} = \frac{I\sqrt{[u^2]}}{\sqrt{(2IT\,[u^2])}} = \sqrt{\frac{I}{2T}}. \tag{22}$$

It is interesting to compare the expression (18) for $[X^2]$ with the expression given in (3) for the standard deviation of X in the special case of discontinuous motion considered there.

In that case R_ξ was shown in (4) to be $e^{-\xi/A}$. In the continuous case if we write $R_\xi = e^{-\xi/A}$, (21) becomes

$$[X^2] = 2[u^2]\int_0^T\int_0^t e^{-\xi/A}\,d\xi\,dt$$

$$= 2[u^2]\int_0^T A(1-e^{-t/A})\,dt$$

$$= 2[u^2]\{AT - A^2(1-e^{-T/A})\}. \tag{23}$$

In the discontinuous case it was shown in (3) that

$$\sqrt{[X^2]} = v\sqrt{\{2AT - 2A^2(1-e^{-T/A})\}}.$$

It is evident that this is exactly the same as (23) except that $\sqrt{[u^2]}$ has been substituted for the constant v which occurred in the discontinuous case.

If as a result of experiments on diffusion, it were possible to obtain a curve representing $[X^2]$ as a function of T, it would be possible to use (18) as a means of discovering something about the nature of the turbulence, for (18) could be written

$$\frac{d^2}{dt^2}[X^2] = 2[u^2]R_\xi,$$

and R_ξ could therefore be found.

In a recent communication to the Royal Society,* Mr. L. F. Richardson has described some experiments on the diffusion of smoke emitted from a fixed point in a wind. Similar observations have been made on the smoke from factory chimneys by Mr. Gordon Dobson.† Both these observers came to the conclusion that, at small distances from the origin of the smoke, the surface containing the standard deviations of the smoke from a horizontal straight line to leeward of the source, is a cone. If the mean velocity of the wind is assumed to be uniform, the standard deviation in a short interval of time is therefore proportional to the time. At greater distances their observations indicate that this surface becomes like a paraboloid, so that the deviation of the smoke is proportional to the square root of the time.

* *Phil. Trans.*, A, Vol. 221, p. 1.

† Advisory Committee for Aeronautics (*Reports*, 1919).

Both these observational data are in agreement with equations (19) and (20).

Mr. Richardson's method consisted in taking a photograph of the smoke leaving a source and drifting down-wind. The exposure was not instantaneous, but extended over such a long period that a kind of composite photogragh was obtained showing the outer limits of the region containing the smoke. The general shape of the outline of this region is shown in Figs. 4 and 5; it is, as has been explained, a parabola with a pointed vertex. In some cases the paraboloidal part of the surface joined straight on to the conical part, as shown in Fig. 4, but in other cases there was a sort of neck between them as shown in Fig. 5. According to the theory set forth above this neck would be anticipated in cases where the R_ξ curve contained negative values as shown in Fig. 3. An R_ξ curve of this type might be due to some sort of regularity in the eddies of which the turbulent motion consists.

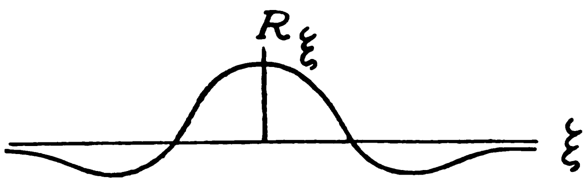

Fig. 3.

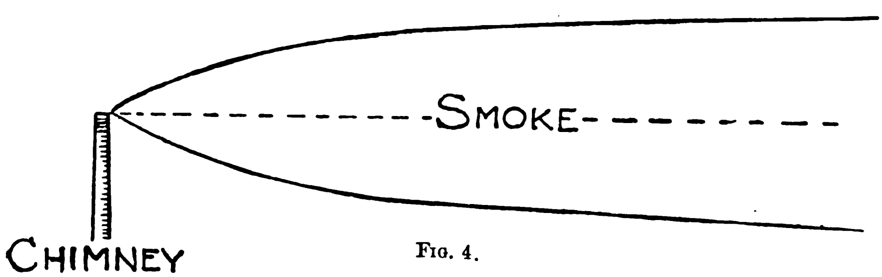

Fig. 4.

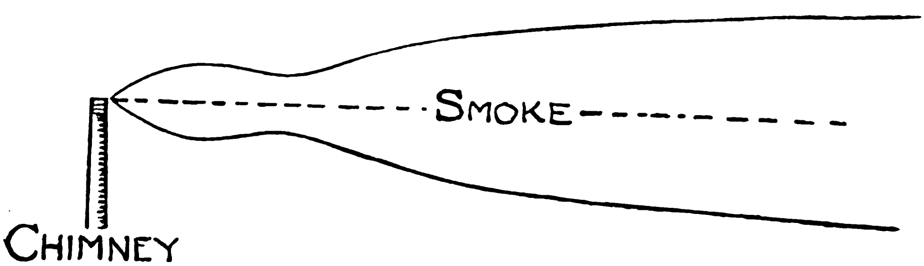

Fig. 5.

It appears that both theory and observation indicate that $[Xu]$ becomes constant after a certain interval of time (which depends of course on the value of ξ at which $\int_0^\xi R_\xi d\xi$ becomes practically constant with increasing values of ξ). This is a matter of considerable interest in the theory of the conduction of heat by means of turbulence, because it indicates a reason why the "diffusing power" of any type of turbulence appears to depend so little on the molecular conductivity and viscosity of the fluid.

After writing this paper I showed it to Mr. Richardson, who informed me that he had already noticed the relations (11), and at my request he sent me his proof which follows.

Note on a Theorem by Mr. G. I. Taylor on Curves which Oscillate Irregularly

By LEWIS F. RICHARDSON.

The theorem referred to is proved on the hypothesis that the standard deviations of p, dp/dt, d^2p/dt^2, ..., d^np/dt^n are constant over any long time. It also follows, as will now be shown, from the rather different hypotheses which may be stated thus:—

(i) No one of p, dp/dt, d^2p/dt^2 has a standard deviation less than a certain lower limit. (1)

(ii) The instantaneous values (t being time) of p, dp/dt, d^2p/dt^2, ..., never exceed in numerical value a certain upper limit. (2)

We might state simple numerical upper and lower limits. But as we are dealing with oscillations, it will be as well to take a hint from the properties of the sine curve. If $p = c \sin sp$, then $|d^np/dt^n|$ is not greater than cs^n, and the standard deviation of d^np/dt^n is $\sqrt{\tfrac{1}{2}}\, cs^n$.

For our irregular curve let us define B and r and A so that $|p| < B$ and $|d^np/dt^n| < Br^n$. (3)

The standard deviation of p is greater than A and that of d^np/dt^n is greater than Ar^n. (4)

It is required to find

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p\,\frac{d^{2n}p}{dt^{2n}}\,dt. \tag{5}$$

Integrate by parts, successively, so as to differentiate the p and to integrate $d^{2n}p/dt^{2n}$ until they both coincide in $d^n p/dt^n$. For example, when $n = 5$, the result is

$$\begin{aligned}&\frac{1}{t_2-t_1}\int_{t_1}^{t_2} p\,\frac{d^{10}p}{dt^{10}}\,dt\\ =\;&\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\left[-\frac{d^9p}{dt^9}\,p+\frac{d^8p}{dt^8}\,\frac{dp}{dt}-\frac{d^7p}{dt^7}\,\frac{d^2p}{dt^2}+\frac{d^6p}{dt^6}\,\frac{d^3p}{dt^3}-\frac{d^5p}{dt^5}\,\frac{d^4p}{dt^4}\right]\\ &+\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\left(\frac{d^5p}{dt^5}\right)^2 dt. \qquad (6)\end{aligned}$$

The expression in square brackets is less than $5B^2r^9$ however long the interval (t_2-t_1) may be, while $\int_{t_1}^{t_2}\left(\frac{d^5p}{dt^5}\right)^2 dt$ is greater than $(t_2-t_1)A^2r^{10}$, and so increases with the interval.

Thus when (t_2-t_1) is large enough, the term in square brackets becomes negligible. Generalizing the example, and taking account of the changes of sign introduced by partial integration,

$$\lim_{t_2-t_1\to\infty}\frac{1}{t_2-t_1}\int_{t_1}^{t_2} p\,\frac{d^{2n}p}{dt^{2n}}\,dt \text{ is } \frac{(-1)^n}{t_2-t_1}\int_{t_1}^{t_2}\left(\frac{d^np}{dt^n}\right)^2 dt. \qquad (7)$$

If in place of $d^{2n}p/dt^{2n}$ in (5) we had had a coefficient of odd order, the partial integrations, when pursued so as to lead back again to the original form, would have produced an arrangement of signs such that like terms were added. So that

$$\lim_{t_2-t_1\to\infty}\frac{1}{t_2-t_1}\int_{t_1}^{t_2} p\,\frac{d^{2n+1}p}{dt^{2n+1}}\,dt = 0. \qquad (8)$$

This depends on the hypothesis (2) only. Hypothesis (1) does not come in here. It was needed in proving (7).

ON DR. SHEPPARD'S METHOD OF REDUCTION OF ERROR BY LINEAR COMPOUNDING

By A. S. EDDINGTON.

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1. Dr. W. F. Sheppard* has recently developed a new method of treating the problem of fitting a smooth curve to a series of observations, which seems to be of considerable importance in the general theory of the subject. The central idea of the new method is not difficult to understand; but the formulæ into which it is translated become very bewildering by their number and absence of any apparent connection with one another, and by the multitude of arbitrary symbols which have to be introduced. I believe that a greater coherence can be obtained by introducing the notation and methods of the tensor calculus, which is now becoming well known in connection with Einstein's theory of gravitation. In this paper I use these methods in order to present Dr. Sheppard's theory in a condensed form in which it may be grasped as a whole—more especially by those who have already become accustomed to the notation as used in Einstein's theory. It is in itself a matter of some interest to exhibit the close similarity of the analysis used in two such widely different subjects.

When the results of a series of observations are represented by a smooth curve, the ordinate of the curve represents, not the ordinate actually measured for that point, but an "improved value", for which the observational error has presumably been reduced by combining in an appropriate way observations of neighbouring ordinates. The problem of finding the improved value is indeterminate unless we have some *a priori* knowledge or expectation as to the degree of smoothness of the true curve. In practice this is usually expressed in the form that the curve shall represent a polynomial of the j-th degree, or, equivalently, that the differences of order greater than j are negligible. The latter is the more natural form of the statement, since "smoothness" refers directly to regularity of differences. Dr. Sheppard's general theory of this process

* "Reduction of Error by Linear Compounding," *Phil. Trans.*, Vol. 221, A, pp. 199-237.

covers cases in which the errors of the original ordinates are partially correlated, thus adding greatly to the complication of the problem.

Part of Dr. Sheppard's paper, especially §§ 17–23, is devoted to the adaptation of the theory to numerical computation; it is beyond my purpose to discuss this. But I believe that the rest of his discussion is fairly well covered in this presentation; conversely, all the formulæ here deduced correspond to results already given by him.

2. Let A_μ ($\mu = 1, 2, \ldots, n$) be a set of n quantities containing errors which may be independent or correlated in any way. Let δA_μ denote the error of A_μ. Let the mean product error of any pair of these quantities A_μ, A_ν be denoted by $g_{\mu\nu}$, so that

$$g_{\mu\nu} = M(\delta A_\mu \delta A_\nu), \tag{2.1}$$

where M denotes "mean value of".

Following the method of the tensor calculus—

Let $g =$ the determinant of n rows and columns formed with the elements $g_{\mu\nu}$.

Let $g^{\mu\nu} =$ the minor of $g_{\mu\nu}$, divided by g.

We shall make the convention that when any Greek suffix appears twice in a term that term is to be summed for values of this suffix from 1 to n. Thus $g_{\mu\nu} g^{\mu\sigma}$ will stand for

$$\sum_{\mu=1}^{\mu=n} g_{\mu\nu} g^{\mu\sigma}. \tag{2.2}$$

It is easily seen that (2.2) reproduces the determinant g, divided by g, if $\nu = \sigma$, and gives a determinant with two rows identical if $\nu \neq \sigma$. We denote (2.2) by g_ν^σ; thus

$$\begin{aligned} g_\nu^\sigma = g_{\mu\nu} g^{\mu\sigma} &= 1 \quad \text{if} \quad \nu = \sigma \\ &= 0 \quad \text{if} \quad \nu \neq \sigma. \end{aligned} \tag{2.3}$$

Evidently g_ν^σ acts as a substitution operator. For example,

$$\begin{aligned} g_\nu^\sigma A_\sigma &= 0+0+\ldots+1 \,.\, A_\nu+\ldots+0 \\ &= A_\nu. \end{aligned} \tag{2.4}$$

We shall need also to introduce corresponding definitions relating to the first j only of the quantities A_μ ($j < n$). Denote the determinant limited to j rows and columns by $(g)_j$; and let $(g^{rs})_j$ be the minor of g_{rs} in this smaller determinant, divided by $(g)_j$. We make the convention that

when an *italic* suffix appears twice, it is to be summed from 1 to j. It follows as before that

$$(g_s^t)_j = g_{rs}(g^{rt})_j = 1 \quad \text{if} \quad s = t$$
$$= 0 \quad \text{if} \quad s \neq t. \tag{2.5}$$

This expression looks more symmetrical if we write $(g_{rs})_j$ for g_{rs}. It will be noticed that $(g_{rs})_j$ and $(g_r^s)_j$ are independent of j; but $(g)_j$ and $(g^{rs})_j$ depend on j.

Corresponding to our use of italic letters for suffixes equal to or less than j, we shall use capital letters for suffixes greater than j; in the latter case the summation indicated by a doubled suffix will be from $j+1$ to n. It follows from this convention that

$$a_\mu b_\mu = a_m b_m + a_M b_M, \tag{2.6}$$

and
$$g_m^M = 0 \tag{2.7}$$

since m cannot be equal to M.

3. Introduce a new set of quantities A^μ defined by

$$A^\mu = g^{\mu\nu} A_\nu. \tag{3.1}$$

so that the new quantities are linear functions of the old. Then

$$g_{\mu\sigma} A^\mu = g_{\mu\sigma} g^{\mu\nu} A_\nu$$
$$= g_\sigma^\nu A_\nu = A_\sigma \text{ by (2.4);}$$

hence (changing the notation)

$$A_\mu = g_{\mu\nu} A^\nu, \tag{3.2}$$

which gives the reverse transformation.

The mean product error of A_μ and A^ν is

$$M(\delta A_\mu \delta A^\nu) = M(\delta A_\mu g^{\nu\sigma} \delta A_\sigma) \text{ by (3.1)}$$
$$= g^{\nu\sigma} . M(\delta A_\mu \delta A_\sigma)$$
$$= g^{\nu\sigma} g_{\mu\sigma} \text{ by (2.1)}$$
$$= g_\mu^\nu. \tag{3.3}$$

Thus the mean product error of corresponding members of the two sets is unity, and of non-corresponding members is zero. Sets related in this way are called by Dr. Sheppard *conjugate sets*.

The mean product errors of $A_\mu A_\nu$, $A_\mu A^\nu$, and $A^\mu A^\nu$,

are respectively $$g_{\mu\nu}, \quad g^\nu_\mu, \quad \text{and} \quad g^{\mu\nu}. \tag{3.4}$$

4. Let w be any given linear function of the A^μ, viz.,

$$w = a_\mu A^\mu.$$

Let Δw be an arbitrary linear function of another set of quantities B^ν (which may or may not form part of the set A^μ), viz.,

$$\Delta w = b_\nu B^\nu.$$

If the coefficients b_ν are determined so as to make the mean square error of $w+\Delta w$ a minimum, then $w+\Delta w$ is called "the *improved value* of w using the quantities B^ν as *auxiliaries*."

In particular we denote by $(w)_j$ the improved value of w using $A^{j+1}, A^{j+2}, \ldots, A^n$ as auxiliaries.

The useful application is when it is known that the *true values* of the B^ν are zero (or negligible), *e.g.* when they represent tabular differences of a reasonably high order. In that case w and $w+\Delta w$ have the same true values, but $w+\Delta w$ is a better approximation than w because its mean square error has been made a minimum.

Dr. Sheppard's fundamental theorem is that $(w)_j$ can be expressed as a linear function of the first j quantities of the conjugate set A_μ.

By means of the n equations (3.1) we can eliminate any n of the $2n$ quantities A_μ and A^μ, leaving w expressed as a linear function of the remaining n quantities. Let w be accordingly expressed as a linear function of

$$A_1, A_2, \ldots, A_j, A^{j+1}, A^{j+2}, \ldots, A^n,$$

viz. $$w = a_r A_r + a_R A^R \tag{4.1}$$

(summed in accordance with the conventions explained in § 2). We have to add an arbitrary function of the auxiliaries $A^{j+1}, \ldots, A^n$, viz.,

$$\Delta w = b_R A^R, \tag{4.2}$$

giving $$w+\Delta w = a_r A_r + c_R A^R, \tag{4.3}$$

where $$c_R = a_R + b_R.$$

Now the arbitrary coefficients c_R must be determined so as to make the mean square error of $w+\Delta w$ a minimum. The mean product error of $a_r A_r$ and $c_R A^R$ is

$$a_r c_R . M(\delta A_r \delta A^R) = a_r c_r g^R_r = 0 \text{ by } (2.7).$$

Hence

$$(\text{m. s. e.})^2 \text{ of } w+\Delta w = (\text{m. s. e.})^2 \text{ of } a_r A_r + (\text{m. s. e.})^2 \text{ of } c_R A^R,$$

and this will be a minimum if we choose c_R so that the second term is zero. We must therefore take $c_R = 0$, and (4.3) becomes

$$(w)_j = a_r A_r, \tag{4.4}$$

which proves the theorem.

The coefficient a_r is equal to the mean product error of $(w)_j$ and A^r; for

$$M[\delta(w)_j \delta A^s] = a_r M(\delta A_r \delta A^s)$$
$$= a_r g_r^s = a_s. \tag{4.5}$$

The mean product error of $(w)_j$ and any of the auxiliaries vanishes.

5. Taking $w = A^h$ in (4.4), the improved value of A^h may then be expressed in the form

$$(A^h)_j = a^{rh} A_r, \tag{5.1}$$

where the coefficients a^{rh} are as yet undetermined. By (3.2) this becomes

$$(A^h)_j = a^{rh} g_{\mu r} A^\mu$$
$$= a^{rh} g_{mr} A^m + a^{rh} g_{Mr} A^M \text{ by } (2.6). \tag{5.2}$$

But by definition $\qquad (A^h)_j = A^h + b_M A^M. \qquad (5.3)$

Comparing (5.2) and (5.3),

$$a^{rh} g_{mr} A^m = A^h = g_m^h A^m.$$

Thus $\qquad a^{rh} g_{mr} = g_m^h.$

But, by (2.5), $\qquad (g^{rh})_j g_{mr} = (g_m^h)_j = g_m^h.$

Hence* $\qquad a^{rh} = (g^{rh})_j, \qquad (5.4)$

and (5.1) becomes $\qquad (A^h)_j = (g^{rh})_j A_r. \qquad (5.5)$

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* In deducing from $\qquad a^{rh} g_{mr} = (g^{rh})_j g_{mr},$

that $\qquad a^{rh} = (g^{rh})_j,$

we perform a kind of pseudo-division by g_{mr} which is legitimate but needs to be justified. Evidently the above values of a^{rh} form one solution, and this is the only solution, because (h being fixed) the different values of m provide j simultaneous linear equations to determine the j unknowns a^{rh}.

From (5 . 1) and (5 . 3) the mean product error of improved values is

$$M[\delta(A^h)_j\,\delta(A^k)_j] = M\{a^{rh}\delta A_r(\delta A^k + b_M\delta A^M)\}$$
$$= a^{rh}g_r^k + a^{rh}b_M g_r^M$$
$$= a^{kh} + 0 \text{ by } (2.7)$$
$$= (g^{kh})_j, \qquad (5.6)$$

and the (mean square error)^2 of $(A^h)_j$ is

$$(g^{hh})_j \quad \text{(not summed)}. \qquad (5.7)$$

6. Let A'_μ be another set of n quantities, linear functions of the A_μ, given by

$$A'_\mu = a_\mu^\sigma A_\sigma, \qquad (6.1)$$

or, reciprocally,

$$A_\mu = b_\mu^\sigma A'_\sigma. \qquad (6.2)$$

Then

$$A'_\mu = a_\mu^\sigma A_\sigma = a_\mu^\sigma b_\sigma^\tau A'_\tau,$$

so that

$$a_\mu^\sigma b_\sigma^\tau = g_\mu^\tau. \qquad (6.3)$$

Similarly

$$b_\mu^\sigma a_\sigma^\tau = g_\mu^\tau.$$

Let A'^μ be the set conjugate to A'_μ, and let the expression in terms of A^μ be

$$A'^\mu = c_\sigma^\mu A^\sigma,$$

then

$$\delta A'^\mu \delta A'_\nu = c_\sigma^\mu \delta A^\sigma \,.\, a_\nu^\tau \delta A_\tau,$$

whence taking mean product errors of both sides

$$g_\nu^\mu = c_\sigma^\mu a_\nu^\tau g_\tau^\sigma$$
$$= c_\sigma^\mu a_\nu^\sigma.$$

Comparing with (6 . 3), it follows that

$$c_\sigma^\mu a_\nu^\sigma = b_\sigma^\mu a_\nu^\sigma,$$

whence (as in § 5, footnote)

$$c_\sigma^\mu = b_\sigma^\mu.$$

The relations can accordingly be written

$$\left.\begin{array}{c} A'_\mu = a_\mu^\sigma A_\sigma, \quad A^\mu = a_\sigma^\mu A'^\sigma, \quad A_\mu = b_\mu^\sigma A'_\sigma, \quad A'^\mu = b_\sigma^\mu A^\sigma, \\ a_\mu^\sigma b_\sigma^\tau = b_\mu^\sigma a_\sigma^\tau = g_\mu^\tau. \end{array}\right\} \qquad (6.4)$$

Denoting the mean product error of A'_μ and A'_ν by $g'_{\mu\nu}$, we shall have

$$\left.\begin{aligned} g'_{\mu\nu} &= a^\sigma_\mu a^\tau_\nu g_{\sigma\tau}, \\ g_{\mu\nu} &= b^\sigma_\mu b^\tau_\nu g'_{\sigma\tau}, \end{aligned}\right\} \qquad (6.5)$$

and similarly

$$\left.\begin{aligned} g'^{\mu\nu} &= b^\mu_\sigma b^\nu_\tau g^{\sigma\tau}, \\ g^{\mu\nu} &= a^\mu_\sigma a^\nu_\tau g'^{\sigma\tau}. \end{aligned}\right\} \qquad (6.6)$$

The transformations obey the same laws as in the tensor calculus.

7. Whether we are dealing with accented or unaccented letters we shall in all cases use the unaccented $A^{j+1} \dots A^n$ as auxiliaries. The improved value of A'^μ is then given by

$$(A'^\mu)_j = (b^\mu_\sigma A^\sigma)_j = b^\mu_\sigma (A^\sigma)_j. \qquad (7.1)$$

The second step (that the improved value of the sum is equal to the sum of the improved values of the separate terms) follows because by § 5 the mean product error of any improved value and any of the auxiliaries vanishes, so that the further addition of a linear function of the auxiliaries could only increase the mean square error.

Now when $\sigma > j$, so that A^σ is itself one of the auxiliaries, its improved value will evidently be $A^\sigma - A^\sigma$, with mean square error zero. Thus

$$(A^\sigma)_j = 0 \quad \text{if} \quad \sigma > j.$$

Hence in (7.1) the summation for σ can be restricted to $\sigma \leqslant j$; so that

$$\begin{aligned} (A'^\mu)_j &= b^\mu_k (A^k)_j \qquad (7.2) \\ &= b^\mu_k (g^{rk})_j A_r \text{ by } (5.5) \\ &= b^\nu_r b^\mu_k (g^{rk})_j A'_\nu \text{ by } (6.2). \end{aligned}$$

Comparing with (6.6) it is natural to write*

$$(g'^{\nu\mu})_j = b^\nu_r b^\mu_k (g^{rk})_j. \qquad (7.3)$$

Then

$$(A'^\mu)_j = (g'^{\nu\mu})_j A'_\nu. \qquad (7.4)$$

* This definition must be specially noted. If we defined $(g'^{\nu\mu})_j$ directly from the determinant of j rows and columns formed with elements $g'_{\mu\nu}$, we should obtain an entirely different quantity. We have to refer back to the limited *unaccented* determinant, because it is the unaccented quantities which have been chosen as auxiliaries.

The mean product error

$$M[\delta(A'^{\mu})_j\,\delta(A'^{\nu})_j] = b^{\mu}_k b^{\nu}_h M[\delta(A^k)_j\,\delta(A^h)_j] \text{ by } (7.2)$$

$$= b^{\mu}_k b^{\nu}_h (g^{hk})_j \text{ by } (5.6)$$

$$= (g'^{\mu\nu})_j \text{ by } (7.3). \qquad (7.5)$$

And the (mean square error)^2 of the improved value is

$$(g'^{\mu\mu})_j \quad \text{(not summed).} \qquad (7.6)$$

To recognise the remarkable symmetry of these results, we may compare (3.1), (5.5), and (7.4), viz.

$$A^{\mu} = g^{\mu\nu} A_{\nu}, \quad (A^h)_j = (g^{hr})_j A_r, \quad (A'^{\mu})_j = (g'^{\mu\nu})_j A'_{\nu},$$

and their respective mean product errors (3.4), (5.6), and (7.5),

$$g^{\mu\nu}, \quad (g^{hk})_j, \quad (g'^{\mu\nu})_j.$$

8. The results (7.4) and (7.6) constitute the solution of our problem. We take A'^{μ} to be n equidistant ordinates through which a smoothed curve is to be drawn; and we take A^{μ} to be differences of successively higher orders starting from any particular A'^{μ}. The coefficients a^{ν}_{μ}, b^{ν}_{μ} of the equations between the A'^{μ} and the A^{μ} are known from the theory of tabular differences. The degree of smoothness imposed on the curve will be taken to correspond to the vanishing of differences of order higher than the j-th, so that $A^{j+1}, \ldots, A^n$ are the auxiliaries. The improved value of any ordinate A'^{μ} is then given by (7.4), and its mean square error by (7.6).

In practice the computation of the $(g'^{\mu\nu})_j$ would be very lengthy. We must suppose that we are given initially the $g'^{\mu\nu}$, *i.e.* the mean square errors and correlations of the original ordinates A'^{μ}. From these we pass to the $g^{\mu\nu}$ by (6.6). Next the $g_{\mu\nu}$ must be found by solving (2.3); in fact $g_{\mu\nu}$ is the minor of $g^{\mu\nu}$ in the determinant of the $g^{\mu\nu}$, divided by that determinant. At this stage we pass to the limited determinant $(g)_j$, and then retrace our steps calculating first $(g^{\mu\nu})_j$ and then $(g'^{\mu\nu})_j$ by (7.3).

The geometrical interpretation of our procedure is of some interest. The A'^{μ} are the components of a contravariant vector (a displacement) referred to certain axes in an n-dimensional space; and the metric associated with these axes is defined by the fundamental tensor $g'_{\mu\nu}$. If α'_{μ}, β'_{μ} are two *unit* covariant vectors, the lengths of a displacement $\delta A'^{\mu}$ resolved orthogonally in these two directions will be the scalar products $\alpha'_{\mu}\delta A'^{\mu}$, $\beta'_{\nu}\delta A'^{\nu}$ respectively, and their product will be $\alpha'_{\mu}\beta'_{\mu}(\delta A'^{\mu}\delta A'^{\nu})$. Hence,

taking mean values, the mean product error in the directions α', β' will be $\alpha'_\mu \beta'_\nu g'^{\mu\nu} = \alpha'_\mu \beta'^\mu = 1$, if the directions agree, and 0 if they are at right-angles. Thus the metric ensures that the errors *in space* will be isotropic and uncorrelated, however the errors of the *components* A'^μ referred to particular (oblique) axes may be correlated. Performing a linear transformation of coordinates we obtain new components A^μ of the same vector, the associated metric being given by $g_{\mu\nu}$. We proceed to use our knowledge that $n-j$ of these new components ought to be zero; that is to say, we know that the true vector A^μ lies on a certain j-dimensional surface. Owing to errors, the observed vector will not in general satisfy this; and it easily follows that the most probable vector is obtained by projecting orthogonally on the j-dimensional surface. This, of course, depends on the result proved above that the errors in space are isotropic. The orthogonal projection is expressed by (5 . 5) according to well-known geometrical principles; and it only remains to apply the usual formulæ to transform the modified vector back to the original coordinates. We have here a proof that our process gives not merely *improved* values but the *most probable* values, subject to the condition that the higher differences vanish; and the results must necessarily agree with those found by any other method of application of the criterion of least squares.

9. For comparison with Dr. Sheppard's paper I add the chief correspondences between the notations. In his § 3,

$$\delta_r,\ \sigma_r,\ u_r,\ y_r \quad \text{correspond to} \quad A_r,\ A^r,\ A'_r,\ A'^r,$$

$$\xi_{r,t},\ \eta_{r,t} \qquad ,, \qquad g_{rt},\ g^{rt},$$

$$Z,\ Z_{p,q} \qquad ,, \qquad g,\ g\cdot g^{pq}.$$

The brackets ( ) and { } in § 4 correspond to a^ν_μ and b^ν_μ. In § 7, δ_r corresponds to A^r, and

$$(\epsilon_f)_j,\ (\lambda_{f,g})_j \quad \text{to} \quad (A^f)_j,\ (g^{fg})_j.$$

EXTENDED MEANING OF CONJUGATE SETS

By W. F. SHEPPARD.

[Received January 2nd, 1921.—Read January 13th, 1921.]

IN connexion with the paper by Prof. Eddington, which precedes this note, I should like to take the opportunity of pointing out that the definition of conjugate sets, in my paper to which he refers, may be made somewhat wider. If A and B are two variable quantities, not necessarily of the same kind, we can denote by $(A\,;\,B)$ some quantity which (1) is determinate when A and B have assigned meanings but is independent of particular values of A and B, and (2) satisfies the laws of arithmetic for multiplication, *i.e.* is such that

$$(A\,;\,B) = (B\,;\,A), \qquad (A\,;\,B+C) = (A\,;\,B)+(A\,;\,C),$$

and, if p is a constant with regard to A and B,

$$(A\,;\,pB) = p\,(A\,;\,B).$$

Usually, if A and B relate to a member of a class, $(A\,;\,B)$ will be something depending on the values of A and B for the class as a whole. If, for instance, as in the original paper, A and B were measurements containing errors, $(A\,;\,B)$ might be the mean product of such pairs of errors; or, if A and B were, say, the height and weight of an individual, $(A\,;\,B)$ might be the mean product of the deviations of these from their respective means. A meaning having been given to $(A\,;\,B)$, the definition of conjugate sets, for this meaning of $(A\,;\,B)$, is to be adapted accordingly. Let $A_\mu \equiv A_0, A_1, \ldots, A_l$ be a set of $l+1$ variable quantities. Then the conjugate set $A^\mu \equiv A^0, A^1, \ldots, A^l$ can be defined either by the condition that $(A^r;\,A_s)$ is 0 if $r \neq s$ or 1 if $r = s$, or, as in Prof. Eddington's paper, directly by the relation $A^\nu = g^{\mu\nu} A^*_\mu$, where $g_{rs} \equiv (A_r;\,A_s)$.

For finding improved values we require the further condition (3) that $(A\,;\,A)$ is positive unless $A = 0$ or a constant, in which case it is $= 0$. If w is any linear function of $A_0, A_1, \ldots, A_l$, its improved value, which I will here call Iw, is defined by the condition that it is the sum of w and a

linear function of $l-j$ specified A's (or specified linear functions of the A's), called auxiliaries, the coefficients in this linear function being chosen so as to make $(Iw\,;\; Iw)$ a minimum.

With these extensions, Prof. Eddington's methods and results still apply, with the substitution of $(A\,;\; B)$ for the mean product of errors of A and B. His notation is adopted in the following supplementary paragraphs; w, x, y denoting definite linear functions of the A's, and B_μ, C_ν, etc. sets of such functions.

(i) We can write w in the form $w = k_\mu A_\mu$. If also $x = k_\mu B_\mu$, then x is related to the B's in the same way that w is related to the A's. We can express this by saying that $w/A_\mu = x/B_\mu$; it being understood that a Greek letter in a denominator is what Prof. Eddington elsewhere calls a "dummy," *i.e.* that the quantity in which it occurs is one of the factors of an "inner product" such as $k_\mu A_\mu$. We may also have relations such as $A_\mu/B_\nu = C_\mu/D_\nu$ or $A_\mu/B_\nu = E_\nu/F_\mu$; in this latter case $A_\mu F_\mu = B_\nu E_\nu$. These relations can be inverted; *e.g.* if $A_\mu/B_\nu = C_\mu/D_\nu$, then $B_\nu/A_\mu = D_\nu/C_\mu$.

(ii) The second and third of the conditions stated under (2) of the first paragraph of this note may be combined in the form

$$(y\,;\; k_\mu A_\mu) = k_\mu\,(y\,;\; A_\mu).$$

If we write $w \equiv k_\mu A_\mu$, so that $k_\mu = w/A_\mu$, this becomes

$$w/A_\mu = (y\,;\; w)/(y\,;\; A_\mu).$$

(iii) If we define a conjugate set in the first of the two ways mentioned in the first paragraph, we easily obtain

$$w = (w\,;\; A^\mu)\,A_\mu = (w\,;\; A_\mu)\,A^\mu,$$

or
$$w/A_\mu = (w\,;\; A^\mu), \qquad w/A^\mu = (w\,;\; A_\mu).$$

Hence
$$B_\nu = (B_\nu\,;\; A^\mu)\,A_\mu = (B_\nu\,;\; A_\mu)\,A^\mu.$$

(iv) From $w = (w\,;\; A^\mu)A_\mu$ it follows that

$$(w\,;\; y) = (w\,;\; A^\mu)(A_\mu\,;\; y),$$

since $(w\,;\; A^\mu)$ is a constant as regards w and y. Hence

$$(B_\nu\,;\; C_\rho) = (B_\nu\,;\; A^\mu)(A_\mu\,;\; C_\rho).$$

(v) For two related sets, and their conjugates, we have four relations

of the form
$$\frac{A^\mu}{B_\nu} = \frac{B^\nu}{A_\mu} = (A^\mu\,;\; B^\nu) = \frac{Q}{A_\mu B_\nu},$$

where $Q \equiv A_\mu A^\mu$. This inner product Q is invariant for any particular original set, and may be expressed either as a quadratic form of the members of a related set, the coefficients being the (;) of members of its conjugate set—*e.g.*

$$Q = (C^0\,;\; C^0)\,C_0 C_0 + 2\,(C^0\,;\; C^1)\,C_0 C_1 + (C^1\,;\; C^1)\,C_1 C_1 + \ldots$$

—or in the more general form $(C^\mu\,;\; D^\nu)\,C_\mu D_\nu$.

(vi) Relations similar to those mentioned above may hold between sets of coefficients or of other constants. If, for instance, A_μ and B_ν are related in a specified way, and if w is a linear function of B_ν which we want to express in terms of A^μ, then, if we write

$$w = p_\mu A^\mu, \qquad q_\nu \equiv (w\,;\; B_\nu),$$

we have $\quad p_\mu = w/A^\mu = (w\,;\; A_\mu) = (w\,;\; A_\mu/B_\nu.B_\nu) = A_\mu/B_\nu\,.\,q_\nu,$

so that
$$p_\mu/q_\nu = A_\mu/B_\nu.$$

Thus the p's are related to the q's in the same way that the A's are related to the B's.

(vii) The relation between improved values and original values is such that
$$Ik_\mu A_\mu = k_\mu I A_\mu, \quad \text{or} \quad Iw/IA_\mu = w/A_\mu.$$

It may be noted that a set of $l+1$ improved values has no conjugate set, as the $l+1$ values are not independent but are connected by $l-j$ relations.

(viii) The problem of finding improved values may be expressed as a problem of finding parts of two conjugate sets in succession. Take $A_\mu \equiv A_F$ & A_R, where A_F are $j+1$ quantities whose improved values IA_F are required, and A_R are the $l-j$ auxiliaries. Let the set conjugate to A_F & A_R be A^F & A^R: and let the set conjugate to A^F & A_R be B_F & B^R. Then $B_F = IA_F$. For finding either or both of A^F and B_F we may replace A_R by any $l-j$ linear functions of A_R, and may add to members of A_F any linear functions of A_R.

ARITHMETIC OF QUATERNIONS

By L. E. Dickson.

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1. The algebra of quaternions is formed of all quaternions

$$q = a+bi+cj+dk$$

whose coordinates a, b, c, d are ordinary complex numbers, while the units i, j, k satisfy the relations

$$i^2=j^2=k^2=-1, \quad ij=k=-ji, \quad jk=i=-kj, \quad ki=j=-ik.$$

The conjugate to q is $\qquad q' = a-bi-cj-dk.$

The norm $N(q)$ of q is $qq' = q'q = a^2+b^2+c^2+d^2.$

The norm of a product of two quaternions equals the product of their norms. The associative law $pq \,.\, r = p \,.\, qr$ holds. No further properties of quaternions are presupposed in this paper.

Quaternions have recently been applied to the solution of several important problems in the theory of numbers. For this purpose it is necessary to make a choice of the quaternions which are to be called integral. R. Lipschitz* quite naturally called only those quaternions integral whose coordinates are integers (whole numbers). His complicated theory of integral quaternions was based upon the solutions of congruences

$$\xi^2+\eta^2+\zeta^2 \equiv 0 \pmod{p^k}.$$

He made no mention of a greatest common divisor process, which in fact is not applicable in general.

A. Hurwitz† succeeded in developing a perfect arithmetic of quaternions by taking as his integral quaternions those whose coordinates are either all integers or all halves of odd integers. But the presence of the

* "Untersuchungen über die Summen von Quadraten," Bonn, 1886. French translation in *Jour. de Math.*, sér. 4, t. 2, 1886, pp. 393–439.

† *Göttingen Nachrichten*, 1896, pp. 311–340. Amplified in his "Vorlesungen über die Zahlentheorie der Quaternionen," Berlin, J. Springer, 1919, 74 pp.

denominators 2 in certain integral quaternions was an inconvenience in the application which I recently* made of Hurwitz's theory to the complete solution in integers of quadratic equations in several variables. Accordingly I shall give here a new theory of the arithmetic of quaternions in which, following Lipschitz, the integral quaternions are those whose coordinates are integers exclusively. Call such a quaternion odd if its norm is odd. I shall prove that, if at least one of two integral quaternions is odd, they have a greatest common divisor which is expressible as a linear combination of them. It is then a simple matter to develop the theory of factorization of integral quaternions.

The limitation that one of the quaternions is odd causes no inconvenience for the applications. Moreover, it is of theoretical interest to know exactly to what extent we can meet the difficulties which arise in the arithmetic of quaternions in which the integral quaternions are defined naturally to be those with integral coordinates exclusively. Furthermore, the present theory is more direct and elementary than the earlier theories.

An integral quaternion whose norm is unity is called a unit. There are only eight units ± 1, $\pm i$, $\pm j$, $\pm k$. A quaternion is said to be associated with its products by the eight units.

2. A quaternion shall be said to be integral if its four coordinates are integers. If a, b, q are integral quaternions such that $a = qb$, a is said to have b as a right-hand divisor. Similarly, if $a = bQ$, where Q is an integral quaternion, a has b as a left-hand divisor.

A quaternion which has $1+i$ (or $1+j$ or $1+k$) as a right-hand divisor has it also as a left-hand divisor and *vice versa*, so that we may say simply that it has $1+i$ as a divisor. In fact,

$$(1) \qquad (1+i)(a+bi+cj+dk) = (a+bi-dj+ck)(1+i).$$

LEMMA 1.—An integral quaternion $A = a+bi+cj+dk$ is divisible by $1+i$, if and only if $a+b$ and $c+d$ are both even; it is divisible by $1+j$, if and only if $a+c$ and $b+d$ are both even; and by $1+k$, if and only if $a+d$ and $b+c$ are both even.

We may write A in the form

$$a-b+b(1+i)+(c-d)j+d(1+i)j.$$

Hence A is divisible by $1+i$, if and only if $\alpha+\beta j$ is divisible by it, where $\alpha = a-b$, $\beta = c-d$. But $\alpha+\beta j = (1+i)Q$ is equivalent to the equa-

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* "Relations between the Theory of Numbers and other branches of Mathematics," *Comptes Rendus Congrès International des Mathématiciens*, Strasbourg, 1920.

tion obtained by multiplying each member by $1-i$ on the left:

$$\alpha-\alpha i+\beta j-\beta k = 2Q.$$

Here Q is an integral quaternion, if and only if α and β are both even. The remaining two parts of Lemma 1 now follow since the multiplication table for quaternions is unaltered by the cyclic permutation (ijk) of the units.

3. THEOREM 1.—*Any integral quaternion whose norm is even can be expressed in one and but one way in the form* $2^e\pi Q$, *where* π *is one of the six quaternions*

(2) $$1,\ 1+i,\ 1+j,\ 1+k,\ (1+i)(1+j),\ (1+i)(1+k),$$

while Q *is an odd quaternion* (*i.e. an integral quaternion whose norm is odd*), *and* $e>0$ *if* $\pi = 1$.

Let the norm of $A = a+bi+cj+dk$ be even so that $a+b+c+d$ is even. If $a+b$ and $a+c$ are both odd, their sum and hence also $b+c$ is even. Hence we have at least one of the three cases in Lemma 1, so that A is divisible by at least one of $1+i$, $1+j$, $1+k$. If, for example, $A = (1+j)q$, where q is of even norm, we remove from q one of the same three factors. Thus A equals αQ, where Q is of odd norm and α is a product of factors $1+i$, $1+j$, $1+k$. All of the factors $1+i$ may be moved to the left in view of the following two cases of (1):

$$(1+i)(1+j) = (1+k)(1+i), \quad (1+i)(1-k) = (1+j)(1+i),$$

and $$1-k = (1+k)(-k).$$

Also $$(1+j)(1+k) = (1+i)(1+j), \quad (1+k)(1+j) = (1+i)(1+k)j,$$

and $$(1+i)^2 = 2i.$$

Hence A may be expressed in the form $2^r\pi Q$.

It remains to prove that A can be expressed in this form in a single way. Since $1+i$, $1+j$ and $1+k$ are of norm 2, there are four cases.

First, if $(1+i)\,Q$ is divisible by $1+j$ or $1+k$, where

$$Q = a+bi+cj+dk,$$

then $$(1+i)\,Q = a-b+(a+b)i+(c-d)\,j+(c+d)k$$

is divisible by $1+j$ or $1+k$, whence, by Lemma 1, $a-b+c-d$ or $a-b+c+d$ is even, and Q would be of even norm.

Second, if

$$(1+j)\,Q = a-c+(b+d)\,i+(a+c)\,j+(d-b)\,k$$

were divisible by $1+k$, then $a-c+d-b$ would be even and Q of even norm.

Third, if $2^e Q = 2^v(1+i)(1+j)q$, where Q and q are odd quaternions, their norms give $2^{2e}=2^{2v}2^2$, whence $e=v+1$. Thus $2Q=(1+i)(1+j)q$. Multiply on the left by $1+i$, and apply $2i(1+j)=2(1+j)k$. We get $(1+i)Q=(1+j)(kq)$, which is impossible by the first case.

Fourth, $2^e Q \neq 2^v(1+i)(1+k)q$, as in the third case.

4. LEMMA 2.—Given any quaternion g and any positive odd integer m, we can find an integral quaternion q such that $N(g-mq) < m^2$.

For, if g_s and q_s are the coordinates of g and q, those of $g-mq$ are g_s-mq_s, each of which can be made numerically less than $m/2$ by choice of integers q_s. Then $N(g-mq) < 4(\frac{1}{2}m)^2$.

THEOREM 2.—*If a is any integral quaternion and b is any odd quaternion, we can find integral quaternions q, c, q_1, c_1 such that*

$$a = qb+c, \qquad N(c) < N(b), \tag{3}$$

$$a = bq_1+c_1, \quad N(c_1) < N(b). \tag{4}$$

To obtain (3), apply Lemma 2 for $g = ab'$, $m = bb'$. Then

$$g-qm = (a-qb)\,b'$$

has the norm $b'bN(a-qb) < m^2$. Noting that $m^2 = b'bN(b)$, and writing c for the integral quaternion $a-qb$, we have (3). To obtain (4), apply Lemma 2 for $g = b'a$, $m = b'b$, $q = q_1$, and write c_1 for $a-bq_1$.

5. Two integral quaternions a and b shall be said to have a right-hand greatest common divisor D, if D is a right-hand divisor of both a and b, and if every common right-hand divisor of them is a right-hand divisor of D. There is a similar definition of a left-hand greatest common divisor.

THEOREM 3.—*Any two integral quaternions a and b, at least one of which is odd, have a right-hand greatest common divisor D which is uniquely determined up to a unit factor. Also*

$$D = Aa+Bb, \tag{5}$$

where A and B are integral quaternions. Similarly, there is a left-hand greatest common divisor δ, unique up to a unit factor, for which

$$\delta = a\alpha+b\beta.$$

Let b be an odd quaternion. In (3) express c in the form $2^{\gamma}\pi C$, where π is one of the six quaternions (2), and C is an odd quaternion. Repeat the process on b and C. Since $N(b)$, $N(c)$, ... form a series of decreasing integers $\geqslant 0$, we must reach a term of norm zero. To simplify the notations, let this happen at the third step, so that

$$(6) \qquad a = qb+2^{\gamma}\pi C, \qquad b = q_1 C+2^{\delta}\pi_1 D, \qquad C = q_2 D,$$

where C and D are odd quaternions, while π and π_1 are quaternions (2). These equations, taken in reverse order, evidently imply that D is a right-hand divisor of both b and a.

Next, let δ be a right-hand divisor of both $a = \alpha\delta$ and $b = \beta\delta$. Then $(\alpha-q\beta)\delta = 2^{\gamma}\pi C$. Since δ and C are odd quaternions, it follows from Theorem 1 that $\alpha-q\beta = 2^{\gamma}\pi Q$, where Q is an odd quaternion such that $C = Q\delta$. Then the second equation (6) gives $(\beta-q_1 Q)\delta = 2^{\delta}\pi_1 D$. As before, $\beta-q_1 Q=2^{\delta}\pi_1 Q_1$, $D=Q_1\delta$. Thus δ is a right-hand divisor of D.

As to the uniqueness of D, let D and E be right-hand divisors of each other, so that $D = dE$, $E = eD$, where d and e are integral quaternions. Then $E = edE$, $1 = N(ed) = N(e)N(d)$, so that e and d are units.

To prove (5), multiply the second equation (6) by $2^{\gamma}\pi$ on the left. We get

$$2^{\gamma+\delta}\pi\pi_1 D = 2^{\gamma}\pi b-2^{\gamma}\pi q_1 C.$$

By (1) and its analogues in $1+j$ and $1+k$, $\pi q_1 = Q\pi$, where Q is an integral quaternion. Next replace $2^{\gamma}\pi C$ by its value from the first equation (6). We get

$$2^{\gamma+\delta}\pi\pi_1 D = -Qa+(Qq+2^{\gamma}\pi)b.$$

Multiply this on the left by the conjugate to $\pi\pi_1$, whose norm is a power 2^{ϵ} of 2. Thus

$$(7) \qquad ED = \rho a+\sigma b, \quad E = 2^{\gamma+\delta+\epsilon},$$

where ρ and σ are integral quaternions. Since E is relatively prime to the odd integer $b'b=I$, there exist integers l and m for which $lE+mI=1$. Multiplying (7) by l and $ID = DI = (Db')b$ by m and adding, we get (5).

6. The limitation made in Theorem 3 that one of the quaternions be odd is essential. In fact, there exists no greatest common divisor of 2 and $q = 1+i+j+k$, each of norm 4. If either 2 or q be a product of two integral quaternions not units, each factor is of norm 2. But $2 = (1+i)^2(-i)$. Hence the only factors of 2 are the quaternions associated with 2, 1, $1+i$, $1+j$, $1+k$ (the last three being indecomposable); while those of q are associated with q, 1, $1+i$, $1+j$, $1+k$. The only

common factors are the last four, no one of which is divisible by all the others. Finally, 2 and q are not associated quaternions.

Note that 2 is divisible by the indecomposable quaternions $1+i$ and $1+j$, but not by their product q.

7. *Relatively prime.*—Two integral quaternions a and b, at least one of which is odd, shall be called right-handed relatively prime if, and only if, they have no right-hand common divisor other than a unit, the condition being that there exist integral quaternions A and B such that

$$1 = Aa+Bb.$$

When neither a nor b is an odd quaternion, the last equation is impossible, as shown by the proof in the second case in § 8. For example, $1+i$ and $1+j$ are right-handed relatively prime, but

$$1 = A(1+i)+B(1+j)$$

is impossible in integral quaternions A and B.

8. Theorem 4.—*Let v denote one of the products* $r\epsilon$, $r(1+i)\epsilon$, $r(1+j)\epsilon$, $r(1+k)\epsilon$, in which r is a rational number and ϵ is a unit. Let a be any integral quaternion such that at least one of v and a is an odd quaternion. Then v and a are right-handed (or left-handed) relatively prime if, and only if, $N(v)$ and $N(a)$ are relatively prime. Hence, if v and a are right-handed relatively prime they are left-handed relatively prime and may be called relatively prime.*

First, let $r\epsilon$ and a be right-handed relatively prime, at least one being an odd quaternion. Then, by § 7, there exist integral quaternions g and h for which $ga+hr\epsilon = 1$. Write $l = h\epsilon$. Then

$$N(g)N(a) = N(1-lr) = 1-(l+l')r+ll'r^2,$$

where $l+l'$ and $ll' = N(l)$ are integers. Thus $N(a)$ and $N(r\epsilon) = r^2$ have no common factor. Conversely, if $N(a)$ and $N(v)$ have no common factor, a and v are right-handed relatively prime. For, if $a = A\delta$, $v = V\delta$, where δ is not a unit, their norms have the common factor $N(\delta) \neq 1$.

Second, let $v = r(1+i)\epsilon$ and an odd quaternion a be right-handed relatively prime. Then there exist integral quaternions g and h for which

* Note that, if q is any integral quaternion, $vq = Qv$, where Q is a suitably chosen integral quaternion. For, if $v = r\epsilon$, then $Q = \epsilon q\epsilon'$, since $\epsilon\epsilon' = 1$. If v is one of the remaining three products, we apply (1) and its analogues.

$ga+hv=1$. Thus

$$N(g)\,N(a)=[1-hv][1-(hv)']=1-t+N(hv),$$

where $t=hv+(hv)'$ is evidently a multiple of $2r$, while $N(hv)$ is a multiple of $N(v)=2r^2$. Hence $N(a)$ is relatively prime to $N(v)$.

COROLLARY.—If v is one of the products in the theorem and if $N(v)$ and $N(a)$ have a common factor >1 and are not both even, then v and a have a common right-hand divisor not a unit.

9. *Prime quaternions.*—An integral quaternion, not a unit, is called prime if it admits only such representations as a product of two integral quaternions in which one of the two factors is a unit.

THEOREM 5.—*Every rational prime p is a product of two integral quaternions neither of which is a unit, so that p is not a prime quaternion.*

Since this is true for $2=(1+i)(1-i)$, let $p>2$. As remarked by Euler, there exist* integral solutions of

$$1+x^2+y^2\equiv 0 \pmod{p}.$$

Hence $q=1+xi+yj$ is an integral quaternion whose coordinates are not all divisible by p, and such that $N(q)$ is divisible by p. Then, by the Corollary in § 8 with $v=p$, there exists a right-hand greatest common divisor δ, not a unit, of $p=P\delta$ and $q=Q\delta$. If P were a unit, p would be associated with δ and hence divide q. But the rational number p does not divide each coordinate of q.

THEOREM 6.—*If $N(\pi)$ is a prime number, π is a prime quaternion and conversely.*

Let $N(\pi)$ be a prime and $\pi=ab$. Then $N(a)\,N(b)$ equals the prime $N(\pi)$, so that either $N(a)=1$ or $N(b)=1$, and either a or b is a unit, whence π is a prime quaternion.

Conversely, let π be a prime quaternion. If $N(\pi)$ is even, the prime π is associated with $1+i$, $1+j$ or $1+k$, by Theorem 1, so that $N(\pi)=2$. Next, let $N(\pi)$ be odd and p a prime factor of it. By the Corollary in

* *Proof.*—If -1 is a quadratic residue of p, we may take $y=0$. Henceforth let -1 be a quadratic non-residue of p. Let a be the first quadratic residue of p among the terms $p-1, p-2, p-3, \ldots$. Then $a+1=b$ is a quadratic non-residue. Hence there exist integers x and y for which $a\equiv x^2$, $-b\equiv y^2 \pmod{p}$.

§ 8 with $v = p$, π and p have a common right-hand divisor which is not a unit. Since π is a prime quaternion, it divides p. Thus $p = \pi\pi_1$, $p^2 = N(\pi)\,N(\pi_1)$. If π_1 were a unit, $p = \pi\pi_1$ would be a prime quaternion, contrary to Theorem 5. Since neither $N(\pi)$ nor $N(\pi_1)$ is unity, each equals p.

Thus every prime quaternion π arises from the factorization $\pi\pi'$ of a rational prime p. Conversely, if p is any rational prime, the proof of Theorem 5 shows that $p = P\delta$, where neither P nor δ is a unit, whence $N(P) = N(\delta) = p$. By Theorem 6, P is a prime quaternion. This proves

THEOREM 7.—*Every rational prime is a product of two conjugate prime quaternions, and all prime quaternions arise as factors of rational primes.*

The first part of this theorem states that every prime number is a sum of four squares. Since the product of any two sums of four squares equals a sum of four squares, we have the

COROLLARY.—Every positive integer is a sum of four integral squares.

10. *Decomposition of quaternions into primes.*—In view of Theorem 1 the decomposition into prime quaternions of any integral quaternion reduces in a definite sense to the decomposition of an odd quaternion.

THEOREM 8.—*If c is any odd quaternion and $N(c) = pqr \ldots$, where $p, q, r, \ldots$ are the (equal or distinct) prime factors of $N(c)$ arranged in an arbitrary, but definite, order, then $c = \pi\kappa\rho \ldots$, where $\pi, \kappa, \rho, \ldots$ are prime quaternions whose norms are $p, q, r, \ldots$ respectively. This decomposition is unique apart from the association of unit factors.*

Let π be a left-hand g.c.d. of $c = \pi c_1$ and p. Then $p = N(\pi)$. Let κ be a left-hand g.c.d. of $c_1 = \kappa c_2$ and q. Then $q = N(\kappa)$. Proceeding in this manner, we get $c = \pi\kappa \ldots$. Also, $\pi, \kappa, \ldots$ are uniquely determined up to unit factors by the g.c.d. process.

THE CLASSIFICATION OF RATIONAL APPROXIMATIONS

By P. J. HEAWOOD.

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1. In the *Proceedings of the London Mathematical Society* for January 14th, 1919,* various interesting questions are raised by Mr. J. H. Grace with respect to the rational approximations x/y, to a given (incommensurable) quantity θ, which satisfy the condition $|x/y-\theta| < 1/ky^2$, where k is a given number, x/y being a fraction in its lowest terms. Among other things he refers to a paper of Markoff's,† on which he bases the statement that, if $k \leqslant 3$, there will be an infinite number of such approximations except only in certain cases where θ is a quadratic surd. As a matter of fact Markoff's results are not precisely on the same footing as those required for our purpose, as will appear in the sequel, and the true condition above is $k < 3$, not $k \leqslant 3$; but the question at once arises whether, if k is only *just* greater than 3, the reverse is the case; *i.e.* whether for such values of k there can be *transcendent* numbers θ, for which there are only a *finite* number of approximations such that $|x/y-\theta| < 1/ky^2$. Mr. Grace proceeds to show that such numbers can be constructed for $k = 3{\cdot}0322$ (or any greater value). The whole theory of such approximations depends on the fact that if $|x/y-\theta| < 1/ky^2$ for such a k, or indeed for any value of $k \geqslant 2$, x/y must be a convergent p_n/q_n to the continued fraction for θ. Then, if

$$\theta = [a_0] + \frac{1}{a_1+}\frac{1}{a_2+}\cdots \quad [a_0 \text{ perhaps zero}],$$

we have
$$|x/y-\theta| = 1/\lambda y^2,$$

where
$$\lambda = (a_{n+1};\ a_{n+2},\ a_{n+3},\ \ldots) + (a_n,\ a_{n-1},\ \ldots,\ a_1); \tag{1}$$

since
$$\theta = (ap_n+p_{n-1})/(aq_n+q_{n-1}),$$

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* Ser. 2, Vol. 17, p. 247.

† *Math. Ann.*, Vol. 15, p. 381.

where $$a = (a_{n+1};\ a_{n+2},\ a_{n+3},\ \ldots),$$

while $$q_{n-1}/q_n = (a_n,\ a_{n-1},\ \ldots,\ a_1);$$

using the abbreviated notation for

$$a_{n+1} + \frac{1}{a_{n+2}+} \cdots \quad \text{and} \quad \frac{1}{a_n+}\frac{1}{a_{n-1}+} \cdots.$$

Supposing then (to take Mr. Grace's application), that in the continued fraction for θ the denominators consist of cycles of m 1's and m 2's, each used any number of times in succession, the greatest values of λ will be numbers of the form $(2;\ 2,\ 2,\ \ldots)+(1,\ 1,\ 1,\ \ldots)$, which by taking m large enough can be brought as near as we please to

$$\sqrt{2}+1+\tfrac{1}{2}(\sqrt{5}-1) = 3{\cdot}032\ldots.$$

If, then, k exceeds this limit, numbers θ can be thus constructed which have *no* approximations x/y such that $|x/y-\theta| < 1/ky^2$. But, as the set of such numbers is unenumerable, it must include non-algebraic numbers.

The author suggests that it ought to be possible (if 3 is really the limiting value) so to choose our cycles that the critical value of λ is brought down to 3, but does not see his way to do so. If, however, we take, instead of cycles of m 1's and m 2's, cycles consisting the one of m 1's, the other of two 2's followed by $m-2$ 1's, used precisely as above, this is accomplished. For the maximum value of λ will now be

$$(2;\ 2,\ 1,\ 1,\ \ldots)+(1,\ 1,\ 1,\ \ldots),$$

or, which is the same thing,

$$(2,\ 1,\ 1,\ \ldots)+(2;\ 1,\ 1,\ 1,\ \ldots);$$

which tends to $\frac{1}{2}(3+\sqrt{5})+\frac{1}{2}(3-\sqrt{5})$, *i.e.* 3, as m is indefinitely increased. If, then, k is ever so slightly greater than 3, we can construct transcendental numbers θ for which there are *no* approximations such that $|x/y-\theta| < 1/ky^2$.

2. The main question, however, is as to the structure of numbers θ for which there are (at most) only a finite number of approximations x/y such that $|x/y-\theta| < 1/ky^2$, when $k \leqslant 3$; *i.e.* where, at least beyond a certain point, when θ is reduced to a continued fraction, every $\lambda < 3$ [$\lambda = 3$, *i.e.* $|x/y-\theta| = 1/3y^2$, is impossible with θ incommensurable]. Towards this, Markoff's assistance is somewhat incidental. His main problem is that of the minima of quadratic forms of positive determinant D; and, imagining from Mr. Grace's paper that Markoff's whole theory

was mixed up with that of these forms, I made an independent analysis. Having now read Markoff's article, I find that his continued fraction analysis, though merely subsidiary to his work on the minima of quadratic forms, is really independent of it, and largely on parallel lines with mine. There are, however, many differences in detail and a certain want of symmetry in his classification of types, which obscures some points of special interest. His notation, too, is rather cumbrous and his logical analysis is not very complete. After showing that certain alternatives must be rejected and that certain others are possible, he is content to conclude "De toutes ces considérations il suit, que la suite cherchée présentera l'une des formes suivantes....", without showing clearly how the result is arrived at. I therefore venture to give my analysis exactly as I worked it out independently, referring in footnotes to any important divergences. Moreover Markoff's work is not exactly on all fours with that required for our purpose. The quantities which he has to examine, as to their being continually equal to or less than 3, are of the same form as λ above, but with the important difference that his a's extend without limit in both directions, so that *neither* of the two fractions

$$(a_{n+1};\ a_{n+2},\ a_{n+3},\ \ldots),\quad (a_n,\ a_{n-1},\ \ldots),$$

whose sum is λ, terminates. One result is that it is possible for his λ* to be equal to 3, in certain cases. This does not affect the general course of the analysis. It has, however, to be borne in mind in the final conclusions.

3. Proceeding then to consider the classes of continued fractions for which, beyond a certain point, *no* λ (as defined above) is greater than 3, we can see at the outset that they will be of very restricted types. To begin with, the expression for λ shows that, from the point in question, no denominators can be as great as 3; they must consist of the digits 1 and 2 only. Again, there cannot be an isolated 2 in the midst of 1's, since $(2;\ 1,\ \ldots)+(1,\ \ldots) > 3$, whatever digits follow; nor can there be an isolated 1 in the midst of 2's, since $(2;\ 1,\ 2,\ \ldots)+(2,\ \ldots) > 3$, whatever digits follow; thus the 2's must occur in groups of 2 or more and likewise the 1's. On the other hand, the "points of danger" will occur only when there is a transition from 1's to 2's or from 2's to 1's. If a_{n+1} be a 1 in the midst of 1's,† or a 2 in the midst of 2's, the corresponding λ *cannot* exceed 3, since $(2;\ 2,\ \ldots)+(2,\ \ldots) < 3$; $(1;\ 1,\ \ldots)+(1,\ \ldots) < 3$,

* $2/L$ in his notation: then the least value of $L\sqrt{D}$ is the minimum of the form.

† Or if $a_{n+1} = 1$, in any case.

whatever digits follow those specified. One case therefore which will do is where (after some point) the digits are all 2's or all 1's. Supposing, however, that both 2's and 1's occur throughout, we have only to examine the values of λ where a_{n+1} is the first or last of a succession of 2's.

Where the digits consist merely of 1's and 2's, we may adopt a still further abridged notation for the continued fractions involved. Let $[p\,|\,q\,|\,r\,\ldots]$ stand for

$$\frac{1}{1+}\,\frac{1}{1+}\cdots+\frac{1}{2+}\,\frac{1}{2+}\cdots,$$

where there are p 1's, followed by q 2's, followed by r 1's, etc.; and $\{p\,|\,q\,|\,r\,\ldots\}$ in like manner for

$$\frac{1}{2+}\,\frac{1}{2+}\cdots+\frac{1}{1+}\cdots,$$

where there are first p 2's, then q 1's, then r 2's, etc. That the λ corresponding to the *first* of a set of r 2's may be less than 3, we have, say,*

$$\left(2+\frac{1}{2+}\,\frac{1}{x}\right)+\left(\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{y}\right)<3,$$

where x and y stand for the aggregates which follow;

i.e. $$\frac{1}{2+}\,\frac{1}{x}+\frac{y+1}{2y+1}<1;$$

i.e. $$\frac{1}{2+}\,\frac{1}{x}<\frac{y}{2y+1},\quad i.e.\ <\frac{1}{2+}\,\frac{1}{y};$$

i.e. $$\frac{1}{x}>\frac{1}{y}.$$

Using the abridged notation just explained, if we suppose the r 2's followed by s 1's, then t 2's, etc., and preceded by q 1's, then p 2's, etc., the condition is

$$\{r-2\,|\,s\,|\,t\ldots\}>[q-2\,|\,p\,|\,o\ldots],\tag{1}$$

the two sides of the inequality being the values of $1/x$ and $1/y$ respectively. Since an interchange of the digits 1 and 2 throughout will make the larger fraction the smaller, this is precisely the same thing as:—

$$\{q-2\,|\,p\,|\,o\ldots\}>[r-2\,|\,s\,|\,t\ldots].$$

Dealing in like manner with the λ corresponding to the *last* of the r 2's,

* We have already seen that both 1's and 2's occur in groups of two or more.

we have a condition which may indifferently be written

$$\{r-2\,|\,q\,|\,p\,...\} > [s-2\,|\,t\,|\,u\,...],$$

or

$$\{s-2\,|\,t\,|\,u\,...\} > [r-2\,|\,q\,|\,p\,...]. \tag{2}$$

If two such conditions are satisfied for each group of 2's,* it will secure that we have a fraction for which every λ is less than 3. Comparing them in the forms numbered (1), (2) above, it will be seen that (2) is exactly the same relatively to r, s, ... that (1) is relatively to q, r, ... If, then, the numbers ... o, p, q, r, s, t, ... are those of the 2's and 1's alternately, *it does not really matter which of the alternate sets are* 2's *and which are* 1's, so far as the conditions for $\lambda < 3$ are concerned. We may take (2) as the typical form of condition, which must be satisfied for each "transition," whether it really be from 2's to 1's or from 1's to 2's.†

4. For the typical condition

$$\{s-2\,|\,t\,|\,u\,...\} > [r-2\,|\,q\,|\,p\,...]$$

to be possible we must have *either* $r = 2$ *or* $s = 2$ (or both), since a fraction beginning $\frac{1}{2+}$ ... cannot be greater than one beginning $\frac{1}{1+}$ ...; *i.e.* of two consecutive sets of digits, one at least must be a doublet, whether of 1's or of 2's. [If *both* $r = 2$ *and* $s = 2$, the condition becomes $[t\,|\,u\,...] > \{q\,|\,p\,...\}$ which is necessarily satisfied.] In any case we have

$$\textit{either } r = 2 \textit{ and } \{s-2\,|\,t\,|\,u\,...\} > \{q\,|\,p\,|\,o\,...\},$$

which is equivalent to $\quad [q\,|\,p\,|\,o\,...] > [s-2\,|\,t\,|\,u\,...];$

$$\textit{or else } s = 2 \textit{ and } \qquad [t\,|\,u\,...] > [r-2\,|\,q\,|\,p\,...].$$

These are the conditions for the r-s "transition." For the q-r transition there will be similar conditions involving *either* $q = 2$ *or* $r = 2$. Suppose $r = 2$. Then the conditions for the q-r *and* r-s transitions,

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* Markoff's analysis is complicated by the fact that, though he begins with a transition from 2's to 1's, he does not confine himself to such crucial points nor treat them comprehensively. [He uses throughout the fullest expressions for the continued fractions involved.]

† This "duality" is not observed by Markoff, and this affects the symmetry of his work. Further his final conclusions do not show explicitly the correspondence throughout of cases in which 1's and 2's are interchanged.

as above, reduce to

$$[s|t|u\ldots] > [q-2|p|o\ldots], \tag{i}$$

and

$$[q|p|o\ldots] > [s-2|t|u\ldots]. \tag{ii}$$

We have to see how these can be simultaneously satisfied. Of the alternatives $q-2 < s$, $s-2 < q$, one (or both) must necessarily be true for any pair of numbers q, s. Suppose $q-2 < s$. Then in the fractions compared in (i), the transition from the 1-digits (with which each begins) to the 2-digits occurs earlier on the right hand than on the left. For the fraction on the right to be smaller, the first 2-digit of the p-set, answering there to 1 in the other, must come in an *odd* place; for

$$\frac{1}{1+}\frac{1}{2+}\cdots > \frac{1}{1+}\frac{1}{1+}\cdots,$$

but

$$\frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\cdots < \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots,$$

whatever digits follow, and so on. Thus $q-2$ must be even and so q must be even. Then (i) is satisfied; but for (ii) to hold, $s-2$ cannot be greater than q, or (q being even) the left-hand fraction in (ii) would be the smaller; but *either* $s-2 < q$ and s even, *which with the preceding entails* $s = q$ *and both even*; or else $s-2 = q$, and then (q being even) (ii) reduces to

$$\{p|o\ldots\} > \{t|u\ldots\},$$

which must also be satisfied. Similarly for the case $q-2 = s$.

By considering a typical transition we thus reach the conclusion that for the requisite conditions to be satisfied throughout:—

(1) All the numbers $\ldots, o, p, q, r, s, t, u, \ldots$ must be even;

(2) Of any two consecutive numbers one at least must $= 2$;

(3) If $r = 2$, we have further:—

either (*a*) $q = s$;

or (*b*) $s = q+2$, $\{p|o\ldots\} > \{t|u\ldots\}$;

or (*c*) $q = s+2$, $\{t|u\ldots\} > \{p|o\ldots\}$;

and similarly for each number $= 2$.

The properties of the numbers concerned indicated by (*b*), (*c*) are fairly obvious. All the *numbers* being *even*, a divergence of digits in the fractions compared, when it occurs, will always come in an *odd* place, and so

the fraction which has to be larger must have 1 there, while the other has 2; therefore if the first divergence between the numbers defining the fractions is between numbers representing 2's* the smaller number must come in the symbol for the larger fraction (this securing a digit 1 in that fraction—in an odd place—where the other still has 2); if the divergence is between numbers representing 1's, the reverse will be the case. Thus a proper arrangement in case (*b*) will be $p < t$, *or* $p = t$ and $o > u$, *or* $p = t$, $o = u$ with a smaller number preceding o, in the direct sequence of numbers as given in (1) above, than that which follows u, and so on; while in case (*c*) we must have $p > t$, *or* $p = t$ and $o < u$, etc. It will ensure that *every* $\lambda < 3$, if the conditions just specified hold for every 2 flanked by unequal numbers, throughout the entire sequence of numbers, it being understood that the numbers are all even, that *one* of every two consecutives is 2 and that the flanking numbers, when unequal, differ by 2.

5. These laws, however, may be simplified when we have considered the possibilities of repeated doublets, $2 = r = s = \ldots$.† Suppose that at any point m consecutive numbers each $= 2$, where $m > 1$; then, by the alternatives in (3) of last section, 4 must stand on each side of this sequence of 2's, and by (2) the next number must again be 2. Suppose that here there are n 2's followed again by 4, so that we have in succession‡ $\ldots 4, 2, 2, 2, \ldots$ (m times), $4, 2, 2, \ldots$ (n times), $4, \ldots$. Identifying the last 2 of the m-set with r in the conditions of last section, 3 (*b*) applies; and identifying the first 2 of the n-set with r (*supposing* $n > 1$) 3 (*c*) applies, giving respectively

$$\{2 \mid 2 \mid 2 \ldots (m-2 \text{ times}) \mid 4 \ldots\} > \{2 \mid 2 \mid 2 \ldots (n \text{ times}), 4 \ldots\}, \qquad \text{(i)}$$

$$\{2 \mid 2 \mid 2 \ldots (n-2 \text{ times}) \mid 4 \ldots\} > \{2 \mid 2 \mid 2 \ldots (m \text{ times}), 4 \ldots\}. \qquad \text{(ii)}$$

Exactly as in the preceding paragraph, if $m-2 < n$, (i) implies that m is odd—the 4 in the first bracket must represent 1's not 2's. So if $n-2 < m$, (ii) implies that n is odd. $m-2 < n$ and $n-2 < m$ may both be true: then $m = n$, since both must be odd. Suppose, however, that not only $m-2 < n$ but $m < n-2$; then, by (i), m should be odd, and by (ii) m should be even, which is impossible. Similarly $n < m-2$ is

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* *I.e.* representing 2's in the fractions compared in (*b*), (*c*), not necessarily in the original fraction owing to transformations. See the end of § 3.

† This is a question not definitely considered by Markoff.

‡ It should hardly be necessary to emphasise that these 2's are not the *digits* of the continued fraction, but the *numbers* of successive 1's and 2's of which these digits consist.

impossible. We may, however, have $m = n-2$ (and so $m-2 < n$) if m be odd, with a further condition, or $n = m-2$ and odd, with a further condition. Since *either* $m-2 < n$ *or* $n-2 < m$ must hold for *any* pair of numbers, these are the only possible cases, except that of $n = 1$, which was put aside to start with. In that case the condition (ii) of this paragraph disappears [since 3 (a) of § 4 holds good, if we take the isolated 2 for r], and (i) becomes $\{2|2|2 \ldots (m-2 \text{ times})|4 \ldots\} > \{2|4 \ldots\}$, ($m$ being by supposition > 1); and this cannot hold unless $m-2 = 1$, $m = 3$; since $\{2|2 \ldots\}$ and $\{4|\ldots\}$ (the fractions which arise when $m-2 > 1$ or $= 0$), are each less than $\{2|4 \ldots\}$. In every case then we have proved that m and n must both be odd. Since we saw before that one (at least) of every two consecutive numbers must be 2, and now that the number of successive 2's must always be odd, it follows that *every alternate place throughout the whole sequence must be occupied by* 2 (though 2 may appear in other places likewise). Thus *either* the 1's *or* the 2's which constitute the digits of the continued fraction must occur in doublets only, from and after some fixed point (though there may also be doublets of the other).* We may therefore fix our attention exclusively on the sequence of alternate numbers, giving the numbers of intermediate 2's or 1's, as the case may be; and the properties enunciated at the end of § 4 are much simplified. In the notation of that section we have, say, $p = r = t = \ldots = 2$, and any possible divergences between the numbers of the ascending and descending sequences which define the fractions compared in 3 (b), (c) will be between those which represent 1's there.† Therefore in accordance with the rules of that section, the larger of the two first diverging numbers must always come in connection with the larger fraction of the two compared. Using ... $2a$, $2b$, $2c$, ... to represent the alternate numbers, so that (understanding that the others are all 2's and that ... a, b, c ... denote integers) law (1) and law (2) of § 4 are necessarily satisfied, we have the following rules *for any two successive numbers* d, e, identifying $2d$, $2e$ with q, s of that section :—

(I) *Either* $d = e$, *or* $e = d+1$, *or* $d = e+1$.

(II) If d, e are unequal, the sequences $c, b, a, \ldots;\ f, g, h, \ldots$, formed by taking the numbers backwards from d and forwards from e, must, when they first diverge, have the *larger* number in the sequence which starts from the *smaller* of the two d, e—that being the sequence corresponding to the fraction which has to be the larger in accordance with (3), (b), (c).

* Markoff only shows explicitly the case of the 2's occurring in doublets, though the possible vanishing of certain numbers which he uses involves the other case.

† Not necessarily representing 1's *in the original fraction*, as noted in § 4.

E.g. if $e = d+1$, either $f < c$, or $f = c$, $g < b$ or $f = c$, $g = b$, $h < a$, and so on; while if $d = e+1$ we have $c < f$, etc. Conversely, if (I), (II) are satisfied for each successive pair of numbers, all the requirements for $\lambda < 3$ are fulfilled.

6. If we consider a little further the implications of the above laws, remarkable consequences follow. To begin with, the numbers in the sequence, even when not all equal, will be very closely restricted. Suppose that, in the sequence ... a, b, c, ..., c differs from b, but is the first of h numbers each equal to c, followed by a different number c', *where* $h > 1$. By § 5 (I) $c' = c \pm 1$: suppose $c' = c+1$. Then we have $c, c+1$ preceded by c (in fact by $h-1$ c's) and therefore by § 5 (II) followed by a number $= c$ or less. By § 5 (I), again, the number following $c+1$ cannot be less than c: it must therefore $= c$, the first (suppose) of k c's. We thus have (α) c, $c+1$ (as indicated by the ordinates at A, B in the graph below) preceded by $h-1$ c's and followed by k c's, and (β) $c+1$, c (as at B, C in the figure) preceded by h c's and followed by $k-1$ c's (Fig. 1).* From (α) by

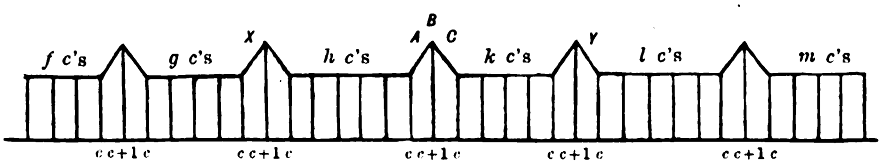

FIG. 1.

§ 5 (II), if $k < h-1$, the (new) number following the k c's should be $< c$ (since the k-th number before A, the smaller of the two values A, B, is c); while by (β) (from a like consideration of the numbers before and after BC) it should be greater, since *a fortiori* $k-1 < h$. Thus we cannot have $k < h-1$; nor (similarly) $h < k-1$: *i.e.* k must $= h-1$ or h or $h+1$.

In the first place suppose that $k = h-1$ or h. Then $k-1 < h$; and so, recurring to the sequences preceding and following BC, the number following the k-th c (which by supposition is not c) must be $c+1$ by § 5(II). If, of the alternatives now being considered, $k = h$, and therefore by hypothesis greater than unity, this $c+1$ must be followed by c, just as $c+1$ at B involved c at C; and similarly $c+1$ and then c will precede the h c's, since $h-1 < k$. If, however, $k = h-1$, it will still follow, from

* The object of the graph is not merely to make clearer to the eye the ups and downs of the sequence of numbers, but also to distinguish by letters A, B, ... certain of these numbers from other equal numbers.

(a), that the $h-1$ c's before A must be preceded by $c+1$, inasmuch as the k c's are followed by $c+1$ (see above), so that by § 5 (II) the term cannot be $< c+1$, while by § 5 (I) it cannot be greater; and again this $c+1$ must be preceded by c, as before. *Therefore* (again recurring to the sequences on each side of AB) *even if* $k \doteq 1$, *so that the original argument for* $c+1$ *at* B *followed by* c *at* C *which was based on* $h > 1$, *does not apply*, the $c+1$ which follows the k c's must be followed by c (as at Y) because the $c+1$ which precedes the $h-1$ c's is preceded by c (at X), k being $= h-1$; for by § 5 (II) that at Y cannot be greater than that at X, while by § 5 (I) it cannot differ by more than a unit from $c+1$, which immediately precedes.

So far as noted, then, the sequences *preceding* and *following* AB, *i.e.* those back to X and on to Y, inclusive, in the graph, Fig. 1, will agree in the case of $k = h-1$, as there shown. Suppose the c at X is the last of g c's and that at y the first of l c's, which g and l c's, again, by a repetition or extension* of the preceding arguments will be preceded and followed by $c+1$, c; and so on, indefinitely (so that the numbers throughout will be limited to the values c and $c+1$). Then in order that, in the sequences following $c+1$ at B and preceding c at A, the larger number, when they diverge, may occur in the sequence starting from the lower, we must have $g < l$, if g, l are unequal; and then $c+1$ before the g c's will answer to c in the other sequence, which is right; or, if $g = l$, then $f < m$, supposing $c+1$ and then f c's precede and $c+1$ and then m c's follow; or else $g = l$, $f = m$, with a like further condition; and so on. Similarly in the case of $h = k-1$, which was left aside, we should have $g > l$; or $g = l, f > m$; or $g = l$, $f = m$, with a like further condition; and so on. The result is that the whole sequence of numbers ... a, b, c, ..., restricted as we have

* An extension of the argument will be required if "singlets" are repeated. Suppose $h = 2$, $k = 1$, $l = 1$, as in Fig. 2; the single c (at Y) given by $l = 1$, being necessarily followed by $c+1$ as before (because $c+1$ and then c precede it). Then we must recur to the sequences preceding and following AB to show that c at X must be preceded by $c+1$, answering to $c+1$ in the following sequence, so that $g = 1$. Then $c+1$, c immediately following X will be the starting point of sequences showing that again c must precede; and then again the sequences based on c, $c+1$ at A, B show that c must follow the $c+1$ after Y, the terms being determined in the order shown by (1), (2), (3), (4) in Fig. 2; and so on.

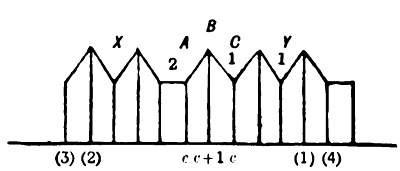

Fig. 2.

seen to sets of c's separated by single $(c+1)$'s, must be such that $\dots f, g, h, k, l, m, \dots$ *the numbers of c's in the successive sets*, obey precisely the same laws as those of the sequence which they thus define, namely, the laws § 5 (I) and (II) which we have been using throughout the present section. This is all on the supposition that at least two consecutive c's occur somewhere and that the next number $c' = c+1$. If $c' = c-1$, we have, by precisely similar arguments, single $(c-1)$'s separating groups of $\dots f, g, h, \dots, c$'s, as indicated in Fig. 3, where $\dots f, g, h, \dots$ obey precisely the same laws as before.

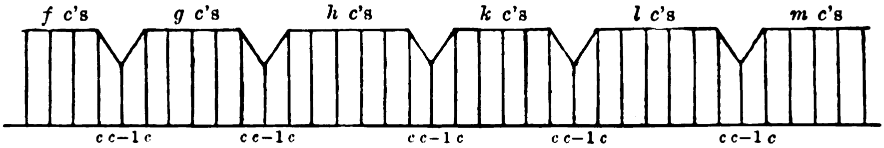

FIG. 3.

We may, however, have single c's alternating throughout with single $(c+1)$'s, which we may consider a special case of either of the above with $f = g = \dots = 1$ (Fig. 4); or the still more rudimentary case where all the numbers are c's (Fig. 5):—

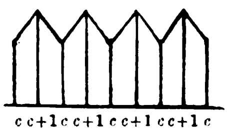

FIG. 4.

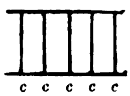

FIG. 5.

But the general result is that the primary sequence $\dots a, b, c, \dots$ consists of at most two different numbers, one of which, occurring singly, differs from the other by unity, while the occurrence of the other in sets of $\dots f, g, h, \dots$ together is "regulated" by the "secondary" sequence of the numbers $\dots f, g, h, \dots$, obeying precisely the same laws as the primary; also, since it obeys the laws § 5 (I) and (II) it must obey the further laws deduced from them in this section and be "regulated" by a like sequence $\dots x, y, z, \dots$, say, and so on; except that we can go no further when we reach a sequence of equal terms. Conversely, if the "regulating" sequence obeys the laws, this will hold also for the sequence which it regulates.

7. So far the work is on parallel lines to that of Markoff. Though his results are less symmetrically formulated, the laws of a sequence of

numbers defining in the way described a sequence of "digits" 1, 2 for which every $\lambda \leqslant 3$, where λ is of the form

$$a_{n+1} + \frac{1}{a_{n+2}+} \frac{1}{a_{n+3}+} \cdots + \frac{1}{a_n+} \frac{1}{a_{n-1}+} \cdots,$$

the a's denoting the digits in order, hold for him and us alike. But for us the successive a's are the digits (*or the digits from and after some fixed point*) of the continued fraction for θ, and therefore unlimited on one side only; and this has to be borne in mind in making our application of the results; whereas for Markoff (as before pointed out) the sequence of a's is unlimited in both directions.

His application to the theory of the (numerical) minima of a set of equivalent quadratic forms such as $ax^2+2bxy+cy^2$, where a, b, c are given numbers, is briefly as follows. [It is supposed that x, y are to receive integral values not both zero.] By linear transformations such a form can always be "reduced" to one

$$a_r x_r^2 + 2b_r x_r y_r + c_r y_r^2,$$

whose roots (*i.e.* the values of x_r/y_r which make it vanish) are one positive and greater than unity $= \xi_r$, say, and the other negative numerically less than unity $= -1/\eta_r$. Supposing that

$$\xi_r = a_r + \frac{1}{a_{r+1}+} \cdots, \quad \eta_r = a_{r-1} + \frac{1}{a_{r-2}+} \cdots$$

(neither terminating), the transformation

$$x_r = a_r x_{r+1} + y_{r+1}, \quad y_r = x_{r+1}$$

will give a new form of which the roots are

$$a_{r+1} + \frac{1}{a_{r+2}+} \cdots, \quad -\frac{1}{a_r+} \frac{1}{a_{r-1}+} \cdots;$$

and similarly through a whole series of transformations. Thus the λ's come in as the differences between the roots of a succession of equivalent reduced forms, got by using such transformations backwards and forwards; the a's here answering to the a's in the preceding paragraphs.

Unless all the a's are 1's we may take for our standard form, of this equivalent set, one $ax^2+2bxy+cy^2$ for which the root $\xi > 2$. For this, *when* $y=0$, the minimum, got by putting $x=1$, has the numerical value $|a|$, and since the form $\equiv ay^2(x/y-\xi)(x/y+1/\eta)$, it can be shown that, when y is different from 0, it can have no lower value than $|a|$ unless, *supposing x/y positive*, $|x/y-\xi| < 1/2y^2$ and then x/y must be a convergent to ξ, say the n-th; and the formulæ of transformation will show that the result of substituting such values of x, y in the standard form is the same as the result of substituting 1, 0 for the variables in the n-th form from the standard one, *i.e.* it is the a of that form. Similarly, if x/y is negative, $-x/y$ (if x, y give a lower value) must be a convergent to $1/\eta$, and we have a value which is the a of some preceding form. Thus the minimum value required is that of the numerically least of the a's. But the difference of the roots of any form, *which we have seen is one of the succession of* λ's, is equal to $2\sqrt{D}/|a|$ where a is the coefficient of x^2 in that form, D being the determinant of the whole set of forms. It follows that the least value of $|a|$ is the same thing as the least value of $2\sqrt{D}/\lambda$, which is therefore the minimum required. In particular, if $\lambda < 3$ throughout, the minimum is greater than $\frac{2}{3}\sqrt{D}$, and this is a critical value. If the *coefficients* a, b, c are rational, the a's must recur. [The case of rational *roots* need not be considered, as the form then has zero for its minimum.]

Our concern is with the fact that so long as the successive λ's belonging to the continued fraction for θ are less than 3, there can be no approximation x/y to θ among the convergents to θ (where alone it could be found) such that $|x/y-\theta| < 1/3y^2$; and these λ's, like Markoff's, *will* be less than 3, provided the digits for θ consist of 1's and 2's obeying the laws formulated in the preceding sections. But as the digits start from a fixed point the numbers which define them are also terminated in one direction. Now the final law of the numbers ... a, b, c, ... which we reached in § 5, is that the sequences taken onwards from the second and backwards from the first of two unequal consecutive terms must, *when* they diverge, diverge in a particular manner, but (though certain consequences have been further developed in § 6) the possibility of continual agreement *without* divergence has not yet been faced, and this introduces quite new considerations. In the sequences belonging to Markoff's problem, this possibility is of no great moment. Continued agreement, with him, is agreement to infinity, and that merely means that the corresponding λ, instead of being definitely less than 3, takes the limiting value 3; a contingency impossible in our case (with n finite) since our λ is the sum of a terminating and an unending fraction. For us continual agreement without divergence can only mean agreement until the backward-reaching sequence terminates with the initial term of the whole series of numbers, and then the result $\lambda < 3$, though not contradicted, is not guaranteed. Further examination is necessary to see whether after all λ may not then be > 3, in any given case.

We have then two or three possibilities to consider. Suppose, in the first place, that the proper divergences do always show themselves and always *within a finite number of terms*, a number $< N$, say, where N is finite. Here Markoff's sequences (except that they have no beginning) are on the same footing as ours, and his digits, starting from some arbitrary point, would equally serve our purpose. Forming the successive derived sequences, as explained in the last section, since each is more "condensed" than the preceding, we shall at length reach a sequence where the "range of fulfilment" reduces to a single term, *i.e.* a sequence of equal numbers. This monotonous repetition involves a corresponding repetition of more extended range in the original sequence, and therefore in the continued fraction for θ, so that θ must equal a quadratic surd. In such a case λ will be always less than 3 *by a certain finite amount at least* (possibly very small) depending on the value of N. Conversely, *any quadratic surd* for which λ beyond a certain point is always definitely less than 3 must depend on a sequence such as that described above, where the laws work

themselves out within a finite range and lead ultimately to a sequence of equal terms. Further, since there will be an infinite number of sequences ... a, b, c, ... of this character, there will be an infinite number of quadratic surds θ (though of narrowly restricted types) for which there are no approximations x/y (or only a finite number) such that

$$|x/y-\theta| < 1/3y^2.$$

But suppose, on the other hand, that the digits are governed by a sequence of numbers for which, although such divergences as there are between the sequences which have to be compared are always in the right direction, instances of complete agreement continually crop up however far we go. In such a case everything may depend on the way in which the digits start. Even if they may be said to be governed from the outset by the sequence of numbers, in the way supposed, it still has to be decided whether a doublet of the one digit, or, say, $2a$ of the other digit (taking a to be the initial number of the sequence) is to come first, and also which of these are 1's and which 2's. And if, as has been contemplated, there are digits preceding those thus determined, everything will depend on the arrangement of such initial digits, which may affect the question of $\lambda >$ or < 3 at any distance ahead, when the sequence a, b, c, ... is of the critical nature supposed. One way of forming a sequence which is certainly *not* of the character considered in the *last* paragraph is by taking for the primary sequence a set of numbers so adjusted that the primary is identical with the secondary. Then so far as the sequence obeys the laws at the start, it will do so throughout, the terms which follow being regulated by initial terms which do, and so on, indefinitely. Since, however, all the successive derived sequences will be identical, we shall never reach a sequence of equal numbers. Therefore (1) the corresponding fraction will not recur, (2) the laws will take longer and longer to work themselves out as we proceed. Such a case is indicated in Fig. 6, where it will be seen that the primary sequence 1 2 1 2 1 1 2 ... is identical with the secondary, determined by the numbers of 1's in the successive sets of 1's in the primary (if in our reckoning of sets we ignore the initial 1 and begin with the first which has 2's on each side of it). Thus the secondary sequence begins with 1 (answering to the singlet in question) and this is followed by 2, answering to the first *doublet* of 1's, and so on. It will be found, however, that the laws § 5 (I), (II) are obeyed throughout only in the sense that divergences never occur in the wrong direction, and however far we go we have pairs of consecutive unequal terms such that the sequences which precede and follow agree without diverging until the backward one terminates; so that the case is such as was proposed for

consideration at the beginning of this paragraph. Thus 2 1 at HK are

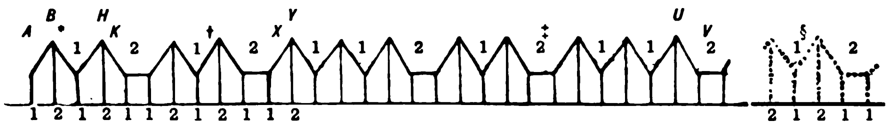

FIG. 6.

preceded by 1 2 1 and followed by 1 2 1 ..., and similarly 1 2 at XY are preceded by 1 2 1 2 1 1 2 1 2 1 (there terminating) and followed by the same numbers (with others beyond); and so continually at successive stages. This agreement is an index to a like agreement in the digits, and everything may therefore depend on how the initial digits are arranged. To examine how this works in detail, suppose that the doubles of the numbers of the sequence, viz. 2, 4, 2, 4, 2, 2, 4, ... indicate numbers of 2's separated by doublets of 1's,* the alternation starting with 1 1; and that (if necessary) these again are preceded by other digits $\alpha, \beta, \gamma, \delta, \ldots$, so that the fraction written at length is

$$\cdots\frac{1}{\delta}+\frac{1}{\gamma}+\frac{1}{\beta}+\frac{1}{\alpha}+\frac{1}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{\underset{A}{2}}+\frac{1}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{\underset{B}{2}}+\frac{1}{2}+\frac{1}{2}+\frac{1}{1}+\cdots,$$

where the *two* 2's preceding A and the *four* 2's divided into pairs at B answer to the initial 1 2 of the number sequence marked AB in Fig. 6. Then the condition to secure $\lambda < 3$ for the first 2 of the B group, which is critical owing to the agreement of digits before A and after B, will be [§ 3 (1)]

$$(2, 2, 1, 1, \alpha, \beta, \gamma, \delta, \ldots) < (2, 2, 1, 1, 2, 2, 1, 1, 2, 2, \ldots),$$

i.e. $$(\alpha, \beta, \gamma, \delta, \ldots) < (2, 2, 1, 1, 2, 2, \ldots). \tag{i}$$

Again, the second 2 1 of the sequence, marked HK in the figure, is the next critical point, the numbers which follow up to the point marked † agreeing, as we have seen, with the 1 2 1 which precede; and so we have the condition [§ 3 (2), first form]

$$(2, 2, \ldots, 1, 1, \alpha, \beta, \gamma, \delta, \ldots) > (2, 2, \ldots, 1, 1, 2, 2, 2, 2, 1, 1, \ldots),$$

i.e. $$(\alpha, \beta, \gamma, \delta, \ldots) > (2, 2, 2, 2, 1, 1, \ldots); \tag{ii}$$

* It will be remembered that the doubles of the sequence numbers may be interpreted as giving *either* numbers of 1's separated by pairs of 2's, *or* numbers of 2's separated by pairs of 1's; but the available initial digits might not correspond in the two cases if they involved other numbers besides 1's and 2's.

$\alpha, \beta, \gamma, \delta, \ldots$ coming now into comparison with the digits defined by the numbers 2 1 which follow † in the sequence. Both conditions are satisfied by taking $\alpha = 2$, $\beta = 2$, $\gamma = 1$, $\delta = 1$, or $\alpha = 2$, $\beta = 2$, $\gamma = 2$, $\delta = 3$, and stopping there. Moreover we can show that if these two conditions are satisfied all subsequent conditions of a like nature will be satisfied, in virtue of the identity of the primary sequence with the secondary, each "critical" point corresponding to the preceding critical point in such a way that the conditions repeat themselves. Thus the 2 1 at HK, already examined, *considered as belonging to the secondary sequence*, answers to 2 *ones* followed by 1 *one* in the primary (in the neighbourhood of XY), after which the grouping of 1's up to the point answering to † will agree with that of those which precede; and this means that again we have 1 2 at XY with the *preceding* sequence of numbers taken right back to the beginning agreeing with the *following* sequence up to ‡, though there will be no other instance of this between HK and XY. Moreover as we have 2, 1 beyond † (the point in the sequence *following* HK which matches that where the backward sequence *preceding* HK ends), so in the sequence following XY we shall have 2 *ones* followed by 1 *one* at the corresponding stage (in the neighbourhood of ‡ in the figure); and that means that we have 1, 2 beyond the point ‡ where the correspondence between the sequence following and that preceding XY ends (owing to the termination of the latter), just as we have 1 2 immediately after AB. $\alpha, \beta, \gamma, \delta, \ldots$ therefore come into comparison as at first with the digits 2 2 1 1 2 2 2 2; and as the sequences now compared are based on 1 2 at XY, we have the same condition (i) as was based on the consideration of the digits defined by 1 2 at AB.

Then, again, corresponding to 1 2 at XY, we have 1 *one* followed by 2 *ones* in the neighbourhood of U, V; and so the sequence following 1 at V will agree with the sequence preceding 2 at U, as far as it goes; and as we have ‡ followed by 1 2, we shall have 1 *one* and then 2 *ones* at the corresponding stage, as indicated by the dotted lines (detached) in the figure, and so giving 2 1 beyond §, the end of the new correspondence based on UV, exactly as we had 2 1 beyond †, in connection with the sequences based on HK, and leading as before to the condition (ii). And so continually the conditions for the critical stages will reduce to either (i) or (ii). Thus the continued fraction, with its initial digits determined as before specified, will have every λ, at least after the very first, less than 3.

We have thus proved that while there are an infinite number of quadratic surds θ, for which there are no approximations such that

$$|x/y - \theta| < 1/ky^2,$$

where $k = 3$, there are also numbers which are *not* quadratic surds of which the same is true. It should be noticed, however, that while, in the case of the surd, every λ is less than 3 by a certain definite amount (at least), λ in the latter case, though always < 3, will approach 3 indefinitely as we proceed; for, at the critical stages, the continued fractions whose difference measures the divergence of λ from 3 will agree to an ever increasing number of digits. And this must hold in the case of any number *not a quadratic surd* of which the statement is true, whether of the kind considered in the present section, or corresponding to a sequence where the proper divergences do (at length) always show themselves. For if agreement always ceases *within a limited number of digits* in the fractions compared, and so within a limited number of terms in the number sequence, while each derived sequence is more "condensed" than the preceding, a sequence of equal numbers must at length be reached and the corresponding fraction will recur.

8. The above disposes of the case of $k = 3$. We have to contrast with it the cases of $k > 3$ and $k < 3$. The former was dealt with at the beginning of the paper (§ 1), where it was shown that however slightly k exceeds 3, there are an unenumerably infinite number of θ's (including therefore transcendental values) for which there are no approximations (or a finite number) with $|x/y - \theta| < 1/ky^2$. The case of $k < 3$ is covered in the main by what we saw incidentally in the last section—that for a quadratic surd of the type specified λ is throughout (or from some point) less than 3 by some definite amount (at least); and conversely that, if λ is always less than 3 by a definite amount, however small, θ must be a quadratic surd. Thus where θ is such a surd, there will be at most only a finite number of approximations x/y such that $|x/y - \theta| < 1/ky^2$, where k is some number < 3, and for this to hold when $k < 3$, θ must be such a surd. It is to be noted, however, that k must be greater than $\sqrt{5}$, the value to which λ tends when all the digits are 1's, or there will be *no* number θ for which the λ's beyond a certain point are all less than k. If, however, $3 > k > \sqrt{5}$, there will always be quadratic surds to which the statement is applicable. Thus the proposed problem is completely solved. In the parallel case Markoff infers that there can be only a finite number of digit-sequences that will do for a given value of $k < 3$, inasmuch as such a value implies a definite limit for N (§ 7), and therefore a limited number of possibilities. He deduces that there can be only a limited number of quadratic forms with minima $\geqslant$ the corresponding fraction of $\sqrt{D}$. We cannot, however, lay down that there will only be a finite number of θ's for such a k, inasmuch as there may be any variety of digits preceding

those which recur in the ways defined by such a sequence. But the recurring part of the continued fraction for θ will admit of only a finite variety of forms. The main conclusion however is that, if there are only a finite number of approximations x/y such that $|x/y-\theta| < 1/ky^2$, θ *must* be a quadratic surd if $k < 3$, but not otherwise.

9. Though it has nothing to do with our own problem, it may be of interest to notice, in conclusion, that in a way parallel to that of § 7 we may construct a sequence such as Markoff uses, unlimited in either direction, where the primary sequence is identical with the secondary and therefore with all that follow, and which strictly obeys the sequence-laws (Fig. 7).

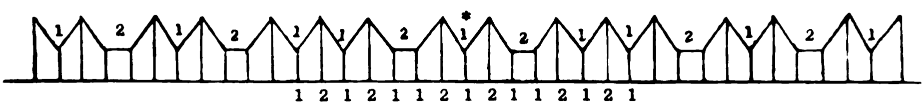

FIG. 7.

Here λ will always be less than 3, approaching however the limiting value 3 as we recede indefinitely from the central point of symmetry, shown by * in the figure. But the coefficients of corresponding quadratic forms would be irrational, not to say transcendental, functions. [See the paragraph in small print under § 7.]

THE DIFFERENTIATION OF THE COMPLETE THIRD JACOBIAN ELLIPTIC INTEGRAL WITH REGARD TO THE MODULUS, WITH SOME APPLICATIONS

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THE first part of this paper contains some results relating to the complete third elliptic integral. They may be regarded as complementary to the formulæ

$$\frac{dK}{dk} = \frac{E-k'^2K}{kk'^2}, \qquad \frac{dE}{dk} = \frac{E-K}{k}.$$

They are followed by examples of their application.

1. The third Jacobian elliptic integral with parameter v and modulus k is defined by

$$\Pi(u, v\,;\ k) \equiv \int_0^u \frac{k^2 \operatorname{sn} v \operatorname{cn} v \operatorname{dn} v \operatorname{sn}^2 u}{1-k^2 \operatorname{sn}^2 v \operatorname{sn}^2 u}\, du,$$

and the identity

$$\Pi \equiv \Pi(K, v\,;\ k) = KZ(v) = KE(v) - vE$$

relates the complete elliptic integral of this kind with those of the first and second kinds.

In the last equation Π is a function of v and of k. If there is a functional relation between v and k it can be proved* by differentiating this equation that

$$\frac{d\Pi}{dk} = (K \operatorname{dn}^2 v - E)\left[\frac{dv}{dk} - \frac{1}{kk'^2}\{E(v) - k'^2 v\}\right] + \operatorname{sn} v \operatorname{cn} v \operatorname{dn} v\, \frac{k}{k'^2} K. \quad (1)$$

Now suppose that $\operatorname{sn} v$ and k are both given as functions of a certain

* *Proc. Edin. Math. Soc.*, Vol. 37, 1918.

number of parameters $\lambda, \lambda', \lambda'', \ldots$, and let it be required to find the derivative of Π with regard to any one of them, say λ. In order to fix our ideas let $\operatorname{sn} v$ and k be given explicitly in the form

$$k = k(\lambda, \lambda', \lambda'', \ldots), \tag{2}$$

$$\operatorname{sn}(v, k) = s(\lambda, \lambda', \lambda'', \ldots). \tag{3}$$

We shall have
$$\frac{\partial \Pi}{\partial \lambda} = \left(\frac{d\Pi}{dk}\right)_\lambda \frac{\partial k}{\partial \lambda},$$

where $\left(\frac{d\Pi}{dk}\right)_\lambda$ means "$\frac{d\Pi}{dk}$ when λ alone varies."

With a similar meaning in the notation, a differentiation of (3) gives

$$\left(\frac{dv}{dk}\right)_\lambda = \frac{1}{kk'^2}\left[E(v) - k'^2 v - k^2 \frac{\operatorname{sn} v \operatorname{cn} v}{\operatorname{dn} v}\right] + \frac{1}{\operatorname{cn} v \operatorname{dn} v} \frac{\frac{\partial s}{\partial \lambda}}{\frac{\partial k}{\partial \lambda}}.$$

This expression being substituted in (1), we obtain $\left(\frac{\partial \Pi}{\partial k}\right)_\lambda$, and on multiplying by $\frac{\partial k}{\partial \lambda}$,

$$\frac{\partial \Pi}{\partial \lambda} = \frac{K \operatorname{dn}^2 v - E}{\operatorname{cn} v \operatorname{dn} v} \frac{\partial s}{\partial \lambda} + \frac{k}{k'^2} \frac{\operatorname{sn} v \operatorname{cn} v}{\operatorname{dn} v} E \frac{\partial k}{\partial \lambda}. \tag{4}$$

If $\operatorname{sn}(v, k)$ be given as a function of k only, this can be written

$$\frac{\partial \Pi}{\partial k} = \frac{K \operatorname{dn}^2 v - E}{\operatorname{cn} v \operatorname{dn} v} \frac{ds}{dk} + \frac{k}{k'^2} \frac{\operatorname{sn} v \operatorname{cn} v}{\operatorname{dn} v} E, \tag{5}$$

where
$$\operatorname{sn}(v, k) = s(k).$$

There are four cases in which the last equation reduces to a very simple form. They occur when $\operatorname{sn}(v, k)$ is given as a function of k by the relations:

(i) $\operatorname{sn}(v, k) = \text{const.}$,

(ii) $\operatorname{sn}(v+K, k) = \text{const.}$,

(iii) $\operatorname{dn}(v, k) = \text{const.}$,

(iv) $\operatorname{dn}(v+K, k) = \text{const.}$

The respective values of $\dfrac{d\Pi}{dk}$ are

$$\left(\frac{d\Pi}{dk}\right)_1 = \frac{\operatorname{sn} v \operatorname{cn} v}{\operatorname{dn} v} \frac{k}{k'^2} E, \tag{6}$$

$$\left(\frac{d\Pi}{dk}\right)_2 = \operatorname{sn} v \operatorname{cn} v \operatorname{dn} v \frac{k}{k'^2} K, \tag{7}$$

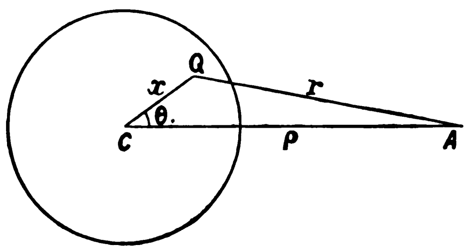
$$\left(\frac{d\Pi}{dk}\right)_3 = \frac{\operatorname{sn} v \operatorname{dn} v}{\operatorname{cn} v} \frac{dK}{dk}, \tag{8}$$

$$\left(\frac{d\Pi}{dk}\right)_4 = -\frac{\operatorname{cn} v \operatorname{dn} v}{\operatorname{sn} v} \frac{1}{k'^2} \frac{dE}{dk}. \tag{9}$$

2. As a first application of the above formulæ, consider the direct determination of the components of electromagnetic force due to a circular current.

If the current be of unit strength the potential at a point P is represented by the solid angle Ω, subtended at P by the circle, and can be expressed in part as a complete elliptic integral of the third kind. The components of magnetic force at P are usually found by differentiating Maxwell's expression for M—the coefficient of mutual induction of two circles which have the same axis and lie in parallel planes—under the sign of integration; M being the Stokes function of Ω.

Suppose that the plane of the paper is horizontal, and that the point



P is at a height z vertically above A. Let C be the centre of the circle, $CQ = x$, $CA = \rho$, $QA = r$, $\angle QCA = \theta$, $\angle QPA = \epsilon$, $QP = R$; so that z, ρ are the usual cylindrical coordinates; then

$$\Omega = \int \frac{dS \cos \epsilon}{R^2},$$

where dS is an element of the area of the circle at Q, and the integral is

taken over the circle. Writing $\cos\epsilon = z/R$, we have

$$\Omega = 2z\int_0^\pi d\theta\int_0^a \frac{x\,dx}{(z^2+\rho^2+x^2-2\rho x\cos\theta)^{\frac{3}{2}}}.$$

If we transform to the notation of elliptic integrals by means of the substitution

$$\cos\frac{\theta}{2} = \operatorname{sn} u, \qquad k^2 = \frac{4\rho a}{z^2+(\rho+a)^2},$$

the value of Ω can be reduced to the form

$$\Omega = 2\pi - \frac{z}{\rho}\,k^2 \operatorname{sn} v\, K - 2i\Pi(K, v;\, k),$$

where v—which is of the form $K+iv'$, where v' is real—is defined by

$$\operatorname{sn} v = \frac{\sqrt{[z^2+(\rho+a)^2]}}{\rho+a},$$

$$\operatorname{cn} v = -i\,\frac{z}{\rho+a}, \qquad \operatorname{dn} v = \frac{a-\rho}{a+\rho},$$

the sign of $\operatorname{dn} v$ being chosen so that its value may be equal to unity on the axis $\rho = 0$, and elsewhere it may be obtained by continuous variation from this value.

For the axial component of magnetic force we then have

$$-\frac{\partial\Omega}{\partial z} = \frac{1}{\rho}\,\frac{\partial}{\partial z}\,(zk^2 \operatorname{sn} v\, K) + 2i\,\frac{\partial\Pi}{\partial z},$$

and in order to find $\partial\Pi/\partial z$ we may quote formula (8), § 1, since $\operatorname{dn} v$ remains constant when z alone varies. The radial component is

$$-\frac{\partial\Omega}{\partial\rho} = z\,\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\,k^2 \operatorname{sn} v\, K\right) + 2i\,\frac{\partial\Pi}{\partial\rho},$$

and to obtain $\partial\Pi/\partial\rho$ it is necessary to fall back upon the general equation (4). After performing the differentiations we find the known values

$$-\frac{\partial\Omega}{\partial z} = \frac{2}{r_1}\left(K - \frac{r^2-a^2}{r_2^2}\,E\right),$$

$$-\frac{\partial\Omega}{\partial\rho} = -\frac{2z}{\rho r_1}\left(K - \frac{r^2+a^2}{r_2^2}\,E\right),$$

where r is now written for CP, and r_1, r_2, are the greatest and least distances of P from a point of the circle.*

3. As a second application, consider the problem of determining, on an ellipsoid of revolution, a particular geodesic whose period in longitude is assigned.

Let a point on the ellipsoid

$$\frac{x^2+y^2}{a^2}-\frac{z^2}{c^2}=1$$

be defined by

$$x = a\sin\theta\cos\psi,\quad y = a\sin\theta\sin\psi,\quad z = c\cos\theta.$$

Suppose, for example, that this surface has the prolate form, so that

$$c^2e^2 = c^2-a^2,\quad\text{and}\quad c > a.$$

Along the geodesic which touches the parallels of latitude $\theta = a$, $\theta = \pi - a$, we know that

$$\frac{d\psi}{d\theta} = \frac{c\sin a\sqrt{(1-e^2\cos^2\theta)}}{a\sin\theta\sqrt{(\cos^2 a-\cos^2\theta)}}.$$

Transform this equation to the notation of Jacobian elliptic integrals by writing

$$\cos\theta = -\cos a\,\mathrm{sn}\,u,\qquad k = e\cos a,$$

the minus sign being chosen so that the value of θ, beginning at $\frac{1}{2}\pi$, will increase with u, and $u = 0$ will correspond to a point where the geodesic crosses the equator. If we introduce the parameter v defined by

$$\mathrm{sn}\,v = \frac{1}{e}\,(>1),\qquad \mathrm{cn}\,v = -i\,\frac{\sqrt{(1-e^2)}}{e},\qquad \mathrm{dn}\,v = \frac{\sqrt{(e^2-k^2)}}{e},$$

we find $$\frac{d\psi}{du} = -i\,\frac{\mathrm{sn}\,v\,\mathrm{dn}\,v}{\mathrm{cn}\,v}+i\,\frac{k^2\,\mathrm{sn}\,v\,\mathrm{cn}\,v\,\mathrm{dn}\,v\,\mathrm{sn}^2 u}{1-k^2\,\mathrm{sn}^2 v\,\mathrm{sn}^2 u},$$

and if Ψ represent one quarter of the longitudinal period,

$$\Psi = \int_0^K \frac{d\psi}{du}\,du = -i\,\frac{\mathrm{sn}\,v\,\mathrm{dn}\,v}{\mathrm{cn}\,v}K + i\Pi(K,\,v;\,k) \tag{1}$$

$$= \frac{\sin a}{\sqrt{(1-e^2)}}K + i\Pi(K,\,v;\,k). \tag{2}$$

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* See Greenhill, *Trans. Amer. Math. Soc.*, Vol. 8 (1907), § 31; Maxwell, *Elec. and Mag.*, Vol. 2, § 701.

Making use of (6) of § 1 (sn v being constant), we find on differentiating

$$\frac{d\Psi}{da} = \frac{ce}{a}\,\frac{K-E}{k} = -\frac{e}{\sqrt{(1-e^2)}}\,\frac{dE}{dk}\,; \tag{3}$$

and

$$\frac{d^2\Psi}{da^2} = -\frac{e}{\sqrt{(1-e^2)}}\,\frac{d^2E}{dk^2}\,\frac{dk}{da} = -\frac{e}{\sqrt{(1-e^2)}}\tan a\,\frac{dK}{dk}\,. \tag{4}$$

Also, if in (1) we write $K-t$ for u, and substitute for v in terms of a, Ψ appears in the form

$$\Psi = \frac{1-e^2\cos^2 a}{\sin a\sqrt{(1-e^2)}}\int_0^K \frac{dt}{1+(1-e^2)\cot^2 a\,\mathrm{sn}^2\,t}, \tag{5}$$

in which the lower limit of integration corresponds to a point where the curve touches a parallel of latitude. Now when $a \to \frac{\pi}{2}$,

$$k \to 0, \qquad K \to \frac{\pi}{2}, \qquad \text{and} \qquad \Psi \to \frac{1}{\sqrt{(1-e^2)}}\,\frac{\pi}{2} = \frac{c}{a}\cdot\frac{\pi}{2}\,. \tag{6}$$

Again,* the value of Ψ given in (5) can be expressed in the form

$$\Psi = \frac{\sqrt{(1-e^2\cos^2 a)}}{\sqrt{(1-e^2)}}\,\frac{\pi}{2} + \frac{\sin a\,(1-e^2\cos^2 a)}{\sqrt{(1-e^2)}}\int_0^K \frac{1-\mathrm{dn}\,t}{\sin^2 a+(1-e^2)\cos^2 a\,\mathrm{sn}^2\,t}\,dt,$$

and when $a \to 0$, the last integral remains finite, and $\Psi \to \frac{1}{2}\pi$.

This result and (3) and (6) show that as a varies from 0 to $\frac{1}{2}\pi$, the longitudinal period of the geodesic defined by a steadily increases from 2π to $c/a\,.\,2\pi$.

The simple forms of (3) and (4) make it easy to compute the numerical value of a for any particular geodesic whose longitudinal period is given in advance. The numerical work was carried out for the case $c = 4a$, and the values of a calculated for those two geodesics which exactly close upon themselves after two and three turns round the ellipsoid. In this case we have, from (5) and (3),

$$4\Psi = \frac{1+15\sin^2 a}{\sin a}\int_0^K \frac{dt}{1+\dfrac{\cot^2 a}{16}\,\mathrm{sn}^2\,t} \tag{7}$$

and

$$\frac{d}{da}(4\Psi) = \frac{16}{\cos a}(K-E). \tag{8}$$

A rough graph was first drawn to represent the variation of 4Ψ as a

* See Forsyth, *Differential Geometry*, p. 141.

varies from 0 to $\frac{1}{2}\pi$. In order to draw it the values of 4Ψ corresponding to

$$\sin^{-1} k = 0°,\ 15°,\ 30°,\ 60°,\ 90°,$$

or

$$a = 90°,\ 74\tfrac{1}{2}°,\ 59°,\ 26\tfrac{1}{2}°,\ 0°,$$

were calculated approximately by means of (7), and the directions of the tangents to the curve at the corresponding points by means of (8). The integral in (7) was evaluated by expanding it in the form

$$4\Psi = \frac{1+15\sin^2 a}{\sin a} \int_0^K (1-\mu \operatorname{sn}^2 t+\mu^2 \operatorname{sn}^4 t-+\ldots)\, dt,$$

where $\mu = \dfrac{\cot^2 a}{16}$, and integrating term by term; the formula of reduction

$$(n+1)k^2 I_{n+2}-n(1+k^2)\, I_n+(n-1)\, I_{n-2} = 0$$

being used to calculate the integrals

$$I_{2n} = \int_0^K \operatorname{sn}^{2n} t\, dt.$$

If the series does not converge fairly rapidly it is more convenient to use some other method of approximate integration.

From the curve thus drawn the two approximate values of a required were obtained, and the accuracy was then improved upon by using Newton's rule in the form:—if a is an approximate solution of the equation

$$f(a) \equiv m\pi-\Psi(a) = 0,$$

a better value for the solution is

$$a+\frac{f(a)}{\Psi'(a)}-\frac{1}{2}\frac{\Psi''(a)}{\Psi'(a)}\left(\frac{f(a)}{\Psi'(a)}\right)^2.$$

The values of a obtained by this process were 13°55′·7 and 33°39′·4 for the two geodesics which close themselves after two and three turns respectively round the ellipsoid.

4. A last application is made to the problem of the Poinsot motion of a rigid body.

It is known that Euler's equations of motion for a rigid body moving about a fixed point under no external forces

$$A\dot{p} = (B-C)qr, \qquad B\dot{q} = (C-A)rp, \qquad C\dot{r} = (A-B)pq,$$

can be solved by writing

$$p = P \operatorname{cn} nt, \quad q = -Q \operatorname{sn} nt, \quad r = R \operatorname{dn} nt,$$

or by

$$p = P' \operatorname{dn} n't, \quad q = -Q' \operatorname{sn} n't, \quad r = R' \operatorname{cn} n't\,;$$

where $P, Q, R, P', Q', R', n, n'$ are certain functions of A, B, C, D; A, B, C being the principal moments of inertia of the body at the point of support, and $D = \Gamma^2/T$, where

Γ = the resultant moment of momentum,

T = twice the kinetic energy;

and we suppose $A > B > C$, and write

$$\Gamma^2 = A^2p^2 + B^2q^2 + C^2r^2 = D^2\mu^2,$$
$$T = Ap^2 + Bq^2 + Cr^2 \quad = D\mu^2,$$

μ having the dimensions of an angular velocity, and D those of a moment of inertia.

In the first case, r never vanishes, and this solution is adapted to the family of polhodes which surround the axis of C; in the second, p never vanishes, and this solution applies to the family of polhodes which surround the axis of A.

The motion being represented in Poinsot's way, by allowing the momental ellipsoid of the body to roll on a plane, let I be the point of contact of ellipsoid and plane, and let OJ be the perpendicular from the centre (O) of the ellipsoid to the plane, so that $OJ = 1/\sqrt{D}$, if the mass of the body be taken as unity.

Let (ρ, ψ) be polar coordinates of a point on the locus of I on the plane (the herpolhode), referred to J as origin. The following summary of results to be used in what follows is made for the case in which $A > B > D > C$, and the polhode cone surrounds the axis of C,

$$\left.\begin{aligned} \frac{p}{\mu} &= \sqrt{\left(\frac{D(D-C)}{A(A-C)}\right)} \operatorname{cn} u, \quad \frac{q}{\mu} = -\sqrt{\left(\frac{D(D-C)}{B(B-C)}\right)} \operatorname{sn} u, \\ \frac{r}{\mu} &= \sqrt{\left(\frac{D(A-D)}{C(A-C)}\right)} \operatorname{dn} u, \\ \text{where } u = nt, \quad \frac{n^2}{\mu^2} &= \frac{(B-C)(A-D)D}{ABC}, \\ k^2 = \frac{(D-C)(A-B)}{(B-C)(A-D)}&, \quad k'^2 = \frac{(A-C)(B-D)}{(A-D)(B-C)}; \end{aligned}\right\} \quad (1)$$

and
$$\frac{d\psi}{du} = \frac{\mu}{n}\,\frac{D}{B} - i\,\frac{d}{du}\,\Pi(u,\ a;\ k), \tag{2}$$

where
$$\mathrm{sn}^2 a = \frac{D}{B}\,\frac{B-C}{D-C} \quad (>1),$$

$$\mathrm{cn}\, a = -i\sqrt{\left(\frac{C(B-D)}{B(D-C)}\right)},$$

$$\mathrm{dn}^2 a = \frac{A}{B}\,\frac{B-D}{A-D} \quad (<1),$$

so that a is of the form $K+ia'$, where a' is real.*

Consider the problem of finding the variation in the apsidal angle of the herpolhode due to increments in A, B, C, D (or, equally well, in the lengths of the axes of the rolling ellipsoid and in OJ). As before, the mass of the body is taken as unity, so that the next paragraph has a purely geometrical meaning, and concerns a rolling ellipsoid, of semi-axes $1/\sqrt{A}$, $1/\sqrt{B}$, $1/\sqrt{C}$, whose centre is fixed at a distance $1/\sqrt{D}$ from the plane on which it rolls.

From (2) the apsidal angle of the herpolhode is defined by

$$\Psi = \int_0^K \frac{d\psi}{du}\,du = \frac{\mu}{n}\,\frac{D}{B}\,K - i\Pi(K,\ a;\ k)$$
$$= \Lambda K - i\Pi(K,\ a;\ k),$$

where
$$\Lambda = \frac{\mu}{n}\,\frac{D}{B} = \sqrt{\left(\frac{ACD}{B(B-C)(A-D)}\right)}.$$

Using the results collected in (1) we notice that

when	A alone varies,		$\mathrm{sn}\, a$	remains constant,
„	B	„	$\mathrm{sn}(a+K)$	„
„	C	„	$\mathrm{dn}\, a$	„
„	D	„	$\mathrm{dn}(a+K)$	„

Consequently, when A, B, C, D receive increments δA, δB, δC, δD, the

* See Greenhill, *Elliptic Functions*, pp. 29 and 109.

corresponding increment in Ψ is

$$\delta\Psi = K\delta\Lambda + \Lambda \frac{dK}{dk}\,\delta k$$

$$-i\left[\left(\frac{d\Pi}{dk}\right)_1 \frac{\partial k}{\partial A}\,\delta A + \left(\frac{d\Pi}{dk}\right)_2 \frac{\partial k}{\partial B}\,\delta B + \left(\frac{d\Pi}{dk}\right)_3 \frac{\partial k}{\partial C}\,\delta C + \left(\frac{d\Pi}{dk}\right)_4 \frac{\partial k}{\partial D}\,\delta D\right].$$

When the right-hand member of this equation is evaluated, we have

$$\frac{2n}{\mu D}\,\delta\Psi = -\frac{1}{A(A-B)}\left(K - \frac{B-C}{A-C}\,E\right)\delta A - \frac{1}{B(A-B)}\left(-K + \frac{A-D}{B-D}E\right)\delta B$$

$$+\frac{1}{C(D-C)}\left(K - \frac{A-D}{A-C}\,E\right)\delta C + \frac{1}{D(D-C)}\left(-K + \frac{B-C}{B-D}\,E\right)\delta D. \tag{3}$$

We observe that, since $K > E$, and $A > B > D > C$, the coefficient of δA is negative, and that of δC is positive. Also in the coefficient of δB,

$$-\left(-K + \frac{A-D}{B-D}E\right) = -\frac{A-D}{B-D}\left(E - \frac{B-D}{A-D}\,K\right)$$

$$= -\frac{A-D}{B-D}\left(kk'^2\,\frac{dK}{dk} + \frac{B-D}{A-D}\,\frac{A-B}{B-C}\,K\right).$$

Both the terms in the bracket are positive, and so the coefficient of δB is negative. Similarly, that of δD is positive. We may write

$$\frac{2n}{\mu D}\,\Psi = -X\delta A - Y\delta B + Z\,\delta C + U\,\delta D, \tag{4}$$

where X, Y, Z, U are all positive and have the values defined by (3). It may at once be verified that

$$\Sigma A\,\frac{\partial\Psi}{\partial A} = 0,$$

as we should expect, since this condition merely corresponds to an alteration in the unit of length.

We pass on to a dynamical application of equation (3). Suppose that the point of support of the moving body is its centre of gravity G, and let this point be suddenly released and the neighbouring point $G'(\xi, \eta, \zeta)$ fixed. A new Poinsot motion will begin about the new point of support, and this motion will depend on values of A, B, C, D, differing slightly from their old values. It is proposed to find the change in the apsidal

angle of the herpolhode. Let the release and fixture be made at any moment when the component angular velocities are p, q, r. The principal axes at G' are known to be the normals to the confocals to the ellipsoid of gyration at G which pass through G'. The mass of the body being unity, the equation of this ellipsoid is

$$\frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C}=1.$$

We find for the new values of the principal moments of inertia

$$A' = A+\eta^2+\zeta^2, \quad B' = B+\zeta^2+\xi^2, \quad C' = C+\xi^2+\eta^2;$$

and for the new values of the components of moment of momentum

$$L' = Ap-Bq\,\frac{\xi\eta}{A-B}+Cr\,\frac{\zeta\xi}{C-A},$$

$$M' = Ap\,\frac{\xi\eta}{A-B}+Bq-Cr\,\frac{\eta\zeta}{B-C},$$

$$N' = -Ap\,\frac{\zeta\xi}{C-A}+Bq\,\frac{\eta\zeta}{B-C}+Cr;$$

and since
$$D = \frac{\Gamma^2}{T},$$

the value of D for the new motion will be

$$D' = D-\frac{\Gamma^2}{T^2}\,\delta T = D-\frac{\delta T}{\mu^2},$$

for the resultant moment of momentum remains unaltered when the point of support is moved from the centre of gravity to any other point.

Also
$$\delta T = \delta\left(\frac{L^2}{A}+\frac{M^2}{B}+\frac{N^2}{C}\right)$$
$$= -\Sigma(q\zeta-r\eta)^2 = -\omega^2 l^2,$$

where l is the distance of (ξ, η, ζ) from the instantaneous axis, and

$$\omega^2 = p^2+q^2+r^2.$$

The small changes in A, B, C, D are therefore given by

$$\delta A = \eta^2+\zeta^2, \quad \delta B = \zeta^2+\xi^2, \quad \delta C = \xi^2+\eta^2,$$

$$\delta D = \frac{\Sigma(q\zeta-r\eta)^2}{\mu^2}.$$

Substituting in (4), we have

$$\frac{2n}{\mu D}\delta\Psi = -X(\eta^2+\zeta^2)-Y(\zeta^2+\xi^2)+Z(\xi^2+\eta^2)+U\frac{\Sigma(q\zeta-r\eta)^2}{\mu^2}.$$

If the expression on the right-hand side vanish, we shall have $\delta\Psi = 0$. Hence, if the point of support be moved from G to a neighbouring point on a certain cone of the second degree, whose apex is at G, the apsidal angle of the herpolhode will remain unchanged. Regarding these terms as the first approximation to the equation of a certain surface, we may surmise that, for given values of p, q, r, the locus of the point P, which is such that, when the centre of gravity is released and the point P fixed, the apsidal angle of the herpolhode is unchanged, is a surface which has a conical point of the second degree at G.

When we substitute for p/μ, q/μ, r/μ, X, Y, Z, U, we obtain the value of $\delta\Psi$ in terms of the time and the dynamical constants of the motion. The resulting expression is rather long, and we confine ourselves to a particular case, in which the moving body is a lamina ($A = B+C$, and $B > C$), and the release and fixture are made at the instant $nt \equiv 0$ (that is to say, when the instantaneous axis is in the plane of A and C).

The equation of the cone then becomes

$$KA\xi^2+E\frac{A-D}{B-D}(B\eta^2-C\zeta^2)+\left(K-\frac{B-C}{B-D}E\right)\sqrt{\left(AC\frac{A-D}{D-C}\right)}\,\zeta\xi = 0.$$

When the same work is carried out for the case in which the polhode surrounds the axis of A the corresponding equation is found to be

$$KA\xi^2+\frac{B}{B-C}\left(K-\frac{C}{B}\,\frac{D-C}{D-B}E\right)(B\eta^2-C\zeta^2)$$
$$-\left(K-\frac{C}{D-B}E\right)\sqrt{\left(AC\frac{D-C}{A-D}\right)}\,\zeta\xi = 0.$$

The equation of the lines in which these cones cut the plane of the lamina ($\xi = 0$) is

$$B\eta^2-C\zeta^2 = 0,$$

this is to say, they coincide with the equi-conjugate diameters of the momental *ellipse* of the lamina at G, no matter what polhode, defined by D, is being described. Hence we may state the particular theorem:—

Let the principal moments of inertia of a lamina about axes in its own plane through its centre of gravity G be B and C ($B > C$), and let

A ($= B+C$) be its moment of inertia about a perpendicular axis through the same point. Let the lamina be moving about G as a point of support under no external forces. At a moment when the instantaneous axis lies in the plane of A and C, let the point of support be displaced from G to a neighbouring point. A new Poinsot motion will begin, and the apsidal angle of the herpolhode in the new motion will be the same as that in the old, to the first order of small quantities, provided that the displacement of the point of support be made in the direction of any generator of a certain quadric cone, one of whose principal axes is the axis of B, and which cuts the plane of the lamina in two lines which coincide with the equi-conjugate diameters of the momental ellipse of the lamina at G.

ON THE MAXIMUM ERRORS OF CERTAIN INTEGRALS AND SUMS INVOLVING FUNCTIONS WHOSE VALUES ARE NOT PRECISELY DETERMINED

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Experimental Setting of the Problem.

1. A type of problem occurring not infrequently in practice may be illustrated by the following example. It is required to determine the path of a sound ray in a moving medium such as the atmosphere. It can be shown that the effect of the motion of the medium (*i.e.* the wind) on the ray largely depends on the *mean value* of the component of wind in a certain horizontal direction, taken with respect to height between the extreme points of the ray. This mean value is thus the principal physical constant to be determined in the course of a practical solution. But the mean value of the wind depends on the values of the wind at intermediate points. These values may not be exactly determined; they may be subject to errors of observation and instrumental errors, or they may be subject (for example) to day-to-day variations. The mean value derived from them will also be subject to errors or variations. It is with this mean value and the errors and the problems arising out of them that this paper is concerned.

Theoretical and Practical Determination of Mean Values.

2. Let A and B be respectively the initial and final points of the ray, and AB the straight line (not the ray) joining them. Take any point P on AB and let v be the velocity of the medium at P. If $AP/AB = x$, then the theoretical value of the mean required is

$$\mu = \int_0^1 v\,dx. \tag{1}$$

Now in practice v is not known at every point of the range (0, 1); in

the majority of cases it can only be determined at a limited number of points of this range, say

$$x_0 = 0,\ x_1,\ x_2,\ \ldots,\ x_s,\ \ldots,\ x_n = 1.$$

To obtain the mean value we make use of some approximate formula giving the value of the integral (1) in terms of the values $v_0, v_1, \ldots, v_n$ of v at the points taken.

A rough and ready approximation is got by treating v as linear between the points x_s. This gives

$$\mu = \tfrac{1}{2} \sum_{s=1}^{n} (v_s + v_{s-1})(x_s - x_{s-1})$$

$$= \tfrac{1}{2} \left[v_0 \delta_1 + v_1 (\delta_1 + \delta_2) + v_2 (\delta_2 + \delta_3) + \ldots + v_n \delta_n \right], \tag{2}$$

where $\delta_1 = x_1 - x_0,\ \delta_2 = x_2 - x_1,\ \ldots,\ \delta_n = x_n - x_{n-1}$.

Better approximations can be obtained by using one or other of the various rules for approximating to the integral (1). Simpson's rule, for instance, gives

$$\mu = \frac{1}{3n} \left[v_0 + v_n + 2(v_2 + v_4 + \ldots + v_{n-2}) + 4(v_1 + v_3 + \ldots + v_{n-1}) \right], \tag{3}$$

it being assumed that n is even and the x's at equal distances apart.

Both (2) and (3), and indeed all the values of μ obtainable by the ordinary rules for approximating to an integral, are of the form

$$\mu = v_0 c_0 + v_1 c_1 + \ldots + v_n c_n, \tag{4}$$

where the c's are constants, depending on the choice of x's and the mode of approximation, and such that

$$c_0 + c_1 + \ldots + c_n = 1. \tag{5}$$

We have now obtained two forms for μ: one, a theoretical form, giving μ as an integral; and the other, a practical form, giving μ as a finite sum.

Error of the Mean Value due to Inaccuracy of Measurement.

3. Take first the mean value as given by (4). It is subject to two kinds of error. There are the errors due to the deviation of the functional form of v from that required for the particular rule taken in making the approximation to (1)—these are errors about which we can say very little;

and there are the errors due to inaccuracies in the values of $v_0, v_1, \ldots, v_n$—it is these with which we are immediately concerned.

To investigate them let

$$a_0, \; a_1, \; \ldots, \; a_n$$

be the deviations of the measured values of $v_0, v_1, \ldots, v_n$ from their true values. The error in μ due to inaccuracy of measurement is evidently given by

$$\Delta\mu = c_0 a_0 + c_1 a_1 + \ldots + c_n a_n. \tag{6}$$

This gives us the definite mathematical expression with which we shall presently be concerned.

Preliminary Assumptions as to the Nature of the Inaccuracies in the Values of v.

4. The deviations of v from its true value will not usually be completely arbitrary. If the errors (instrumental or observational) are of the usual type occurring *in practice*, then we say (a) that they will all be within a certain standard, *i.e.* there is some positive number ρ which the absolute value of a never exceeds;* and (b) that they are just as likely to occur one way as another, and so if our mean is based on a good many determinations (*i.e.* we take a fairly large value of n), then their total will be nil, *i.e.*

$$a_0 + a_1 + \ldots + a_n = 0.$$

* It may be pointed out here that in this paper we are not concerned with mean Gaussian errors: we are attempting to find out the worst that is likely to happen in practice. There is no question of random distributions of error, and so on. We assume that what we find happening in practice is so, namely, that we hardly ever make very big errors. And what we want to know is the worst that may happen if these small errors occur in bad places. We allow ourselves certain limits of individual error and then determine the worst that may happen, *subject to these limits.* In the Gaussian theory it is assumed that big errors are not so likely to happen as the smaller ones. We assume that they do not happen at all. Of course exceptional cases will occur when our maximum values are exceeded. But for a good many practical purposes we may allow ourselves exceptions. If our gun does shoot badly now and again it does not matter. What we really want is to know that apart from these exceptions it is quite certain to shoot up to a certain standard; and it is precisely this kind of information that work based on our lines gives. See also, N. R. Campbell, *Physics, The Elements*, p. 487.

Maximum Error of the Sum.

5. We have to find the maximum value of (6) subject to the conditions

$$\text{(i)} \quad |a_s| \leqslant \rho,$$

$$\text{(ii)} \quad \Sigma a_s = 0,$$

where s takes all integral values from 0 to n inclusive.

To do so we observe that in virtue of (ii) the value of (6) is unaltered by an alteration in the c's, provided each c is altered by the same amount.

Choose this amount so that half the resultant c's are greater than or equal to 0, half are less than or equal to 0 and (when n is odd) the one left over is equal to 0. The maximum value of (6) is now evidently obtained by taking $a = \rho$ for the c's of the first set, $a = -\rho$ for the c's of the second, and $a = 0$ for the odd one over if it exists.

Let us consider one or two particular cases.

I. *The c's are all equal.*

The sum (6) reduces to $$c\Sigma a_s,$$

which is zero in virtue of condition (ii). This shows that, with the instruments working as assumed above, the error to be expected due to inaccuracies of observation is zero, when the values of x for which the observations are made are so chosen that the factors c are the same throughout—a fact of no small practical importance.

II. *The c's are obtained by making use of the approximation* (2) *with the points x_0, x_1, ..., x_n equidistant.*

We have $$c_0 = \tfrac{1}{2}\delta,$$

$$c_1 = c_2 = \ldots = c_{n-1} = \delta,$$

$$c_n = \tfrac{1}{2}\delta.$$

The maximum is best obtained as follows.

By (ii), $$\Sigma \delta a_s = 0.$$

Therefore $$c_0 a_0 + c_1 a_1 + \ldots + c_n a_n = -\tfrac{1}{2}\delta(a_0 + a_n),$$

the maximum value of which is

$$-\tfrac{1}{2}\delta(-\rho-\rho) = \rho\delta = \frac{\rho}{n},$$

showing that for this case the maximum error is directly proportional to the distance between the points $x_0, x_1, \ldots, x_n$.

III. *The c's are obtained by making use of Simpson's rule.*

x is even in this case, $= 2m$, say. The c's are given by

$$c_0 = c_{2m} = \frac{1}{6m},$$

$$c_2 = c_4 = \ldots = c_{2m-2} = \frac{1}{3m},$$

$$c_1 = c_3 = \ldots = c_{2m-1} = \frac{2}{3m}.$$

To obtain the maximum by the standard procedure, decrease each c by $1/3m$. We get m new c's greater than or equal to 0, namely,

$$c_1' = c_3' = \ldots = c_{2m-1}' = \frac{1}{3m},$$

and m less than or equal to 0, namely,

$$c_2' = c_4' = \ldots = c_{2m-4}' = 0, \quad c_0' = c_{2m}' = -\frac{1}{6m},$$

and an odd c over, which is zero, namely c_{2m-2}.

The maximum is given by

$$\rho\,(c_1'+c_3'+\ldots+c_{2m-1}')-\rho\,(c_2'+c_4'+\ldots+c_{2m-4}'+c_0'+c_{2m}') = \frac{\rho}{3}+\frac{\rho}{3m}.$$

In this case there is in the maximum error a constant term of value $\rho/3$, together with a term varying inversely as the number of intervals between the points of observation. The term $\rho/3$ shows that Simpson's rule, though excellent from the point of view of avoiding errors due to the deviation of the form of v from the standard, is not at all good for avoiding errors due to inaccuracies of observation.

The approximation (2), although inferior in the usual way to that obtained by Simpson's rule, is here very much better: the maximum value error of (2) is ρ/n; that of (3), obtained by Simpson's rule, is $\rho/3+2\rho/3n$, which is considerably greater.

The Analogous Problem for Integrals.

6. There is no question of obtaining the errors of (1) because the analogue of (ii) gives

$$\int_0^1 a\,dx = 0,$$

and this is equivalent to saying that the error of (1) is zero.

(1) is, however, only the *physical* analogue of (4). If we omit altogether the way in which (4) was derived and consider it merely as a certain kind of sum, we find that its analogue (which we may term the *mathematical* analogue) is

$$\mu = \int_0^1 v(x)\,\gamma(x)\,dx, \tag{7}$$

where $v(x)$ is a function subject to variations, and $\gamma(x)$ depends only on x.

If $a(x)$ is the variation in $v(x)$ at any point x, then the variation in μ is given by

$$\Delta\mu = \int_0^1 a(x)\,\gamma(x)\,dx.^*$$

This is the quantity whose maximum we wish to find.

The appropriate restrictions analogous to those of § 5 are, of course,

$$\text{(i)}' \quad |\,a(x)\,| \leqslant \rho,$$

$$\text{(ii)}' \quad \int_0^1 a(x)\,dx = 0.$$

To find the maximum value of $\Delta\mu$ we increase or decrease $\gamma(x)$ by a constant amount adjusted so that the function $\gamma_1(x)$ obtained is such that the interval (0, 1) can be divided into two sets E_1 and E_2 of equal measure in the one of which $\gamma_1(x) \geqslant 0$ and in the other $\gamma_1(x) \leqslant 0$. The maximum

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* It may be remarked that this expression is not altogether devoid of practical significance. Suppose we are concerned, not with the mean value of $v(x)$ but with the mean value of some expression based upon $v(x)$, *i.e.* with that of some function $F[v(x)]$, which is

$$\int_0^1 F[v(x)]\,dx.$$

The variation in this due to a variation $a(x)$ in $v(x)$ is approximately

$$\int_0^1 a(x)\,F'[v(x)]\,dx,$$

which is of the form given.

value of $\Delta\mu$ is then given by

$$\rho \int_{E_1} \gamma_1(x)\,dx - \rho \int_{E_2} \gamma_1(x)\,dx.$$

It is not, however, obvious that E_1 and E_2 can be obtained as stated. To establish their existence let $E(t)$ be the set of points in (0, 1) for which

$$\gamma(x) + t \geqslant 0.$$

$\gamma(x)$ being supposed a measurable function,* $E(t)$ is a measurable set and its measure gives us a function $m(t)$ of t.

$m(t)$ evidently increases with t. Thus there is a unique value τ of t such that

$$m(t) < \tfrac{1}{2} \quad \text{for} \quad t < \tau,$$

$$m(t) \geqslant \tfrac{1}{2} \quad \text{for} \quad t > \tau.$$

Let us now make use of the idea of the limit of a variable set. Just as numbers depending on a variable parameter may tend to a limiting number as the parameter tends to some particular value, so sets of points depending on a variable parameter may tend to a limiting set as the parameter tends to the particular value. As in the case of numbers, when the set either increases steadily or decreases steadily as the parameter increases, the limiting set always exists.†

In the case in point

$$\lim_{t \to \tau + 0} E(t), \quad i.e.\ E(\tau + 0),$$

exists.

Since $$mE(t) = m(t) > \tfrac{1}{2} \quad \text{for} \quad t > \tau,$$

$$mE(\tau + 0) \geqslant \tfrac{1}{2}.$$

Also, since $E(t)$ includes $E(\tau)$ for $t > \tau$,

$E(\tau + 0)$ „ $E(\tau)$.

Now let ξ be any point such that

$$\gamma(\xi) + \tau < 0.$$

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* In all modern work dealing with integrals it is almost invariably assumed that the functions concerned are *measurable* functions (in Lebesgue's sense).

† De la Vallée Poussin, *Intégrales de Lebesgue* (*Borel Tracts*), p. 9; Carathéodory, *Vorlesungen über Reelle Funktionen*, pp. 113–119.

Then there is a value t_1 of t greater than τ for which

$$\gamma(\xi)+t_1 < 0.$$

Hence ξ does not belong to $E(t_1)$, and therefore does not belong to $E(\tau+0)$.

Thus in $E(\tau+0)$, $\qquad \gamma(x)+\tau \geqslant 0.$

But every point for which $\qquad \gamma(x)+\tau > 0$

belongs to $E(\tau)$. Thus in the difference $E(\tau+0)-E(\tau)$,

$$\gamma(x)+\tau = 0.$$

If we take E_1 to be $E(\tau)$ together with sufficient points of

$$E(\tau+0)-E(\tau)$$

to bring its measure up to $\frac{1}{2}$ it is easily seen that E_1 has the required property. E_2 is then the remainder left after taking E_1 from the interval $(0, 1)$.*

Example on the above.

7. As a simple case take that in which $\gamma(x)$ steadily decreases.† E_1 is evidently the interval $(0, \frac{1}{2})$, and E_2 the interval $(\frac{1}{2}, 1)$, $\frac{1}{2}$ being included in either E_1 or E_2, but not in both. $\gamma_1(x)$ is $\gamma(x)-\gamma(\frac{1}{2})$. The maximum value of $\Delta\mu$ is

$$\rho\int_0^{\frac{1}{2}}\{\gamma(x)-\gamma(\tfrac{1}{2})\}\,dx-\rho\int_{\frac{1}{2}}^1\{\gamma(x)-\gamma(\tfrac{1}{2})\}\,dx=\rho\int_0^{\frac{1}{2}}\gamma(x)\,dx-\rho\int_{\frac{1}{2}}^1\gamma(x)\,dx,$$

the terms in $\gamma(\frac{1}{2})$ cutting each other out.

Imposition of an Additional Restriction.

8. In the above nothing has been said about the total numerical error which may be committed, *i.e.* $\Sigma\,|\alpha|$. In extreme cases, when the maximum error ρ is made at every observation, this will be either $n\rho$ or

* The above result being of a certain amount of interest we formulate it as a general proposition. It runs as follows:—*If $f(x)$ is measurable in any measurable set G, then a constant k can be found and G divided into two parts of equal measure in such a way that $f(x) \geqslant k$ in the first and $f(x) \leqslant k$ in the second.*

† For instance, when $\gamma'(x)$ is negative.

$(n-1)\rho$, when n is the number of observations; $n\rho$ being taken when n is even, and $(n-1)\rho$ when n is odd. But in a good many cases it will be nothing like as much. So it is not without interest to consider what happens when the additional restriction is made that the total numerical error is not to exceed a fixed amount σ.*

The a's are now subject to (i), (ii), and

$$\text{(iii)} \quad \Sigma\,|\,a_s\,| \leqslant \sigma.$$

To find the maximum error we may, in the first place, proceed exactly as in § 5, obtaining new c's which divide into two groups, the first containing no negative members and the second no positive. Only we do not take $a = \rho$ for the first group and $a = -\rho$ for the second. This is prohibited by the new restriction. Instead we take $a = \rho$ for the greatest positive c, then $a = -\rho$ for the greatest negative c; then $a = \rho$ for the next greatest positive c, $a = -\rho$ for the next greatest negative c; and so on—proceeding in this way until another step would make the total numerical error exceed σ.

We are now left with the problem of distributing the remainder R of the permissible total numerical error (of an amount less than 2ρ, as otherwise we could take another step) among the remaining c's. To see how it must be distributed, observe that from (i) it follows that half of it must go to the positive group and half of it to the negative group. This being so, the maximum is evidently obtained by taking $a = R/2$ with the greatest positive c's yet unused and $= -R/2$ with the greatest† negative.

Since the c's used above may be altered back into the original c's without affecting the value of μ, we can state a practical rule as follows:—

Arrange the c's in descending order of magnitude. Take $a = \rho$ with the first and $= -\rho$ with the last, then $a = \rho$ with tho second and $= -\rho$ with the last but one, and so on, until another step would make the total numerical error exceed σ. Let q be the number of steps taken. Take $a = \sigma/2 - q\rho$ (*i.e.* $R/2$) with the $(q+1)$-th and $= -(\sigma/2 - q\rho)$ with the last but q. The maximum value of μ is obtained.

The actual value of the maximum, it should be noted, is

$$\rho\left(\sum_{s=0}^{q-1} c_s - \sum_{s=n-q+1}^{n} c_s\right) + f\rho(c_q - c_{n-q}),$$

where q is the greatest integer such that $2q\rho \leqslant \sigma$, *i.e.* the greatest integer in $\sigma/2\rho$, and $f = \sigma/2\rho - q$, a positive proper fraction.

* σ is evidently less than $(n+1)\rho$.

† "Greatest" is used here as above, as is evident from the context, in the sense of greatest numerically.

Maximum Error of the Integral subject to the New Restriction.

9. The analogue of (iii) for integrals is

$$\text{(iii)}' \quad \int_0^1 |a(x)|\,dx \leqslant \sigma,$$

where σ is some positive number less than ρ.

To find the maximum value of $\Delta\mu$ we first of all obtain the sets E_1, E_2 as before, in the first of which $\gamma_1(x)$ is never negative, and in the second of which it is never positive. If, now, we can establish the existence of sub-sets e_1 of E_1 and e_2 of E_2 such that

$$\rho m e_1 = \rho m e_2 = \tfrac{1}{2}\sigma,$$

and no value of $\gamma_1(x)$ in $E_1 - e_1$ exceeds any value in e_1, nor is any value in $E_2 - e_2$ less than any value in e_2, then the maximum of $\Delta\mu$ subject to (i)′, (ii)′, and (iii)′ is given by taking $a(x) = \rho$ in e_1, $= -\rho$ in e_2, and $= 0$ elsewhere. Its value is

$$\rho \int_{e_1} \gamma_1(x)\,dx - \rho \int_{e_2} \gamma_1(x)\,dx,$$

i.e. since $\gamma_1(x) - \gamma(x)$ is constant,

$$\rho \int_{e_1} \gamma(x)\,dx - \rho \int_{e_2} \gamma(x)\,dx.$$

The existence of the sub-sets e_1 and e_2 is established as follows:—

Take any positive number ϵ and form the infinite scale

$$0,\ \epsilon,\ 2\epsilon,\ \ldots,\ (\nu-1)\epsilon,\ \nu\epsilon,\ \ldots.$$

Let S_ν be the sub-set of E_1 for which

$$(\nu-1)\,\epsilon \leqslant \gamma_1(x) < \nu\epsilon.$$

The sets $S_1, S_2, \ldots$ so obtained do not overlap and together make up the set E_1. Since, as we have already supposed, $\gamma(x)$ is a measurable function, so is $\gamma_1(x)$, and these sets S are measurable. This gives, from the above, by means of a well known theorem in the theory of measure,

$$mS_1 + mS_2 + \ldots + mS_\nu + \ldots = mE_1.$$

Every term of the series on the left is positive, and so there is a first number N such that

$$\rho\,\{mS_{N+1} + mS_{N+2} + \ldots\} \leqslant \tfrac{1}{2}\sigma.$$

Write now
$$S_{N+1}+S_{N+2}+\ldots = J,$$
$$S_1+S_2+\ldots+S_N = K,$$
$$S_N = \omega.$$

Then
$$\rho m J \leqslant \tfrac{1}{2}\sigma, \tag{8}$$
$$\rho\,(mJ+m\omega) > \tfrac{1}{2}\sigma, \tag{9}$$
and no value of $\gamma_1(x)$ in K exceeds any value in J.

Now let ϵ assume in turn each of the values
$$1/2,\ 1/2^2,\ \ldots,\ 1/2^r,\ \ldots,$$
and let J_r, K_r, ω_r be the determinations of J, K, ω, where $\epsilon = 1/2^r$.

From the inequalities
$$(\nu-1)\frac{1}{2^{r-1}} = 2(\nu-1)\frac{1}{2^r} < (2\nu-1)\frac{1}{2^r} < 2\nu\frac{1}{2^r} = \nu\frac{1}{2^{r-1}}$$
it follows that the sets S for $\epsilon = 1/2^r$ are all sub-sets of the sets S for $\epsilon = 1/2^{r-1}$. No one of the former overlaps two of the latter. And it is quite easy to show that
$$J_r \supset J_{r-1}, \quad K_r \subset K_{r-1}, \quad \omega_r \subset \omega_{r-1}.^*$$

Thus the sequence of sets
$$J_1,\ J_2,\ \ldots,\ J_r,\ \ldots$$
is an increasing sequence, and those of sets
$$K_1,\ K_2,\ \ldots,\ K_r,\ \ldots,$$
$$\omega_1,\ \omega_2,\ \ldots,\ \omega_r,\ \ldots,$$
are decreasing. All three, therefore, by the theorem quoted in § 6, tend to limits, which we will denote by
$$J',\ K',\ \omega',$$
respectively.

From (8) and (9) it follows that
$$\rho m J' \leqslant \tfrac{1}{2}\sigma, \tag{10}$$
$$\rho\,(mJ'+m\omega') \geqslant \tfrac{1}{2}\sigma. \tag{11}$$

* $\supset$ denotes "contains" and $\subset$ "is contained in."

Also $\gamma_1(x)$ is constant in ω'. For its oscillation in ω_r does not exceed $1/2^r$. Since ω_r decreases, ω' is contained in ω_r, and therefore the oscillation in ω' does not exceed $1/2^r$, *i.e.* it must be zero.

Let us now show that no value of $\gamma_1(x)$ in K' exceeds any value in either J' or ω'.

The first part is evident, for no value in K_r exceeds any value in J_r. Therefore no value in K' exceeds any value in J_r, for $K' \subset K_r$. But every value in J' is a value in at least one J_r. Therefore no value in K' exceeds any value in J'.

For the second, no value in K_r exceeds any value in ω_r, by the definition of K_r and ω_r. But $\omega' \subset \omega_r$. Therefore no value in K_r exceeds any value (which is really *the* value) in ω'. As before, every value in K' is a value in some K_r. Thus no value in K' can exceed any value in ω'.

It follows from the above that if we take e_1 to be J' together with any part of ω', then no value in $E_1 - e_1$ (which is contained in K') can exceed any value in e_1. It is now only a matter of choosing the portion of ω' so that $\rho m e_1 = \frac{1}{2}\sigma$ to obtain e_1 as required. And this can be done in virtue of (10) and (11). e_1 is thus obtainable.

In exactly the same way so is e_2.

Example.

10. Suppose $\gamma(x)$ is decreasing. Then e_1, e_2 are evidently given by

$$e_1 = \left(0, \frac{\sigma}{2\rho}\right), \quad e_2 = \left(1 - \frac{\sigma}{2\rho}, 1\right),$$

and the maximum value of $\Delta\mu$ is given by

$$\rho \int_0^{\sigma/2\rho} \gamma(x)\,dx - \rho \int_{1-\sigma/2\rho}^{1} \gamma(x)\,dx. \tag{12}$$

Alternative Method for Sums, based on Abel's Transformation.

11. The maximum value of the sum (6) can also be obtained in an entirely different way, as follows.

Write $$A_s = a_0 + a_1 + \ldots + a_s.$$

Then, by Abel's transformation,

$$\begin{aligned} \Delta\mu &= c_0 a_0 + c_1 a_1 + \ldots + c_n a_n \\ &= A_0(c_0 - c_1) + A_1(c_1 - c_2) + \ldots + A_{n-1}(c_{n-1} - c_n) + A_n c_n. \end{aligned}$$

As in § 8 let q be the greatest integer such that

$$q\rho \leqslant \sigma/2,$$

and
$$f = \sigma/2\rho - q.$$

We have, by (i),

$$|A_0| \leqslant \rho, \quad |A_1| \leqslant 2\rho, \quad \ldots, \quad |A_{q-1}| \leqslant q\rho,$$

$$|A_{n-1}| \leqslant \rho, \quad |A_{n-2}| \leqslant 2\rho, \quad \ldots, \quad |A_{n-q}| \leqslant q\rho,$$

the latter since $\quad |A_{n-r}| = |A_n - A_{r-1}| \leqslant |A_{r-1}|.$

Also, in all cases,
$$\begin{aligned} 2|A_s| &= |A_s - (A_n - A_s)| \\ &\leqslant |A_s| + |A_n - A_s| \\ &\leqslant |a_0| + |a_1| + \ldots + |a_n| \\ &\leqslant \sigma. \end{aligned}$$

Now, supposing, as we may, that the c's are arranged in descending order of magnitude, so that the factors $c_0 - c_1$, $c_1 - c_2$, ..., $c_{n-1} - c_n$ affecting the A's are all positive or zero, we have

$$\begin{aligned} |\Delta\mu| &\leqslant \rho(c_0 - c_1) + 2\rho(c_1 - c_2) + \ldots + q\rho(c_{q-1} - c_q) + \tfrac{1}{2}\sigma(c_q - c_{q+1}) + \ldots \\ &\qquad + \tfrac{1}{2}\sigma(c_{n-q-1} - c_{n-q}) + q\rho(c_{n-q} - c_{n-q+1}) + \ldots + \rho(c_{n-1} - c_n) \\ &\leqslant \rho(c_0 + c_1 + \ldots + c_{q-1}) + (\sigma/2 - q\rho)\, c_q - (\sigma/2 - q\rho)\, c_{n-q} \\ &\qquad - \rho(c_{n-q+1} + c_{n-q+2} + \ldots + c_n) \\ &= \rho\left(\sum_{s=0}^{q-1} c_s - \sum_{s=n-q+1}^{n} c_s\right) + f\rho(c_q - c_{n-q}). \end{aligned}$$

Taking
$$\begin{aligned} a_0 = a_1 = \ldots = a_{q-1} &= \rho, \\ a_{n-q+1} = a_{n-q+2} = \ldots = a_n &= -\rho, \\ a_q &= f\rho, \\ a_{n-q} &= -f\rho, \end{aligned}$$

which values evidently satisfy (i), (ii), and (iii), we see that $\Delta\mu$ actually may be equal to the bound given above. This bound is therefore its maximum value. This is the result of § 8, and it has been obtained by an entirely different method.

Alternative Method for Integrals, based on Integration by Parts.

12. For the special case in which $\gamma(x)$ has a derivative of constant sign the result of § 9 can be obtained as follows.

Write
$$A(\xi) = \int_0^\xi a(x)\,dx.$$

As in § 11, we can show that
$$\begin{aligned} |A(\xi)| &\leqslant \rho\xi \quad \text{for} \quad 0 \leqslant \xi \leqslant \sigma/2\rho, \\ &\leqslant \rho(1-\xi) \quad \text{for} \quad 1-\sigma/2\rho \leqslant \xi \leqslant 1, \\ &\leqslant \tfrac{1}{2}\sigma \ \text{ in all cases.} \end{aligned}$$

Now
$$\begin{aligned} \int_0^1 a(x)\,\gamma(x)\,dx &= \Big[A(x)\,\gamma(x)\Big]_0^1 - \int_0^1 A(x)\,\gamma'(x)\,dx \\ &= -\int_0^1 A(x)\,\gamma'(x)\,dx, \end{aligned}$$

since
$$A(1) = A(0) = 0.$$

To fix the ideas, take the case in which $\gamma'(x)$ is constantly negative or zero. We get

$$\begin{aligned} |\Delta\mu| &= \left|\int_0^1 a(x)\,\gamma(x)\,dx\right| \\ &\leqslant \rho\left[-\int_0^{\sigma/2\rho} x\gamma'(x)\,dx - \int_{1-\sigma/2\rho}^1 (1-x)\,\gamma'(x)\,dx\right] - \tfrac{1}{2}\sigma\int_{\sigma/2\rho}^{1-\sigma/2\rho} \gamma'(x)\,dx. \end{aligned}$$

But
$$-\int_0^{\sigma/2\rho} x\gamma'(x)\,dx = -\Big[x\gamma(x)\Big]_0^{\sigma/2\rho} + \int_0^{\sigma/2\rho} \gamma(x)\,dx,$$

$$-\int_{1-\sigma/2\rho}^1 (1-x)\gamma'(x)\,dx = -\Big[(1-x)\,\gamma'(x)\Big]_{1-\sigma/2\rho}^1 - \int_{1-\sigma/\rho}^1 \gamma(x)\,dx,$$

and these give $\quad |\Delta\mu| \leqslant \rho\left[\int_0^{\sigma/2\rho} \gamma(x)\,dx - \int_{1-\sigma/2\rho}^1 \gamma(x)\,dx\right].$

As before, we show that $\Delta\mu$ may take its bound, which is therefore the required maximum. This is the result of § 10.

The Constants K_1 and K_2.

13. For the integral
$$\Delta\mu = \int_0^1 a(x)\,\gamma(x)\,dx,$$

the maximum value is, by § 9, given by an expression

$$\rho K_1 - \rho K_2,$$

where K_1 and K_2 are independent of $u(x)$, *i.e.* for any given function $\gamma(x)$ they are constants.

Now we have shown that these constants exist, but we have not exhibited in any very obvious way their relation to the function $\gamma(x)$. This relation we proceed to investigate.

THEOREM I.—*If e is any sub-set of* (0, 1) *of measure* $\sigma/2\rho$, *then*

$$K_1 \geqslant \int_e \gamma(x)\,dx \geqslant K_2.$$

Since $\gamma_1(x)$ differs from $\gamma(x)$ only by a constant, it is sufficient to show that

$$\int_{e_1} \gamma_1(x)\,dx \geqslant \int_e \gamma_1(x)\,dx \geqslant \int_{e_2} \gamma_1(x)\,dx.$$

For this gives
$$\int_{e_1} \gamma(x)\,dx \geqslant \int_e \gamma(x)\,dx \geqslant \int_{e_2} \gamma(x)\,dx,$$

since e_1, e_2, e are all of the same measure, and these inequalities are the inequalities required.

Let CE denote the complement of any set E with respect to the interval (0, 1), *i.e.* the points of (0, 1) which do not belong to E.

From the definition of e_1 it is evident that no value of $\gamma_1(x)$ in ce_1 exceeds any value in e_1. Thus, if ee_1 is the common part of e and e_1, no value of $\gamma_1(x)$ in $e-ee_1$, the remainder of e, exceeds any value in e_1-ee_1. But $e-ee_1$ and e_1-ee_1 are of the same measure. Hence

$$\int_{e_1-ee_1} \gamma_1(x)\,dx \geqslant \int_{e-ee_1} \gamma_1(x)\,dx,$$

i.e.
$$\int_{e_1} \gamma_1(x)\,dx \geqslant \int_e \gamma_1(x)\,dx.$$

In exactly the same way

$$\int_{e_2} \gamma_1(x)\,dx \leqslant \int_e \gamma_1(x)\,dx,$$

and the theorem is proved.

From this result it follows that K_1 and K_2 can be defined as follows:

DEFINITION.—*Let* Σ *be the aggregate of values of* $\int_e \gamma(x)\,dx$ *where* e *is any sub-set of* (0, 1) *of measure* $\sigma/2\rho$. *Then* K_1 *is the upper and* K_2 *the lower bound of* Σ.

Elementary Determination of K_1 *and* K_2.

14. THEOREM II.—*In forming the aggregate* Σ *we need only consider such sets e as consist of a finite number of non-overlapping intervals.*

For since e_1 is measurable there is a set $\mathcal{E}$ consisting of a finite number of non-overlapping intervals such that

$$e_1 = \mathcal{E} + e' - e'', \tag{13}$$

where e' and e'' are measurable sets of measure less than any positive number ϵ given in advance.* (13) gives

$$|m\mathcal{E} - me_1| \leqslant me' + me'' = 2\epsilon,$$

i.e.

$$|m\mathcal{E} - \sigma/2\rho| \leqslant 2\epsilon.$$

Thus we can make $\mathcal{E}$ of the right measure by the addition or subtraction of intervals of total length not exceeding 2ϵ. Let F be the set thus obtained. Then

$$m(e_1 \sim F) \leqslant 4\epsilon,\dagger$$

and therefore, by a well known theorem on summable functions,‡

$$\left| \int_{e_1} \gamma(x)\,dx - \int_F \gamma(x)\,dx \right|$$

is arbitrarily small with ϵ, *i.e.* $\int_F \gamma(x)\,dx$ can be made as near K_1 as we please.

It follows that K_1 can be obtained by considering only sets of the manner described. Similarly for K_2. The result is obtained.

* De la Vallée Poussin, *Cours d'Analyse Infinitésimale*, t. 1, 3rd ed., p. 63.

† $E \sim F$ is the *complete difference* between E and F, *i.e.* the points of E which do not belong to F together with those of F which do not belong to E.

‡ De la Vallée Poussin, *ibid.*, p. 260. Also, *Intégrales de Lebesgue*, p. 48.

Application of an Approximation Theorem.

15. The result of the preceding section may become a little plainer if we make use of the fact that we can approximate to summable functions in a certain way.

Call a function which is equal to a constant in some interval and zero elsewhere a function of zero type, and a function consisting of the sum of a finite number of functions of zero type a function of simple type. Then, if $f(x)$ is any function which is summable in a measurable set E, we can find a function of simple type $\psi(x)$ such that

$$\int_E |f(x)-\psi(x)|\,dx$$

is as small as we please.*

Take E to be (0, 1) and $f(x)$ our function $\gamma(x)$, so that

$$\int_0^1 |\gamma(x)-\psi(x)|\,dx < \epsilon.$$

Let K_1' and K_2' be the constants for $\psi(x)$ corresponding to K_1 and K_2 for $\gamma(x)$. Then K_1' differs from K_1 and K_2' from K_2 by less than ϵ.† But K_1' and K_2' are evidently of the form

$$\int_{e'} \psi(x)\,dx,$$

where e' consists of a finite number of intervals of total length $\sigma/2\rho$.

Now an integral of the form $\int_{e'} \psi(x)\,dx$ differs from an integral of the form $\int_{e'} \gamma(x)\,dx$ by less than ϵ. Thus K_1 and K_2 can be approached to by integrals of the form

$$\int_{e'} \gamma(x)\,dx,$$

within the standard 2ϵ. This is substantially the result required.

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* This result is substantially obtained by de la Vallée Poussin on p. 106 of the second volume of his *Cours d'Analyse*.

† To every member of the aggregate of which K_1 is the upper bound there corresponds a member of the aggregate of which K_1' is the upper bound differing from it by less than ϵ.

Converse of the Fundamental Theorem.

16. We shall show:

THEOREM III.—*If* K_1 *and* K_2 *are the upper and lower bounds of* $\int_e \gamma(x)\,dx$ *for all possible sub-sets e of* (0, 1) *with measure* $\sigma/2\rho$, *then the maximum of* $\Delta\mu$ *is* $\rho K_1 - \rho K_2$.

For suppose it is not. Then it differs from $\rho K_1 - \rho K_2$ by an amount of positive absolute value ω. By properly choosing $\psi(x)$ we can ensure that

(*a*) The maximum of $\Delta\mu_1 = \int_0^1 a(x)\,\psi(x)\,dx$ differs from that of $\Delta\mu$ by less than $\omega/3$.

(*b*) The constants K_1', K_2' for $\psi(x)$ differ from those of $\gamma(x)$ by less than $\omega/3\rho$.

Now $\rho(K_1' - K_2')$ is evidently the maximum of $\Delta\mu_1$. Thus the maximum of $\Delta\mu$ differs from $\rho K_1 - \rho K_2$ by less than

$$\frac{\omega}{3} + \rho\left(\frac{\omega}{3\rho} + \frac{\omega}{3\rho}\right) = \omega,$$

i.e.

$$\omega < \omega,$$

which is impossible.

REMARK.—The proof given above may at first sight seem unnecessary, as the converse follows at once from the way in which the theorem was proved in § 9. There we showed that the maximum was given by

$$\rho\int_{e_1} \gamma(x)\,dx - \rho\int_{e_2} \gamma(x)\,dx,$$

where e_1 and e_2 are sets obtained in a certain way. In § 13 we show that

$$\int_{e_1} \gamma(x)\,dx, \quad \int_{e_2} \gamma(x)\,dx$$

are the bounds by which K_1 and K_2 are subsequently defined. Thus the converse must be true.

But if we examine the proof of § 9 we find that it depends on the two following propositions:—

(A) If $$S_1 + S_2 + \ldots + S_\nu + \ldots = E_1,$$

$$mS_1 + mS_2 + \ldots + mS_\nu + \ldots = mE_1.$$

(B) If $J_1, J_2, \ldots$ increase to J', then

$$mJ_\nu \to mJ'.$$

Neither of these propositions have yet been demonstrated apart from the use of the so-called Multiplicative Axiom,* a result which as far as can be seen does not follow from the ordinary axioms of mathematics and whose truth at present we do not know how to ascertain.

It is desirable, if possible, to establish the existence of a solution of the problem apart from this axiom. And this is precisely what the proof of § 16 enables us to do. For K_1 and K_2 can be defined as upper and lower bounds apart from the Multiplicative Axiom. And the approximation theorem of § 15 can be established without it. The problem is thus solved without it.

Determination of K_1 and K_2 when $\gamma(x)$ is Riemann-integrable.

17. When $\gamma(x)$ is integrable in Riemann's sense then K_1 and K_2, as we are about to show, can be obtained by a direct process as limits of certain approximative sums, just as a Riemann integral can be obtained as a limit of sums. From a theoretical point of view, of course, this fact is not of any great importance. But from a practical point of view it is the one thing that matters, because it provides us with a reasonable means of calculating the constants required.

Contrast, for the moment, a limit of a sequence with a bound of an aggregate as regards practicability of calculation. Suppose that a number L is given

(*a*) As the limit of the sequence

$$a_1,\ a_2,\ \ldots,\ a_n,\ \ldots\,.$$

(*b*) As the bound of an aggregate of numbers a;

where the a_n's and the a's are not explicitly given, but have to be calculated according to certain rules; and that we try to find L approximately by calculating the a's.

We are at once faced with the fact that we can only calculate a finite number of the a's. Consequently, as far as (*b*) goes, we cannot in general get anywhere near determining L. For there is no reason at all for supposing that the greatest of the a's calculated is anywhere near L. If

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* See Russell, *Introduction to Mathematical Philosophy*, Ch. xii.

the a's in (b) form a more than enumerable set, the odds are that we shall never hit on a's near L. For instance, if the a's consisted of all the numbers between 0 and 1, together with the number 100 aggregated together in some random manner, the likelihood of our calculating the a which is 100 is nil. We should almost invariably deduce from our calculations that L was some number not greater than 1. Even if we could calculate an enumerable infinity of a's we should not in general include 100 and would still get $L \leqslant 1$.

But when we come to L as given by (a) we are on different ground. We know, by the definition of convergence, that by calculating sufficient a's we can get as near L as we please. It is always possible to determine L* by calculating a finite number of a's, and the only difficulty is, when we have obtained some likely value, to verify—by means of the law defining the sequence—that this value is a proper approximation. Whereas before we were outside the bounds of even probability, now we are within the bounds of the actually possible. The only difficulty that remains, is, as we have stated, the difficulty of verification.

Consequently if we can show that K_1 and K_2 are capable of exhibition as the limits of definite sequences, then, from the practical point of view, we have made enormous strides towards numerically obtaining them.†

Approximative Sums for K_1 and K_2.

18. Take any set of dividing points

$$x_0 = 0, \quad x_1, x_2, \ldots, x_{k-1}, x_k, \ldots, x_n = 1$$

in the interval (0, 1). $\gamma(x)$, being Riemann-integrable in (0, 1), is bounded in (0, 1), and therefore bounded in each of the intervals (x_{k-1}, x_k). Let M_k, m_k be its upper and lower bounds in the representative interval (x_{k-1}, x_k), which we will denote by δ_k.

Rearrange the intervals δ_k so that the corresponding upper bounds M_k form a descending series. In their new order let the intervals be

$$\eta_1, \eta_2, \ldots, \eta_k, \ldots, \eta_n,$$

the corresponding upper bounds

$$P_1, P_2, \ldots, P_k, \ldots, P_n,$$

* Approximately, that is to say.

† It may make matters clearer if we state that one of the great advantages of the Riemann integral is that the value of any given definite Riemann integral is within the bounds of practical calculability. That of a Lebesgue integral does not in general seem to be so.

and the corresponding lower bounds

$$p_1,\ p_2,\ \ldots,\ p_k,\ \ldots,\ p_n.$$

Find the first integer λ satisfying

$$\eta_1+\eta_2+\ldots+\eta_\lambda \geqslant l,$$

where $l=\sigma/2\rho$, and form the sums

$$S = P_1\eta_1+P_2\eta_2+\ldots+P_\lambda\eta_\lambda,$$

$$s = p_1\eta_1+p_2\eta_2+\ldots+p_{\lambda-1}\eta_{\lambda-1}.$$

Again, rearrange the intervals δ_k so that the corresponding *lower* bounds m_k form an ascending series. In the new order let the intervals be

$$\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_k,\ \ldots,\ \zeta_n,$$

the corresponding lower bounds

$$q_1,\ q_2,\ \ldots,\ q_k,\ \ldots,\ q_n,$$

and the corresponding upper bounds

$$Q_1,\ Q_2,\ \ldots,\ Q_k,\ \ldots,\ Q_n.$$

Find the first integer μ satisfying

$$\zeta_1+\zeta_2+\ldots+\zeta_\mu \leqslant l,$$

and form the sums

$$\sigma = q_1\zeta_1+q_2\zeta_2+\ldots+q_\mu\zeta_\mu,$$

$$\Sigma = Q_1\zeta_1+Q_2\zeta_2+\ldots+Q_{\mu-1}\zeta_{\mu-1}.$$

The sums S, s are approximative sums for K_1, and Σ, σ approximative sums for K_2 analogous to the approximative sums for an integral obtained by Darboux in his account of Riemann integration.*

We proceed to show that, as the length of the maximum interval δ tends to zero, S and s tend to K_1 and Σ and σ to K_2. First of all, however, we shall prove one or two subsidiary results.

* Goursat, *Cours d'Analyse*, t. i, 2nd ed., pp. 171-176; Whittaker and Watson, *Modern Analysis*, 3rd ed., p. 96.

Inequalities for the Approximative Sums when $\gamma(x)$ satisfies certain Conditions.

19. THEOREM.—*If $\gamma(x) \geqslant 0$ throughout, then $S \geqslant K_1 \geqslant s$. If $\gamma(x)$ is negative throughout, then $\sigma \leqslant K_2 \leqslant \Sigma$.*

Let us prove the first.

In the first place, if ω is the set consisting of $\eta_1, \eta_2, \ldots, \eta_{\lambda-1}$ together with any part of (0, 1) outside the intervals of length such as to make $m\omega = l$,* then

$$K_1 \geqslant \int_\omega \gamma(x)\,dx \geqslant \int_{\eta_1+\eta_2+\ldots+\eta_{\lambda-1}} \gamma(x)\,dx$$
$$= \int_{\eta_1} \gamma(x)\,dx + \int_{\eta_2} \gamma(x)\,dx + \ldots + \int_{\eta_{\lambda-1}} \gamma(x)\,dx$$
$$\geqslant p_1\eta_1 + p_2\eta_2 + \ldots + p_{\lambda-1}\eta_{\lambda-1}$$
$$= s.$$

Again, denoting by $I(E)$ the integral of $\gamma(x)$ in any interval E we have, if e is any sub-set of (0, 1) of measure l,

$$\int_e \gamma(x)\,dx = I(e)$$
$$= I\{e\eta_1 + e\eta_2 + \ldots + e\eta_\lambda + e(\eta_{\lambda+1} + \ldots + \eta_n)\}$$
$$\leqslant P_1 m(e\eta_1) + P_2 m(e\eta_2) + \ldots + P_\lambda m(e\eta_\lambda) + P_\lambda m\{e(\eta_{\lambda+1} + \ldots + \eta_n)\}.$$

Now
$$S = P_1\eta_1 + P_2\eta_2 + \ldots + P_\lambda\eta_\lambda.$$

Thus

$$S - \int_e \gamma(x)\,dx \geqslant P_1 m(\eta_1 - e\eta_1) + P_2 m(\eta_2 - e\eta_2) + \ldots + P_\lambda m(\eta_\lambda - e\eta_\lambda) - P_\lambda m\{e(\eta_{\lambda+1} + \ldots + \eta_n)\}$$
$$\geqslant P_\lambda[m(\eta_1 - e\eta_1) + \ldots + m(\eta_\lambda - e\eta_\lambda) - m\{e(\eta_{\lambda+1} + \ldots + \eta_n)\}]$$
$$= P_\lambda[\eta_1 + \eta_2 + \ldots + \eta_\lambda - m(e\eta_1 + e\eta_2 + \ldots + e\eta_n)]$$
$$= P_\lambda[\eta_1 + \eta_2 + \ldots + \eta_\lambda - me]$$
$$\geqslant P_\lambda[l - l]$$
$$\geqslant 0,$$

* m is here the sign of measure.

and so $$S \geqslant \int_e \gamma(x)\,dx,$$

whence, e being arbitrary, $$S \geqslant K_1.$$

This completes the result.

The Fundamental Property of the Approximative Sums.

20. Lemma.—*As the maximum interval tends to zero, $S-s$ and $\Sigma-\sigma$ tend to* 0.

For let Δ be the difference between the Darboux upper and lower sums for $\int_0^1 \gamma(x)\,dx$ for the same mode of sub-division, *i.e.* let

$$\Delta = \Sigma\,(M_k - m_k)\,\delta_k.$$

Then evidently $$|S-s|,\quad |\Sigma-\sigma| \leqslant \Delta + G.\bar{\delta},$$

where G is the upper bound of $|\gamma(x)|$ in (0, 1) and $\bar{\delta}$ is the length of the maximum sub-interval. Darboux's theorem* in the theory of integration states that $\Delta \to 0$ as $\bar{\delta} \to 0$, and we have our result at once.

Cor.—*If $\gamma(x)$ is positive, $S, s \to K_1$*; if $\gamma(x)$ *is negative, $\Sigma, \sigma \to K_2$.*

Theorem.—*Whatever be the sign of $\gamma(x)$, $S, s \to K_1$ and $\Sigma, \sigma \to K_2$.*

For suppose $\gamma(x)$ is not positive. By the addition of a suitable constant C we can make it positive.

If K_1', K_2', S', s' are the respective constants and sums for $\gamma(x)+C$, then, by what has gone before,

$$S',\ s' \to K_1'.$$

But $$K_1' = K_1 + C.$$

Also S' and s' evidently differ from S and s by an amount which tends to C as $\bar{\delta} \to 0$. Thus

$$S+C,\quad s+C \to K_1 + C,$$

i.e. $$S,\ s \to K_1.$$

In exactly the same way we show that

$$\Sigma,\ \sigma \to K_2.$$

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* Goursat, *loc. cit.*

Generation of K_1 and K_2 as Limits of Sequences.

21. Although we know that $S, s \rightarrow K_1$, this is not yet sufficient for the exhibition of K_1 as the limit of a sequence of calculable terms. For S and s being expressed in terms of bounds are not themselves calculable. But if we replace them by a sum of the form

$$T = \gamma_1 \delta_1' + \gamma_2 \delta_2' + \ldots + \gamma_\lambda \delta_\lambda',$$

where
$$\gamma_k = \gamma(\xi_k),$$

the intervals being rearranged in descending order of γ_k and ξ_k being some point in δ_k', then T is calculable; and if $T \rightarrow K_1$ as $\bar{\delta}$ (the maximum sub-interval) $\rightarrow 0$, then what we set out to do, namely, to exhibit K_1 as the limit of a sequence of calculable numbers, has been achieved.

To show that we can get a sequence of T's tending to K_1 observe that for any given sub-division of (0, 1) we can find three sums:

(*a*) By rearranging the δ's in descending order of upper bounds; this gives a sum S.

(*b*) By rearranging them in descending order of lower bounds; this gives a sum we will call $\bar{s}$.

(*c*) By rearranging them in descending order of $\gamma(\xi)$; this gives the sum T.

In each case, of course, we take just enough sub-intervals to obtain a total length not less than l, and in forming the sum the length of each sub-interval is to be multiplied by the appropriate factor, whether upper bound, as in (*a*); or lower bound, as in (*b*); or $\gamma(\xi)$, as in (*c*).

Now make the sub-division so that all the sub-intervals are equal. Then

$$\bar{s} \leqslant T \leqslant S.$$

For, if δ be the length of a sub-interval, s consists of δ multiplied by the sum of the λ greatest upper bounds, $\bar{s}$ of δ multiplied by the sum of the greatest lower bounds, and T of δ multiplied by the sum of the λ greatest intermediate values. But S, as has been shown, tends to K_1. $\bar{s}$, in a similar way, tends to K_1. Thus $T \rightarrow K_1$. Our objective is reached.

NOTE.—*On the Solution of the Problem by Elementary Methods.*

It should be stated that the method given above, depending essentially on the theory of sets of points, for the solution of the problem in the case

of an integral is not the only one. When the functions concerned are Riemann-integrable the solution can be obtained, in a form not differing in any vital way from that actually given, by means of arguments which are throughout of an elementary character. The problem was, in fact, solved first in this way ; the solution by means of sets of points and arguments in the Lebesgue theory only occurring to one of the writers after he had become acquainted with the original solution. We omit the elementary method, which formed the subject of the paper as first communicated,* simply because the paper is already long enough.

* By the first author alone.

ON THE RECIPROCITY FORMULA FOR THE GAUSS'S SUMS IN THE QUADRATIC FIELD

By L. J. MORDELL.

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SOME account of the Gauss's sums, that is series of the type

$$G\left(\frac{a}{b}\right) = \sum_{s=0}^{|b|-1} e^{\pi i a s^2/b}, \tag{1}$$

where a and b are integers, not necessarily positive, is to be found in text books on the theory of numbers as part of the fundamental elements of the subject. It is well* known that (if b or a is even, or if the summation for s extends to $2|b|-1$)

$$G\left(\frac{a}{b}\right) = \left(\frac{bi}{a}\right)^{\frac{1}{2}} G\left(-\frac{b}{a}\right), \tag{2}$$

where the radical is taken with a positive real part, and that this formula contains implicitly not only the sum of the series (1) but also the ordinary law of quadratic reciprocity.

2. The last twenty-five years, however, have seen the laws of not only quadratic reciprocity, but also of l-ic reciprocity (where l is any prime) for the general algebraic field, investigated with complete success by Hilbert and Furtwängler, in a series of memoirs of the greatest importance in the advancement of mathematical knowledge.† The latter writer, de-

* See Bachmann, *Zahlentheorie*, Vol. 3, p. 160.

† For an interesting résumé of the subject and references, see the paper by Fueter "Die Klassenkörper der komplexen Multiplikation und ihr Einfluss auf die Entwicklung der Zahlentheorie," *Jahresberichte der Deutschen Mathematiker-Vereinigung*, Vol. 20 (1911). Hilbert's chief papers are his well known "Bericht über die Theorie der algeb. Zahlkörper," of which there is a French translation published by A. Hermann, "Über die Theorie des relativ quadratischen Zahlkörpers," *Math. Annalen*, Vol. 51 (1898), and "Über die Theorie der relativ-Abelschen Zahlkörper," *Gött. Nachr.*, 1898, or *Acta Math.*, Vol. 26. Furtwängler's chief papers are in the *Math. Annalen*, Vols. 58, 63, 67, 72, 74, and in the *Gött. Nachr.*, 1911.

veloping and extending the ideas initiated by Hilbert, proved the general law for any algebraic field about ten years ago.

Under these circumstances, it seems rather surprising that the Gauss's sums were not also generalized for an algebraic field at the same time. This, however, was done as follows, only in the last few years, by Prof. Hecke, in a very interesting paper,* reminding us what vast mathematical treasures are still at hand if we could only find them.

Let K be the quadratic field of discriminant $-d$, and suppose $\sqrt{d}$ is taken with a plus sign if d is positive, and with a positive imaginary part if d is negative. If Ω is any number in K, we write

$$S(\Omega) = \Omega + \Omega',$$

where Ω' is the conjugate of Ω. It is easily seen then that $S(\omega/\sqrt{d})$ is a rational integer if ω is an integer in K. If, however, ω is fractional, we remove the common ideal factors from its numerator and denominator, put

$$\omega = A/B,$$

and refer to the ideal B as the denominator of ω. It is now clear that if b is any rational integer divisible by the ideal B, then $bS(\omega/\sqrt{d})$ is an integer. Hence if k is any integer in K, $e^{2\pi i S(\omega k/\sqrt{d})}$ is a b-th root of unity depending upon the residue of k (mod B).

The Gauss's sum for the quadratic field is then defined by

$$G(\omega) = \sum_{\rho} e^{2\pi i S(\rho^2\omega/\sqrt{d})},$$

where ρ takes all the values of any complete set of residues (mod B). Prof. Hecke then proves a number of results very similar to those for the ordinary Gauss's sum, and in particular that

$$G(\omega k) = \left(\frac{k}{B}\right) G(\omega), \tag{3}$$

if the ideal B is prime to 2, and k is an integer in K prime to B and where $\left(\frac{k}{B}\right)$ is the symbol of quadratic reciprocity in the quadratic field K.

* "Reziprozitätsgesetz und Gauss'sche Summen in quadratischen Zahlkörpern," *Gött. Nachr.*, 1919.

All this, of course, applies to the general algebraic field; and it is all the more surprising that it has not been discovered sooner, when we note that sums involving an exponent similar to $S(\omega)$ had already been considered by Stickleberger* in a paper of exceptional beauty, wherein he generalized some results of Eisenstein, who had proved† some formulæ such as: if p is a prime of the form $7n+2$, then from

$$x^2+7y^2 = p = 7n+2$$

we have
$$x \equiv \tfrac{1}{2}\frac{(3n)!}{n!\,(2n)!} \pmod{p}, \quad x \equiv 3 \pmod{7}.$$

Having defined the Gauss's sum, Prof. Hecke, who had previously discovered a method of associating a theta function‡ with an ideal, an idea through which he has already considerably enriched mathematics, deduced in the case of a real quadratic field, from the transformation formula for the theta function with two variables, the formula

$$G\left(\frac{b}{a}\right) = e^{\frac{1}{4}\pi i\,(\operatorname{sgn} ab - \operatorname{sgn} a'b')}\, 2\left|\frac{bb_1}{aa_1}\right|^{\frac{1}{2}} \frac{N(A)}{N(B_1)}\, G\left(-\frac{a}{4b}\right), \tag{4}$$

where A is the denominator of b/a, and B_1 the denominator of $-a/4b$. Also $N(A)$ is the norm of the ideal A, while $\operatorname{sgn} ab = \pm 1$ according as ab is positive or negative. He then applies this formula to the proof of the law of quadratic reciprocity in the real quadratic field K.

In a recent paper,§ I gave a very simple method for summing the series (1) in the particular case when $a = 2$. The same method, however, applied to the general series (1) gives at once the reciprocity formula (2), as I noticed when writing that paper, though I did not mention it at the time. In reading Prof. Hecke's paper, I saw at once that my method gives immediately the reciprocity formula for any quadratic field, real or imaginary. This I shall now prove.

Let a function $f(z)$ be defined by

$$(e^{2\pi i\mu z}-1)f(z) = \sum_{\xi,\,\eta} \exp \pi i S\left[(z+\rho)^2\,\omega/\sqrt{d}\right],$$

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* "Ueber eine Verallgemeinerung der Kreisteilung," *Math. Annalen*, Vol. 37 (1890).

† *Crelle's Journal*, Vol. 37, or H. J. S. Smith, *Collected Works*, Vol. 1, p. 280.

‡ It seems difficult to realize that as long ago as 1845 Hermite, in his first letter to Jacobi (Hermite, *Œuvres*, t. 1, p. 100), gave a method for associating a definite quadratic form with an algebraic number.

§ "On a Simple Summation of the Series $\sum_{s=0}^{n-1} e^{2s^2\pi i/n}$," *Messenger of Mathematics*, Vol. 48 (1918).

where $\mu = \pm 1$ will be fixed later, and

$$\rho = \xi + \eta\theta,$$

where the numbers $(1, \theta)$ form the base of the quadratic field K, so that

$$\theta = \tfrac{1}{2}\sqrt{d} \quad \text{if} \quad d \equiv 0 \pmod 4,$$

or

$$\theta = \tfrac{1}{2}(-1+\sqrt{d}) \quad \text{if} \quad d \equiv 1 \pmod 4.$$

Also

$$S\left[(z+\rho)^2\,\Omega\right] = (z+\rho)^2\,\Omega + (z+\rho_1)^2\,\Omega_1,$$

where ρ_1, Ω_1 are the conjugates of ρ, Ω respectively.

The summation is extended to the values

$$\left.\begin{array}{l} \xi = 0,\ 1,\ 2,\ \ldots,\ 2\tau\,|bb_1|-1 \\ \eta = 0,\ 1,\ 2,\ \ldots,\ 2\tau M-1 \end{array}\right\}, \tag{5}$$

where

$$M = \left| \frac{ab_1 - a_1 b}{\sqrt{d}} \right|,$$

b_1 is the conjugate of b, a_1 the conjugate of a, and $\omega = a/b$, $\tau = |aa_1bb_1|$.

Consider now the integral

$$\int f(z)\,dz$$

taken around the parallelogram $ABCD$ where the parallel sides AD, BC cut the real axis of z at $z = -\tfrac{1}{2}$, $z = \tfrac{1}{2}$, respectively, and are inclined to its positive direction at an acute or obtuse angle, according as $S(\omega/\sqrt{d})$ is positive or negative. The sides DC and AB respectively are at an infinite distance above and below the real axis. The integral around the sides AB, DC obviously* vanishes, since if we put $z = x+iy$,

$$\left| e^{\pi i z^2 S(\omega/\sqrt{d})} \right| = e^{-2\pi xy\, S(\omega/\sqrt{d})},$$

and the direction of the sides DA, BC is such that $xy\,S(\omega/\sqrt{d})$ is positive.* The only singularity of the integrand is a simple pole at $z = 0$. Hence,

* Provided that ω is not rational, for then $S(\omega/\sqrt{d}) = 0$. The results of the paper are trivial in this case.

by Cauchy's theorem,

$$\int_A^D [f(z+1)-f(z)]\,dz = \mu \sum_{\xi,\eta} \exp \pi i\, S(\rho^2\omega/\sqrt{d}). \tag{6}$$

We now take the standard expression for the Gauss's sum in a form slightly different from that used by Prof. Hecke, and write

$$G\left(\frac{a}{b}\right) = \sum_\rho \exp \pi i\; S(\rho^2\omega/\sqrt{d}),$$

where ρ runs through a complete set of residues (mod B), and B is the denominator of $\omega/2 = a/2b$. Hence the right-hand side of (6), when we adopt the limits of summation given by (5), can be written as

$$\frac{\tau^2 4\mu M\,|\,bb_1\,|}{N(B)}\, G\left(\frac{a}{b}\right). \tag{6a}$$

The success of my method depends upon the fact that $f(z+1)-f(z)$ is an integral function of z, really a sum of exponentials of the form $\exp(mz^2+nz)$. Hence as the path of integration can be deformed into the real axis of z from either $-\infty$ to ∞ or ∞ to $-\infty$ according as

$$S\left(\frac{\omega}{\sqrt{d}}\right) = \frac{ab_1-a_1b}{bb_1\sqrt{d}}$$

is positive or negative, we can evaluate the left-hand side of (6) which then becomes, except for unimportant factors, a sum which is symmetrical in a/b and $-b/a$.

For we have

$$(e^{2\pi i\mu z}-1)[f(z+1)-f(z)] = \sum_\eta \exp \pi i [S(z+2\tau\,|\,bb_1\,|+\eta\theta)^2\,\omega/\sqrt{d}]$$
$$-\sum_\eta \exp[\pi i S(z+\eta\theta)^2\,\omega/\sqrt{d}],$$

where the summation refers to $\eta = 0, 1, \ldots, 2\tau M-1$. The general term on the right-hand side is the product of two factors of which the first is

$$\exp[\pi i S(z+\eta\theta)^2\,\omega/\sqrt{d}],$$

while the second is

$$-1+\exp \pi i\, S[4\tau\,|\,bb_1\,|\,(z+\eta\theta)\,\omega/\sqrt{d}+4\tau^2b^2b_1^2\omega/\sqrt{d}].$$

But $$S(4\,|\,bb_1\,|\,\theta\omega/\sqrt{d}) \quad\text{and}\quad S(4b^2b_1^2\,\omega/\sqrt{d})$$

are even integers. Also

$$S(4\tau\,|\,bb_1\,|\,\omega z/\sqrt{d}) = 4\tau\frac{|\,bb_1\,|\,(ab_1-a_1b)}{bb_1\sqrt{d}}z = 4\mu\tau Mz,$$

if we take $$\mu = \operatorname{sgn}[bb_1(ab_1-a_1b)/\sqrt{d}].$$

Hence since $\exp(4\pi i\mu\tau Mz)-1$ is divisible by $\exp(2\pi i\mu z)-1$, we have

$$f(z+1)-f(z) = \sum_{\eta,\zeta}\exp\{\pi iS[(z+\eta\theta)^2\,\omega/\sqrt{d}]+2\pi i\mu\zeta z\} = \sum_{\eta,\zeta}\exp\pi iV \text{ say}, \tag{7}$$

where η and ζ also take the values $0, 1, 2, \ldots, 2\tau M-1$.

Now it is well known that

$$\int_{-\infty}^{\infty} e^{fz^2+2gz}\,dz = \left(-\frac{\pi}{f}\right)^{\frac{1}{2}}e^{-g^2/f},$$

where the radical is taken with a positive real part. In evaluating the left-hand side of (6), we change z into $z+\frac{1}{2}\eta$ when $2\theta = -1+\sqrt{d}$, and hence we have

$$V = z^2\frac{(ab_1-a_1b)}{bb_1\sqrt{d}}+z\frac{\eta(ab_1+a_1b)}{bb_1}+\frac{\eta^2(ab_1-a_1b)\sqrt{d}}{4bb_1}+2\mu\zeta z+\nu,$$

where $\nu = \mu\eta\zeta$ or 0 according as $d\equiv 1$ or 0 (mod 4). Hence the integral (6), remembering that the path of integration is deformed into the real axis from $-\infty$ to ∞ or ∞ to $-\infty$, becomes

$$\operatorname{sgn}\left(\frac{ab_1-a_1b}{bb_1\sqrt{d}}\right)\left(\frac{ibb_1\sqrt{d}}{ab_1-a_1b}\right)^{\frac{1}{2}}\sum_{\eta,\zeta}(-1)^{\nu}$$
$$\times\exp\left[\pi i\eta^2\frac{(ab_1-a_1b)\sqrt{d}}{4bb_1}-\pi ibb_1\sqrt{d}\frac{[\mu\zeta+(ab_1+a_1b)\eta/2bb_1]^2}{ab_1-a_1b}\right]$$

and this reduces to

$$\operatorname{sgn}\left(\frac{ab_1-a_1b}{bb_1\sqrt{d}}\right)\left(\frac{ibb_1\sqrt{d}}{ab_1-a_1b}\right)^{\frac{1}{2}}\sum_{\eta,\zeta}(-1)^{\nu}$$
$$\times\exp\left(\frac{-\pi i\sqrt{d}}{ab_1-a_1b}\right)[aa_1\eta^2+\mu\eta\zeta(ab_1+a_1b)+bb_1\zeta^2]. \tag{9}$$

Now it is clear, by putting $\zeta+2\tau M$ for ζ, that the summation for ζ (also for η) need only refer to any complete set of residues (mod $2\tau M$), that is to say we can replace ζ in the summation by $\mu\zeta$.

It is then obvious that the sum of the series $\sum\limits_{\eta, \zeta}$ is unaltered if we replace a, b by $-b$, a. Hence noting (6), (6a), and (9), we have at once

$$\frac{\operatorname{sgn}(bb_1) bb_1 G\left(\dfrac{a}{b}\right)}{N(B)} \left(\frac{ab_1 - a_1 b}{ibb_1\sqrt{d}}\right)^{\frac12} = \frac{\operatorname{sgn}(aa_1) aa_1 G\left(-\dfrac{b}{a}\right)}{N(A)} \left(\frac{ab_1 - a_1 b}{ibb_1\sqrt{d}}\right)^{\frac12},$$

where A is the denominator of $-b/2a$. Since

$$\left(\frac{ab_1 - a_1 b}{ibb_1\sqrt{d}}\right)^{\frac12} = \left|\left(\frac{ab_1 - a_1 b}{bb_1\sqrt{d}}\right)\right|^{\frac12} e^{-\frac14 \pi i \operatorname{sgn} [(bb_1\sqrt{d})/(ab_1 - a_1 b)]},$$

as the left-hand radical is taken with a positive real part, we have

$$|bb_1|^{\frac12} G\left(\frac{a}{b}\right) e^{-\frac14 \pi i \operatorname{sgn} [(bb_1\sqrt{d})/(ab_1 - a_1 b)]} / N(B)$$
$$= |aa_1|^{\frac12} G\left(-\frac{b}{a}\right) e^{-\frac14 \pi i \operatorname{sgn} [(aa_1\sqrt{d})/(ab_1 - a_1 b)]} / N(A), \qquad (10)$$

for the final result.

In the case of an imaginary field, aa_1 and bb_1 are both positive, and we have

$$|bb_1|^{\frac12} G\left(\frac{a}{b}\right) \Big/ N(B) = |aa_1|^{\frac12} G\left(-\frac{b}{a}\right) \Big/ N(A). \qquad (11)$$

In the case of a real field, if aa_1 and bb_1 have the same sign

$$|bb_1|^{\frac12} G\left(\frac{a}{b}\right) \Big/ N(B) = |aa_1|^{\frac12} G\left(-\frac{b}{a}\right) \Big/ N(A). \qquad (12)$$

If, however, aa_1 and bb_1 have opposite signs so that

$$\operatorname{sgn}(aa_1 bb_1) = -1,$$

that is
$$\operatorname{sgn}(ab_1) = -\operatorname{sgn}(a_1 b),$$

and hence
$$\operatorname{sgn}\left(\frac{bb_1\sqrt{d}}{ab_1 - a_1 b}\right) = \operatorname{sgn}\left(\frac{bb_1\sqrt{d}}{ab_1}\right) = \operatorname{sgn}(ab),$$

$$\operatorname{sgn}\left(\frac{aa_1\sqrt{d}}{ab_1 - a_1 b}\right) = \operatorname{sgn}\left(\frac{aa_1\sqrt{d}}{ab_1}\right) = \operatorname{sgn}(a_1 b_1),$$

we have

$$|bb_1|^{\frac12} G\left(\frac{a}{b}\right) \Big/ N(B) = e^{\frac14 \pi i (\operatorname{sgn} ab - \operatorname{sgn} a_1 b_1)} |aa_1|^{\frac12} G\left(-\frac{b}{a}\right) \Big/ N(A). \qquad (13)$$

This formula, which also includes (12), is equivalent to Prof. Hecke's formula (4).

I need hardly remark that we can prove the law of quadratic reciprocity in the imaginary field just as Prof. Hecke has done in the case of the real field from (13). The details are now rather simpler (as is known to be the case in the general investigations of Hilbert and Furtwängler) because of the absence of the factor $\exp\left[\frac{1}{4}\pi i\,(\operatorname{sgn} ab - \operatorname{sgn} a_1 b_1)\right]$. Thus if a and b are two co-prime numbers of odd norms (*i.e.* aa_1 and bb_1 both odd), and if one of them is a primary number, that is a quadratic residue (mod 4), then

$$\left(\frac{a}{b}\right) = \left(\frac{b}{a}\right),$$

where $\left(\frac{a}{b}\right)$ is the symbol of quadratic reciprocity in the imaginary field.

SUR UNE SÉRIE DE POLYNOMES DONT CHAQUE SOMME PARTIELLE REPRÉSENTE LA MEILLEURE APPROXIMATION D'UN DEGRÉ DONNÉ SUIVANT LA MÉTHODE DES MOINDRES CARRÉS*

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1. En statistique mathématique on rencontre souvent le problème suivant: étant données certaines valeurs $x_0, x_1, \ldots, x_{n-1}$ par ordre de grandeur de la variable x auxquelles correspondent respectivement les fréquences $y_0, y_1, \ldots, y_{n-1}$, le nombre n étant généralement grand, il s'agit de reproduire ces résultats aussi bien que possible à l'aide d'un polynome $f_m(x)$ de degré m plus petit que n. Les écarts ou erreurs étant $\delta_i = y_i - f_m(x_i)$, il faut déterminer les coefficients c_ν du polynome $f_n(x) = \Sigma\, c_\nu x^\nu$ conformément à la théorie des moindres carrés, en rendant la somme des carrés des erreurs δ_i minimum.

Les calculs ne présentent pas de difficultés, mais ils sont longs et pénibles; en effet, les valeurs de $m+1$ déterminants du m-ième ordre doivent être calculées.

Les constantes c_ν évaluées, on peut, pour se rendre compte de la précision obtenue, déterminer d'après la théorie des moindres carrés, la somme des carrés des écarts δ_i par la formule suivante:

$$\Sigma\delta_i^2 = \Sigma y_i^2 - c_0 . \Sigma y_i - c_1 . \Sigma y_i x_i - c_2 . \Sigma y_i x_i^2 - \ldots - c_m . \Sigma y_i x_i^m .$$

Si l'on trouve que l'approximation obtenue n'est pas suffisante, pour en avoir une plus grande, on est obligé de refaire le calcul, et de déterminer les coefficients d'un polynome de degré $m_1 > m$; le grand inconvénient de la méthode est que dans ce cas, tout est à recommencer, car les $m+1$ constantes obtenues précédemment ne conservent pas leurs valeurs.

* L'origine de ce travail est dans un cours de statistique mathématique et de probabilités, que j'ai fait en 1919 à l'Université de Budapest.

Si, au lieu de développer le polynome $f_m(x)$ suivant les puissances de x, on fait ce développement suivant les factorielles de x, $x(x-h)$, etc. c.-à-d. si l'on pose:

$$f_m(x) = \Sigma c_\nu x(x-h)(x-2h)\ldots(x-\nu h+h),$$

et qu'on détermine les coefficients c_ν d'après le principe des moindres carrés, on rencontre les mêmes difficultés.

Par contre, si l'on fait l'approximation à l'aide d'une série de Fourier, cette difficulté ne se présente pas ; en effet dans une telle série les $2m+1$ constantes étant calculées, si l'on veut obtenir celles d'une série à $2m_1+1$ termes ($m_1 > m$), il suffit de déterminer les constantes supplémentaires, car les premiers $2m+1$ coefficients restent les mêmes.

Tchebichef a considéré la première fois une série de polynomes tels que le développement d'une fonction suivant ces polynomes possède la propriété précieuse des séries de Fourier.*

Étant données les valeurs $y_0, y_1, \ldots, y_{n-1}$ qu'une fonction y prend pour les valeurs $x_0, x_1, \ldots, x_{n-1}$ de la variable x, il s'agit de déterminer une suite de polynomes $\phi_1(x), \phi_2(x), \ldots, \phi_m(x)$, le polynome $\phi_\nu(x)$ étant de degré ν, tels que si l'on représente y par la somme de degré m ($m < n$)

$$f_m(x) = c_0 + c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_m\phi_m(x),$$

la quantité†
$$\sum_{i=0}^{n} [y_i - f_m(x_i)]^2,$$

soit minimum pour toutes les valeurs de m.

Tchebichef a montré que les polynomes $\phi_\nu(x)$ sont proportionnels aux dénominateurs $\psi_\nu(x)$ des réduites de la fraction continue suivante:

$$\sum_{i=0}^{n} \frac{1}{x-x_i} = \cfrac{a_1}{x-b_1+\cfrac{a_2}{x-b_2+\cfrac{a_3}{x-b_3+\ldots}}}.$$

* "Sur une formule d'Analyse," *Bull. Phys. Math. de l'Académie Impériale des Sciences de St. Pétersbourg*, t. 13 (1854), p. 210; "Sur les fractions continues," *Journal de mathématiques pures et appliquées*, 2 série, t. 3 (1855), p. 289; "Sur l'interpolation par la méthode des moindres carrés," *Mémoires de l'Acad. Imp. des Sciences de St. Pétersbourg*, 7 série, t. 1 (1859), p. 1.

† Dans ce travail, conformément aux principes du calcul des différences finies, la variable x ne prend pas la valeur de la limite supérieure de la somme définie, c.-à-d.

$$\sum_{x=1}^{n+1} f(x) = f(1) + f(2) + \ldots + f(n).$$

Le facteur de proportionnalité étant quelconque on peut choisir :

$$\phi_\nu(x) = (-1)^\nu \psi_\nu(x) \begin{vmatrix} n & \Sigma x_i & \Sigma x_i^2 & \dots & \Sigma x_i^{\nu-1} \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 & \dots & \Sigma x_i^\nu \\ \dots & \dots & \dots & & \dots \\ \Sigma x_i^{\nu-1} & \dots & \dots & & \Sigma x_i^{2\nu-2} \end{vmatrix}.$$

Alors en déterminant les réduites on trouve

$$\phi_\nu(x) = \begin{vmatrix} 1 & x & x^2 & x^3 & \dots & x^\nu \\ n & \Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 & \dots & \Sigma x_i^\nu \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 & \Sigma x_i^4 & \dots & \Sigma x_i^{\nu+1} \\ \dots & \dots & \dots & \dots & & \dots \\ \Sigma x_i^{\nu-1} & \dots & \dots & \dots & & \Sigma x_i^{2\nu-1} \end{vmatrix}.$$

Tchebichef a obtenu la formule suivante donnant les coefficients c_ν,

$$c_\nu = \frac{\Sigma y_i\, \phi_\nu(x_i)}{\Sigma \phi_\nu^2(x_i)}.$$

Dans le mémoire de 1859, mentionné ci-dessus, il a traité en outre le cas particulier dans lequel les valeurs $x_0, x_1, \dots, x_{n-1}$ sont équidistantes, dans ce cas entre les fonctions $\phi_\nu(x)$ définies dans son mémoire et les polynomes $\psi_\nu(x)$ précédents il y a la relation :

$$\phi_\nu(x) = \frac{(2\nu)!}{\nu!}\, \psi_\nu(x).$$

Tchebichef a donné de plus une formule de récurrence pour déterminer les polynomes $\phi_\nu(x)$.

Poincaré dans son *Calcul des Probabilités*,* a repris la question et il est arrivé, par le développement de $\Sigma\, 1/(x-x_i)$ en fraction continue, à des polynomes $D_\nu(x)$ proportionnels aux polynomes $\phi_\nu(x)$ de Tchebichef.

A. Quiquet, dans les *Proceedings of the Fifth International Congress of Mathematicians*, Cambridge,† a indiqué une méthode d'application de ces polynomes aux fonctions de survie.

L'emploi de ces polynomes est certainement avantageux, malgré qu'il nécessite la détermination préalable des valeurs de $\phi_\nu(x_i)$ et de $\Sigma \phi_\nu^2(x_i)$ qui

* 1 éd., 1896, p. 251 ; 2 éd., 1912, p. 280.

† Cambridge Press, Vol. 2 (1913), p. 385.

est généralement laborieuse. Leur avantage ressort surtout dans le cas où l'on a plusieurs approximations à faire, dans lesquelles les grandeurs $x_0, x_1, \ldots, x_{n-1}$ sont les mêmes, en effet les valeurs de $\phi_\nu(x_i)$, etc., peuvent être calculées alors une fois pour toute. C'est ce qui a lieu si les valeurs de $x_0, x_1, \ldots$, etc., sont équidistantes, ce qui arrive très souvent en statistique mathématique.

Dans ce travail nous allons déduire directement la forme générale des polynomes possédant les propriétés mentionnées des séries de Fourier, et étudier leurs propriétés ; puis en supposant les valeurs de x_i équidistantes, nous allons donner des formules simples et des tables permettant de déterminer les valeurs de ces polynomes ; enfin, on va montrer sur un exemple l'emploi de ces formules et de ces tables.

2. *Déduction des polynomes.*—Etant donnés n points de coordonnées x_0, y_0 ; x_1, y_1 ; ... ; x_{n-1}, y_{n-1}, soit $y = f_{n-1}(x)$ l'équation d'une courbe de degré $n-1$ passant par ces points. Développons $f_{n-1}(x)$ en une série de polynomes :

$$y = f_{n-1}(x) = \sum_{\nu=0}^{n} A_\nu G_\nu, \tag{1}$$

où A_ν est un coefficient constant, G_ν un certain polynome de degré ν.

Considérons une somme partielle de l'expression précédente

$$y = f_m(x) = \sum_{\nu=0}^{m+1} A_\nu G_\nu.$$

Disposons des coefficients A_ν de manière que la courbe $y = f_m(x)$ de degré m passe aussi près que possible des n points donnés, suivant le principe des moindres carrés c.-à-d. que la somme (2) des carrés des écarts soit minimum :

$$\sum_{i=0}^{n} [y_i - f_m(x_i)]^2. \tag{2}$$

A cet effet, il faut égaler à zéro les $m+1$ dérivées par rapport à $A_0, A_1, \ldots, A_m$ de cette somme.

Cela nous donne les $m+1$ équations :

$$\sum_{i=0}^{n} [y_i - (A_0 + A_1 G_1 + A_2 G_2 + \ldots + A_m G_m)] G_\nu = 0, \tag{3}$$

pour $\nu = 0, 1, 2, 3, \ldots, m.$

Si les polynomes G_ν sont des polynomes quelconques, les valeurs de A_ν obtenues à l'aide de ces équations dépenderont en général du degré m

de la courbe. Notre but est de *déterminer les polynomes G_ν de manière que les constantes A_ν soient indépendantes de m.*

La résolution des équations précédentes pour $m = 0$ donne :

$$A_0 = \Sigma\, y_i / n G_0 ;$$

en remplaçant A_0 par cette valeur dans les deux équations obtenues de (3) en posant $m = 1$, on a $\Sigma G_0 G_1 = 0$ (équation de condition) et

$$A_1 = \Sigma\, y_i G_1 / \Sigma G_1^2 .$$

Si nous posons dans (3) $m = 2$ et si nous y remplaçons A_0 et A_1 par les valeurs précédentes, nous trouvons :

$$\Sigma G_0 G_2 = 0, \quad \Sigma G_1 G_2 = 0, \quad A_2 = \Sigma y_i G_2 / \Sigma G_2^2 .$$

En procédant de la même manière, on arrive aux $\binom{n}{2}$ équations de conditions

$$\sum_{i=0}^{n} G_\nu(x_i)\, G_\mu(x_i) = 0 \quad \text{pour} \quad \nu \neq \mu. \tag{4}$$

Les polynomes $G_0, G_1, \ldots, G_{n-1}$ contiennent en tout $\binom{n+1}{2}$ constantes arbitraires, on peut donc satisfaire à ces équations et, en plus, on peut choisir arbitrairement dans chaque polynome un des coefficients.

Outre les équations de condition, on trouve encore :

$$A_\nu \sum_{i=0}^{n} [G_\nu(x_i)]^2 = \sum_{i=0}^{n} y_i\, G_\nu(x_i).$$

Par suite, le développement en série de polynomes G possède non seulement l'avantage, vis à vis d'un développement suivant les puissances de x, qu'en poussant l'approximation plus loin, les coefficients déjà obtenus conservent leurs valeurs, mais encore l'extrême simplicité de la détermination de ces coefficients A_ν.

On peut donner une autre expression aux équations de condition (4) ; en supposant $\mu > \nu$ on peut mettre à la place de G_ν un polynome quelconque $F_\nu(x_i)$ de degré ν. En effet ce dernier peut être considéré comme la somme de plusieurs polynomes G_s tels que $s \leqslant \nu$; il en résulte que les conditions (4) sont équivalentes à

$$\sum_{i=0}^{n} F_\nu(x_i)\, G_\mu(x_i) = 0 \quad \text{si} \quad \mu > \nu. \tag{4'}$$

Les polynomes mentionnés de Tchebichef et de Poincaré satisfont à cette équation.

3. Nous nous proposons de résoudre les équations (4′) en supposant les valeurs de x_i équidistantes :

$$x_i = a+\left(\frac{b-a}{n}\right) i = a+hi.$$

C'est dans ce cas, comme nous l'avons remarqué que l'utilisation de ces polynomes est particulièrement avantageuse. Dans ce cas particulier nous désignerons les polynomes G_ν par Q_ν. Pour abréger l'écriture dans la résolution des équations (4′), introduisons une notation nouvelle; les sommes indéfinies de Q_m seront désignées, comme il suit :

$$\Sigma Q_m h = {}^1Q_m, \quad \Sigma\Sigma Q_m h^2 = \Sigma(\Sigma Q_m h)\, h = {}^2Q_m, \quad \text{etc.},$$

de manière que la μ-ième somme indéfinie de Q_m sera ${}^\mu Q_m$; et la μ-ième différence de Q_m sera

$$Q_m^{(\mu)} = \Delta^\mu Q_m.$$

Pour déterminer les polynomes Q_m, nous allons partir de la somme indéfinie qui correspond à (4′),

$$\Sigma F_s(x)\, Q_m(x)\, h.$$

En utilisant la méthode de la sommation par parties, de manière à prendre la somme de Q_m et la différence de F_s,* puis en répétant l'opération $s-1$ fois, jusqu'à arriver à $F_s^{(s)}=$ constante, notre somme indéfinie deviendra :

$$\Sigma Q_m F_s h = {}^1Q_m F_s - {}^2Q_m(x+h) F_s^{(1)} + {}^3Q_m(x+2h) F_s^{(2)} - \dots + (-1)^s\, {}^{s+1}Q_m(x+sh) F_s^{(s)}. \tag{5}$$

Les sommes ${}^\nu Q_m$ ci-dessus ne sont pas complètement déterminées; en effet, on peut leur ajouter sans inconvénient un polynome arbitraire de degré $\nu-1$ sans changer Q_m; nous pouvons donc disposer de ces polynomes arbitraires de manière à annuler les expressions ${}^\nu Q_m(x+\nu h-h)$ pour $x=a$ ou $i=0$ c.-à-d. pour avoir quel que soit ν,

$$ {}^\nu Q_m(a+\nu h-h) = 0, \qquad \nu \leqslant m. \tag{6}$$

Il y aura ainsi m conditions et m constantes arbitraires disponibles dans ${}^m Q_m$.

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* $\Sigma Q(x) F(x) h = {}^1Q(x) F(x) - \Sigma\, {}^1Q(x+h) F^{(1)}(x)\, h.$

Comme dans ces conditions la somme indéfinie précédente est nulle à la limite inférieure de (4'), pour que cette condition soit satisfaite, il faut que (5) soit aussi nulle à la limite supérieure, c.-à-d. pour $x = b$ ou $i = n$; le polynome F_s ainsi que ses différences étant arbitraires, la somme (5) ne peut être nulle pour $x = b$ que si chaque terme est séparément nulle. Il faut donc avoir pour toutes les valeurs de ν,

$$ {}^{\nu}Q_m(b+\nu h-h)\,h = 0. \tag{7}$$

De (6) on conclut que l'on a ${}^1Q_m(a) = 0$, ce qui veut dire que $(x-a)$ doit être un facteur de ${}^1Q_m(x)$ c.-à-d.,

$$ {}^1Q_m(x) = (x-a)\,f_m(x). $$

De cette relation on arrive, en appliquant la méthode de la sommation par parties, à ${}^2Q_m(x)$,

$$ {}^2Q_m(x) = \frac{1}{2!}\,(x-a)(x-a-h)\,f_m(x) - \Sigma\,\frac{1}{2!}\,(x-a+h)(x-a)f_m^{(1)}(x)\,h. $$

Pour abréger les formules, nous allons adopter la notation suivante pour la factorielle*:

$$ (x+k)(x+k-h)(x+k-2h)(x+k-3h)\ldots(x+k-mh+h) = (x+k)_m. $$

Revenons à notre expression de ${}^2Q_m$ et répétons la sommation par parties $m-1$ fois pour avoir :

$$ {}^2Q_m(x) = \sum_{\nu=0}^{m+1} (-1)^{\nu}\,\frac{(x-a+\nu h)_{\nu+2}}{(\nu+2)!}\,f_m^{(\nu)}(x). $$

On en conclut que $(x-a)$ et $(x-a-h)$ doivent être des facteurs de ${}^2Q_m(x)$ de manière que

$$ {}^2Q_m(x) = (x-a)(x-a-h)\,g_m(x) = (x-a)_2\,g_m(x). $$

Par sommations successives on démontre de la même manière que

$$ {}^mQ_m(x) = (x-a)_m\,\omega_m(x). $$

Cette grandeur satisfait à la condition (6) pour qu'elle satisfasse aussi

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* Cette notation fait bien ressortir l'analogie entre les puissances et les factorielles; p. ex. on a :

$$ \Sigma\,(x+k)_m\,h = \frac{1}{m+1}\,(x+k)_{m+1} \quad \text{et} \quad \int (x+k)^m\,dx = \frac{1}{m+1}\,(x+k)^{m+1}. $$

à la condition (7) on part de ${}^1Q_m(b) = 0$ et en procédant de la même manière on est conduit à la formule

$$ {}^mQ_m(x) = C(x-a)_m(x-b)_m, $$

C étant un facteur constant arbitraire. Le polynome Q_m se trouve donc déterminé, c'est *la m-ième différence de l'expression ci-dessus*, c.-à-d.,

$$ Q_m = C.\Delta^m(x-a)_m(x-b)_m. $$

En prenant successivement les différences de $C(x-a)_m(x-b)_m$ on est conduit sans difficulté à la formule suivante où $\nu \leqslant m$,

$$ (8) \quad \Delta^\nu C(x-a)_m(x-b)_m = \nu!\, h^\nu C \sum_{s=0}^{\nu+1} \binom{m}{s}\binom{m}{\nu-s}(x-a+sh)_{m-\nu+s}(x-b)_{m-s}. $$

La même formule peut servir pour les différences d'ordre supérieure à m p. ex. pour $\nu = m+\mu$, mais dans ce cas s ne varie que de μ à $m+1$; en effet, les termes sous le signe Σ sont nuls si $s<\mu$ ou si $s>m$.

Si nous posons $\nu = m$ et $C = 1/2^m.m!.h^m$, les polynomes cherchés deviennent:

$$ (9) \qquad Q_m(x) = (\tfrac{1}{2})^m \sum_{s=0}^{m+1} \binom{m}{s}^2 (x-a+sh)_s(x-b)_{m-s}. $$

La formule (8) donne les différences des polynomes Q_m; citons comme exemple:

$$ (10) \qquad \Delta^m Q_m = \frac{(2m)!\, h^m}{2^m m!}. $$

De la relation (9) on peut déduire les valeurs des polynomes Q_n correspondant à des cas particuliers; p. ex. en posant $a=-1$ et $b=1$, il résulte:

$$ Q_0 = 1, $$

$$ Q_1 = x + \tfrac{1}{2}h, $$

$$ Q_2 = \frac{3}{2}x^2 + \frac{3}{2}hx + \tfrac{1}{2}h^2 - \tfrac{1}{2}, $$

$$ Q_3 = \frac{5}{2}x^3 + \frac{15}{4}hx^2 + \frac{(11h^2-6)}{4}x + \frac{3h(h^2-1)}{4}. $$

4. Pour pouvoir effectuer les calculs indiqués au commencement du no. 2, il faut encore connaître la valeur de

$$ S_m = \sum_{x=a}^{b} Q_m^2(x)\,h. $$

On peut déterminer cette somme par la méthode des sommations successives par parties, on trouve un résultat analogue à celui obtenu par Tchebichef dans son mémoire *Sur une méthode d'interpolation* deja cité.

$$S_m = \frac{n \, . \, h^{2m+1}}{4^m(2m+1)}(n^2-1)(n^2-2^2)(n^2-3^2)\ldots(n^2-m^2). \tag{11}$$

5. Si l'on veut calculer les valeurs des polynomes Q_m correspondant à des grandeurs données de n et de x, on peut bien se servir de la formule (9), mais il est préférable de déduire d'autres formules plus commodes et plus maniables. Pour y arriver nous allons développer $(x-a)_m(x-b)_m$ en série de factorielles de $(x-b)$, $(x-b)_2$, $(x-b)_3$, ... etc., en employant la formule d'interpolation de Newton, qui remplace la formule de Taylor, lorsque au lieu de développer suivant des puissances on veut développer suivant des factorielles. On a :

$$(x-a)_m(x-b)_m = \sum_{\nu=0}^{2m+1} \frac{(x-b)_\nu}{\nu! \, h^\nu}\left[\Delta^\nu (x-a)_m(x-b)_m\right]_{(x=b)}.$$

D'après notre formule (8), la ν-ième différence de $(x-a)_m(x-b)_m$ est égale à zéro pour $x=b$ si $\nu < m$; par contre, si $\nu > m$ cette différence est égale pour $x=b$ à :

$$\nu! \, h^\nu \binom{m}{\nu-m}(b-a+mh)_{2m-\nu}.$$

Il résulte de là

$$(x-a)_m(x-b)_m = \sum_{\nu=m}^{2m+1} \binom{m}{\nu-m}(b-a+mh)_{2m-\nu}(x-b)_\nu. \tag{12}$$

De la même manière en développant $Q_m(x)$ suivant les factorielles de $(x-b)$, on aura

$$Q_m(x) = \sum_{\nu=0}^{m+1} (\tfrac{1}{2})^m \binom{m}{\nu}\binom{m+\nu}{\nu}(b-a+mh)_{m-\nu}(x-b)_\nu. \tag{13}$$

Comme $(x-a)_m(x-b)_m$ est symétrique par rapport à a et b et par suite Q_m aussi, il existe un développement de Q_m en factorielles de $(x-a)$ analogue à l'expression (13) ; on l'obtient de cette dernière en changeant a en b et inversement.

De (13), on peut déduire directement la différence d'ordre μ de Q_m,

$$\Delta^\mu Q_m(x) = (\tfrac{1}{2})^m \frac{(m+\mu)! \, h^\mu}{m!} \sum_{\nu=\mu}^{m+1} \binom{m}{\nu}\binom{m+\nu}{\nu-\mu}(b-a+mh)_{m-\nu}(x-b)_{\nu-\mu}. \tag{14}$$

Voici quelques valeurs particulières de Q_m tirées de (13):

$$Q_1 = \tfrac{1}{2}(b-a+h)+(x-b),$$

$$Q_2 = \tfrac{1}{4}(b-a+2h)_2+\tfrac{3}{2}(b-a+2h)(x-b)+\tfrac{3}{2}(x-b)_2,$$

$$Q_3 = \tfrac{1}{8}(b-a+3h)_3+\tfrac{3}{2}(b-a+3h)_2(x-b)+\tfrac{15}{4}(b-a+3h)(x-b)_2+\tfrac{5}{2}(x-b)_3.$$

6. Les polynomes Q_m montrent une certaine symétrie. En effet, si dans la formule (9) on remplace x par $a+b-h-x$, on obtient:

$$Q_m = (\tfrac{1}{2})^m \sum_{s=0}^{m+1} \binom{m}{s}^2 (b-x+sh-h)_s\,(a-x-h)_{m-s},$$

et en changeant le signe de chaque facteur, on trouve

$$Q_m = (-\tfrac{1}{2})^m \sum_{s=0}^{m+1} \binom{m}{s}^2 (x-b)_s\,(x-a+mh-sh)_{m-s},$$

résultat identique à (9), seul le signe est devenu $(-1)^m$; on en conclut

$$Q_m(x) = (-1)^m Q_m(a+b-h-x)\,; \tag{15}$$

et si nous introduisons une nouvelle variable x_1 telle que:

$$x = x_1+\tfrac{1}{2}(a+b-h)$$

nous aurons $$Q_m(x_1) = (-1)^m Q_m(-x_1).$$

Par conséquent, si m est paire, le polynome Q_m ne contient que des puissances paires de x_1; et si m est impaire, $Q_m(x_1)$ ne contient que des puissances impaires de x_1. Dans le cas particulier de $a=-1$, $b=1$, on a

$$Q_1 = x_1,$$

$$Q_2 = \tfrac{3}{2}x_1^2+\tfrac{1}{8}h^2-\tfrac{1}{2},$$

$$Q_3 = \tfrac{5}{2}x_1^3+\tfrac{1}{8}(7h^2+12)x_1.$$

7. Nous allons maintenant introduire au lieu de x une nouvelle variable ξ, cette dernière prendra les valeurs entières 0, 1, 2, 3, ..., $n-1$ définies par la relation

$$x = a+h\xi \quad \text{où} \quad h=\frac{b-a}{n}. \tag{16}$$

Nos formules établies précédemment deviennent :

$$(9') \qquad q_m = \frac{1}{h^m} Q_m = (\tfrac{1}{2})^m \sum_{\nu=0}^{m+1} \binom{m}{\nu}^2 (\xi+\nu)_\nu (\xi-n)_{m-\nu},$$

$$(13') \qquad q_m = \frac{1}{h^m} Q_m = (\tfrac{1}{2})^m \sum_{\nu=0}^{m+1} \binom{m}{\nu} \binom{m+\nu}{\nu} (m+n)_{m-\nu} (\xi-n)_\nu,$$

$$(13'') \qquad q_m = \frac{1}{h^m} Q_m = (\tfrac{1}{2})^m \sum_{\nu=0}^{m+1} \binom{m}{\nu} \binom{m+\nu}{\nu} (m-n)_{m-\nu} (\xi)_\nu,^*$$

$$(15') \qquad q_m(\xi) = (-1)^m q_m(n-1-\xi).$$

La relation (13″) donne immédiatement le développement de $q_m(\xi)$ suivant les factorielles de ξ. [Remarquons que dans ces factorielles, la différence étant l'unité, $(\xi)_\nu$ signifie $\xi(\xi-1)(\xi-2)\dots(\xi-\nu+1)$.]

$$q_1 = \tfrac{1}{2}(1-n)+\xi,$$
$$q_2 = \tfrac{1}{4}(2-n)_2+\tfrac{3}{2}(2-n)\xi+\tfrac{3}{2}(\xi)_2,$$
$$q_3 = \tfrac{1}{8}(3-n)_3+\tfrac{3}{2}(3-n)_2\,\xi+\tfrac{15}{4}(3-n)(\xi)_2+\tfrac{5}{2}(\xi)_3.$$

En remplaçant dans la formule (13″) les factorielles par les puissances de ξ, on trouve†

$$q_2 = \tfrac{3}{2}\xi^2-\tfrac{3}{2}(n-1)\xi+\tfrac{1}{4}(n^2-3n+2),$$
$$q_3 = \tfrac{5}{2}\xi^3-\tfrac{15}{4}(n-1)\,\xi^2+\tfrac{1}{4}(6n^2-15n+11)\xi-\tfrac{1}{8}(n^3-6n^2+11n-6),$$
$$q_4 = \tfrac{35}{8}\xi^4-\tfrac{35}{4}(n-1)\,\xi^3+\tfrac{5}{8}(9n^2-21n+17)\,\xi^2-\tfrac{5}{8}(2n^3-9n^2+17n-10)\,\xi$$
$$+\tfrac{1}{16}(n^4-10n^3+35n^2-50n+24),$$
$$q_5 = \tfrac{63}{8}\xi^5-\tfrac{315}{16}(n-1)\,\xi^4+\tfrac{35}{8}(4n^2-9n+8)\,\xi^3-\tfrac{105}{16}(n^3-4n^2+8n-5)\,\xi^2$$
$$+\tfrac{1}{16}(15n^4-105n^3+365n^2-525n+274)\,\xi$$
$$-\tfrac{1}{32}(n^5-15n^4+85n^3-225n^2+274n-120).$$

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* On obtient la formule (13″) en partant de (13), si, avant d'introduire la variable ξ, on y change a en b et inversement.

† Cette substitution est faite par la formule connue :

$$(x)_m = \sum_{\mu=1}^{m+1} (-1)^{m-\mu} C_m^{m-\mu} x^\mu,$$

où les coefficients $C_m^{m-\mu}$ sont les nombres de Stirling de première espèce (Voir Nielsen, *Gammafunktionen*, p. 67).

8. Maintenant nous sommes en état de pouvoir développer un polynome $F(x)$ de degré $n-1$ en séries de polynomes $Q_\nu(x)$, ou un polynome $f(\xi)$ en séries de polynomes $q_\nu(\xi)$:

$$F(x) = \sum_{\nu=0}^{n} A_\nu Q_\nu(x), \quad f(\xi) = \sum_{\nu=0}^{n} a_\nu q_\nu(\xi). \tag{1'}$$

Pour déterminer les coefficients A_ν il suffit de multiplier la première équation par Q_ν et de faire la somme des quantités obtenues, x variant de a à b, ou x_i de x_0 à x_{n-1}. D'après la formule (4), tous les termes du second membre disparaissent sauf le terme en A_ν et l'on trouve :

$$A_\nu = \frac{1}{S_\nu} \sum_{i=0}^{n} F(x_i)\, Q_\nu(x_i) \quad \text{où} \quad S_\nu = \sum_{i=0}^{n} [Q_\nu(x_i)]^2.$$

De la même manière, on aura

$$a_\nu \,.\, \sum_{\xi=0}^{n} [q_\nu(\xi)]^2 = \sum_{\xi=0}^{n} f(\xi)\, q_\nu(\xi). \tag{16'}$$

Remarquons en passant que $a_\nu = h^\nu . A_\nu$ et que $q_\nu h^\nu = Q_\nu$.

Si la fonction $F(x)$ est donnée par n points de coordonnées x_0, y_0; x_1, y_1; ...; x_{n-1}, y_{n-1} on peut considérer (1') comme une formule d'interpolation que l'on pourrait appeler formule d'interpolation de Tchebichef; cette formule présente de grands avantages sur les autres formules semblables ; les calculs sont plus simples et plus rapides, surtout dans le cas considéré dans ce mémoire où les grandeurs x_i sont équidistantes. Alors non seulement on peut utiliser les formules simples que nous venons de donner, mais encore comme nous le verrons, on peut construire des tables abrégeant beaucoup les calculs.

Il n'est pas sans intérêt de comparer les diverses formules d'interpolation :

(1) La formule de Lagrange :

$$y = \sum_{\nu=0}^{n} y_\nu L_\nu(x),$$

où $$L_\nu(x) = \frac{\omega(x)}{(x-x_\nu)\left[\dfrac{d\omega}{dx}\right]_{x=x_\nu}} \quad \text{et} \quad \omega(x) = (x-x_0)(x-x_1)\ldots(x-x_{n-1}).$$

Si nous considérons une somme partielle de y, dans laquelle ν varie de 0 à k, l'équation obtenue représente une courbe de degré $n-1$, passant

par les premiers k points de coordonnées x_0, y_0 ; x_1, y_1 ; ... ; x_{k-1}, y_{k-1} et par les points de coordonnées $x_k, 0$; $x_{k+1}, 0$; ... ; $x_{n-1}, 0$.

(2) La formule d'Ampère :

$$y = y_0 + \sum_{\nu=1}^{n} B_\nu (x-x_0)(x-x_1)\dots(x-x_{\nu-1}).$$

Les coefficients B_ν sont déterminés successivement en remplaçant x et y par les valeurs correspondantes de x_0, y_0 ; x_1, y_1 ; ... ; x_{n-1}, y_{n-1}. La somme partielle de y, où ν varie de 0 à k, représente une courbe de degré $k-1$ passant par les premiers k points.

(3) Formule de Tchebichef :

$$y = \sum_{\nu=0}^{n} A_\nu G_\nu(x),$$

où A_ν est donnée par :

$$A_\nu \sum_{i=0}^{n} [G_\nu(x_i)]^2 = \sum_{i=0}^{n} y_i G_\nu(x_i),$$

et où $G_\nu(x)$ est le dénominateur de la ν-ième réduite de $d/dx[\log \omega(x)]$ développé en fraction continue. La somme partielle de y, où ν varie de 0 à k, représente une courbe de degré $k-1$ passant aussi près des n points donnés que possible selon la théorie des moindres carrés.

Nous avons vu que cette dernière formule ne devient réellement pratique que si les valeurs de x sont équidistantes. Dans ce cas, on a

$$y = \sum_{\nu=0}^{n} a_\nu q_\nu(\xi),$$

où $q_\nu(\xi)$ est donnée par la formule (13″), et a_ν par (16′) ; la valeur de s_ν est conformément à la relation (11) :

$$\text{(11′)} \qquad s_\nu = \sum_{\xi=0}^{n} [q_\nu(\xi)]^2 = \frac{n}{4^m(2m+1)}(n^2-1)(n^2-2^2)\dots(n^2-\nu^2).$$

En outre, on peut construire une fois pour toutes des tables donnant les valeurs de $q_\nu(\xi)$ et s_ν.

9. Les tables les plus importantes pour le travail statistique en vue de l'utilisation de nos formules sont les suivantes :

(A) Des tables à double entrée donnant les valeurs de $q_\nu(n, \xi)$; une

table pour chaque valeur de ν variant de 1 à 6 ou tout au plus jusqu'à 10. L'interpolation à l'aide de polynomes de degré supérieur à 6 ne se fait que très rarement. On fera varier dans ces tables n de $\nu+1$ à 20 ou à 50 selon les besoins.

Nous avons vu que ξ doit varier de zéro à n, mais on peut réduire les tables de moitié en tenant compte de la symétrie des polynomes $q_\nu(\xi)$ selon (15').

Cette formule, comme celle de

$$\sum_{\xi=0}^{n} q_\nu(\xi) = 0,$$

peut servir comme vérification aux calculs des tables. Pour déterminer les valeurs numériques de $q_\nu(\xi)$ on se servira de la formule (13'').

(B) Une table à double entrée donnant les grandeurs $s_\nu(n)$ pour les valeurs de ν et de n qui figurent dans les tables précédentes. On utilisera la formule (11').

A titre d'exemple, nous avons joint à ce mémoire six tables. Les Tables I–V donnent les valeurs de $q_\nu(n, \xi)$ pour $\nu = 1, 2, 3, 4, 5$ et pour n jusqu'à 20; la Table VI donne $s_\nu(n)$ pour les mêmes valeurs de ν et de n.

Nous allons montrer sur un exemple la facilité avec laquelle les constantes a_ν se déterminent en se servant de ces tables.

La somme des carrés des erreurs mesurant la précision obtenue est

$$\begin{aligned}\Sigma\delta_\xi^2 &= \Sigma(y_\xi - a_0 - a_1 q_1 - a_2 q_2 - \ldots)^2 \\ &= \Sigma y_\xi^2 - 2a_0\Sigma y_\xi - 2a_1 \Sigma y_\xi q_1 - \ldots + a_0^2 n + a_1^2 \Sigma q_1^2 + a_2^2 \Sigma q_2^2 + \ldots.\end{aligned}$$

En y substituant $\Sigma y_\xi q_\nu$ à $a_\nu . \Sigma q_\nu^2$ on trouve:

$$\Sigma\delta_\xi^2 = \Sigma y_\xi^2 - a_0 \Sigma y_\xi q_1 - a_2 \Sigma y_\xi q_2 - \ldots - a_m \Sigma y_\xi q_m, \tag{17}$$

dans ces sommes, ξ varie de 0 à n. De la relation

$$\Sigma y_\xi q_\nu = a_\nu s_\nu,$$

il résulte:

$$\sum_{\xi=0}^{n} \delta_\xi^2 = \sum_{\xi=0}^{n} y_\xi^2 - \sum_{\nu=0}^{m+1} s_\nu a_\nu^2. \tag{17'}$$

Notons que tous les termes du second membre sauf le premier sont négatifs, ce qui n'était pas nécessairement vraie dans le cas de la formule, du no. 1 donnant la somme des carrés des erreurs. C'est un point im-

portant, en effet si l'approximation obtenue à l'aide d'un polynome de degré m est insuffisante, pour avoir une meilleure approximation à l'aide d'un polynome de degré $m+1$, les constantes $a_0, a_1, \ldots, a_m$ obtenues précédemment conservent leurs valeurs et il suffit de calculer la constante a_{m+1}. La somme des carrés des erreurs sera diminuée de $a_{m+1}^2 . s_{m+1}$.

Par suite, en calculant ces termes au cours des calculs on se rend toujours compte de l'approximation déjà obtenue, cela permet de juger, si la nécessité de continuer s'impose.

Les calculs sont si simples et peuvent être exécutés si rapidement que même lorsque un développement suivant des *puissances* de x est nécessaire, il y a avantage à passer par les polynomes q_ν.

Les polynomes q_ν sont utiles non seulement aux statisticiens, mais encore aux physiciens, quand il s'agit d'interpréter par un polynome les résultats numériques des expériences.

10. Nous allons montrer un exemple d'interpolation appuyé sur les polynomes q_ν, en partant des données empreuntées à Bowley, *Elements of Statistics* (3-ième éd., p. 91).

TABLE DES SALAIRES JOURNALIERS DANS L'ANNÉE 1891 EN AMÉRIQUE.

Salaires.	Nombre d'ouvriers.	ξ	y^2
0.50	317	0	100489
1.00	1472	1	2166784
1.50	1297	2	1682209
2.00	970	3	940900
2.50	506	4	256036
3.00	198	5	39204
3.50	254	6	64516
4.00	96	7	9216
4.50	4	8	16
5.00	9	9	81
	$\Sigma y = 5123$		$\Sigma y^2 = 5259451$

Dans l'exemple ci-dessus $a = 50$ cents, $h = 50$ cents, $n = 10$. On en tire immédiatement

$$a_0 = \frac{\Sigma y}{n} = 512{\cdot}3.$$

Si l'on s'arrêtait à ce terme, la somme des carrés des erreurs serait :

$$\Sigma \delta_0^2 = \Sigma y_i^2 - a_0 \Sigma y_i = 2634938.$$

Pour déterminer la constante a_1 écrivons en utilisant la Table I pour $q_1(10, \xi)$ et la Table VI pour $s_1(10)$,

ξ	$y(\xi)-y(9-\xi)$	$q_1(\xi)$	$q_1(\xi)[y(\xi)-y(9-\xi)]$
0	308	$-4{\cdot}5$	-1386
1	1468	$-3{\cdot}5$	-5138
2	1201	$-2{\cdot}5$	$-3002{\cdot}5$
3	716	$-1{\cdot}5$	-1074
4	308	$-0{\cdot}5$	-154
			$\Sigma y q_1 = -10754{\cdot}5$

Il en résulte:

$$a_1 = \frac{\Sigma y q_1}{\Sigma q_1^2} = -130{\cdot}357 \quad \text{et} \quad \Sigma\delta_1^2 = \Sigma\delta_0^2 - a_1 \Sigma y q_1 = 1233015.$$

Pour déterminer a_2 et $\Sigma\delta_2^2$, on procède de la même manière:

ξ	$y(\xi)+y(9-\xi)$	$q_2(\xi)$	$q_2(\xi)[y(\xi)+y(9-\xi)]$
0	326	18	5868
1	1476	6	8856
2	1393	-3	-4179
3	1224	-9	-11016
4	704	-12	-8448
			$\Sigma y q_2 = -8919$

On en tire

$$a_2 = -7{\cdot}51, \qquad \Sigma\delta_2^2 = 1166981.$$

Détermination du coefficient a_3 et de la somme $\Sigma\delta_3^2$,

ξ	$y(\xi)-y(9-\xi)$	$q_3(\xi)$	$q_3(\xi)[y(\xi)-y(9-\xi)]$
0	308	-63	-19404
1	1468	21	30828
2	1201	$52{\cdot}4$	$63052{\cdot}5$
3	716	$46{\cdot}5$	33294
4	308	18	5544
			$\Sigma y q_3 = 113314{\cdot}5$

$$a_3 = 5{\cdot}87, \qquad \Sigma\delta_3^2 = 500879.$$

Détermination du coefficient a_4 et de la somme $\Sigma\delta_4^2$,

	$y(\xi)+y(9-\xi)$	$q_4(\xi)$	$q_4(\xi)[y(\xi)+y(9-\xi)]$
0	326	189	61614
1	1476	-231	-340956
2	1393	$-178{\cdot}5$	$-248650{\cdot}5$
3	1224	$31{\cdot}5$	38556
4	704	189	133056

$$\Sigma yq_4 = -356380{\cdot}5$$

$$a_4 = -1{\cdot}13, \qquad \Sigma\delta_4^2 = 98527.$$

Détermination de a_5 et de $\Sigma\delta_5^2$,

ξ	$y(\xi)-y(9-\xi)$	$q_5(\xi)$	$q_5(\xi)[y(\xi)-y(9-\xi)]$
0	308	$-\ 472{\cdot}5$	$-\ 145530$
1	1468	$1102{\cdot}5$	1618470
2	1201	$-\ 78{\cdot}5$	$-\ 94578{\cdot}75$
3	716	$-\ 866{\cdot}25$	620235
4	308	$-\ 472{\cdot}5$	$-\ 145530$

$$\Sigma yq_5 = 612596{\cdot}25$$

par suite $\qquad a_5 = 0{\cdot}1268, \qquad \Sigma\delta_5^2 = 20850.$

Il en résulte que si nous nous arrêtons à a_5, l'erreur moyenne ϵ sera :

$$\epsilon = \left[\frac{\Sigma\delta^2}{n}\right]^{\frac{1}{2}} = 45{\cdot}7.$$

La fonction y cherchée est la suivante :

(*a*) $$y = 512{\cdot}3 + 130{\cdot}4q_1 - 7{\cdot}51q_2 + 5{\cdot}87q_3 - 1{\cdot}13q_4 + 0{\cdot}127q_5,$$

ou en substituant aux polynomes q leurs développements suivant les puissances de ξ, on trouve :

(*b*) $$y = 320{\cdot}54 + 2144{\cdot}76\xi - 1271{\cdot}25\xi^2 + 280{\cdot}35\xi^3 - 27{\cdot}45\xi^4 + \xi^5.$$

Comme vérification, déterminons à l'aide de la formule (*a*) et les Tables I–V les écarts δ correspondant aux valeurs de $\xi = 0, 1, \ldots, 9$; on a

très rapidement:

ξ	δ
0	3·54
1	−24·05
2	63·71
3	−67·15
4	− 6·29
5	79·99
6	−68·27
7	16·35
8	4·59
9	− 2·44

La somme de ces erreurs devrait être égale à zéro, et la somme de leur carré à 20810 ; effectivement nous avons $\Sigma\delta = -0{\cdot}02$ et $\Sigma\delta^2 = 20565$, ce qui prouve que les calculs ont été exécutés avec une précision suffisante.

II Partie.

Propriétés mathématiques des polynomes $Q_\nu(x)$.

11. Examinons la limite des polynomes $Q_\nu(x)$ lorsque l'intervalle h tend vers zéro. Comme

$$\lim_{h=0}(x-a)_m = (x-a)^m \quad \text{et} \quad \lim \frac{\Delta^m}{h^m} F = \frac{d^m}{dx^m} F,$$

on conclut:
$$\lim_{h=0} Q_m(x) = \frac{1}{2^m m!}\frac{d^m}{dx^m}(x-a)^m (x-b)^m.$$

Posons $a=-1$, $b=1$, nous trouverons $\lim Q_m = P_m$; où P_m est le m-ième polynome de Legendre.

En égalant dans nos formules (9) et (13) h à zéro, nous obtenons des expressions donnant les polynomes de Legendre.

$$(18)\quad \left\{\begin{aligned} P_m &= \sum_{s=0}^{m+1}\binom{m}{s}^2\left(\frac{x+1}{2}\right)^s\left(\frac{x-1}{2}\right)^{m-s},\\ P_m &= \sum_{s=0}^{m+1}\binom{m}{s}\binom{m+s}{s}\left(\frac{x-1}{2}\right)^s,\\ P_m &= \sum_{s=0}^{m+1}(-1)^{m-s}\binom{m}{s}\binom{m+s}{s}\left(\frac{x+1}{2}\right)^s.\end{aligned}\right.$$

Si nous remplaçons x dans les formules (18) par $\cos \vartheta$ elles deviennent identiques aux développements des polynomes de Legendre suivant les puissances de $\tan^2 \frac{1}{2}\vartheta$, de $\sin^2 \frac{1}{2}\vartheta$ et de $\cos^2 \frac{1}{2}\vartheta$ donnés par Dirichlet.*

Remarquons que l'on a aussi d'après la formule (11),

$$\lim_{h=0} \Sigma (Q_m)^2 h = \int_{-1}^{1} (P_m)^2 \, dx = \frac{2}{2m+1}.$$

12. Nous allons déduire maintenant une équation aux différences finies, dont la solution est le polynome Q_m. Désignons $(x-a)_m (x-b)_m$ par $u_m(x)$ et déterminons la première différence de $u_{m+1}(x)$:

$$\Delta u_{m+1}(x) = (m+1)\, h \left[2x - mh + h - a - b\right] u_m(x).$$

Écrivons la différence de m-ième ordre de ce produit en utilisant la formule suivante analogue à celle de Leibnitz

$$\Delta^n \left[U(x)\, V(x)\right] = \sum_{s=0}^{n+1} \binom{n}{s} \Delta^s V(x+nh-sh)\, \Delta^{n-s} U(x), \tag{18a}$$

d'après cette formule on a :

$$\Delta^{m+1} u_{m+1} = (m+1)\, h \left[(2x+mh+h-a-b)\, \Delta^m u_m + 2mh\, \Delta^{m-1} u_m\right]. \tag{19}$$

On peut encore exprimer cette quantité autrement, en considérant u_{m+1} comme le produit des deux facteurs $u_m(x)$ et $u_1(x-mh)$. On trouve alors à l'aide de (18a),

$$\begin{aligned} \Delta^{m+1} u_{m+1} = u_1(x+h)\, \Delta^{m+1} u_m + (m+1)\, \Delta u_1(x)\, \Delta^m u_m(x) \\ + \binom{m+1}{2} \Delta^2 u_1(x-h)\, \Delta^{m-1} u_m(x). \end{aligned} \tag{20}$$

En remarquant que $\Delta u_1(x) = (2x+h-a-b)h$ et que $\Delta^2 u_1(x) = 2h^2$ on tire de (19) et de (20),

$$u_1(x+h)\, \Delta^{m+1} u_m(x) - m(m+1) h^2\, \Delta^m u_m(x) - m(m+1) h^2 \Delta^{m-1} u_m(x) = 0.$$

La première différence de cette expression donne, si l'on remplace $\Delta^m u_m(x)$ par $1/C . Q_m$, *l'équation aux différences cherchée* :

$$\begin{aligned} (x-a+2h)(x-b+2h)\, \Delta^2 Q_m + \left[2x+3h-m(m+1)-a-b\right] h . \Delta Q_m \\ - m\,(m+1)\, h^2 . Q_m = 0. \end{aligned} \tag{21}$$

---

* *Crelle Journal*, Bd. 17, pp. 39, 40.

Si nous posons $a=-1$, $b=1$ et si nous faisons tendre h vers zéro. l'équation (21) est transformée en une équation différentielle du second ordre admettant comme solution le polynome de Legendre de degré m.

En introduisant la variable ξ au lieu de x d'après (16) on a :

$$(22)\quad (\xi+2)(\xi-n+2)\Delta^2 q_m+[2\xi-n+3-m(m+1)]\,\Delta q_m-m(m+1)q_m=0.$$

La solution de cette équation est donnée par la méthode de Boole (*Treatise on Finite Differences*, 1860, p. 176),

$$q_m(\xi)=\sum_{\nu=0}^{m+1} b_\nu(\xi+\nu)_\nu,$$

$$b_\nu=(-1)^\nu\binom{m}{\nu}\binom{m+\nu}{\nu}\frac{b_0}{(n+\nu)_\nu},$$

où b_0 est une constante arbitraire. De la première de ces deux relations il résulte que $b_0=q_m(-1)$. En remplaçant ξ par -1 dans notre formule (9′) nous obtenons :

$$(23)\quad \begin{cases} b_0=(-\tfrac{1}{2})^m(m+n)_m \\ \text{et}\ \ q_m=\displaystyle\sum_{\nu=0}^{m+1}(-1)^{m-\nu}(\tfrac{1}{2})^m\binom{m}{\nu}\binom{m+\nu}{\nu}(m+n)_{m-\nu}(\xi+\nu)_\nu. \end{cases}$$

C'est une formule semblable à celle de (13′); elle est aussi très commode pour le calcul des polynomes $q_m(\xi)$. En y remplaçant n par $2/h$, la variable ξ par $(x+1)/h$, et $q_m(\xi)$ par $Q_m(x)h^m$ et en faisant tendre h vers zéro, la formule (23) coïncide à la limite avec la troisième formule (18) donnant les polynomes de Legendre.

13. Pour établir *l'équation fonctionnelle* qui relie les polynomes Q de divers degrés, il suffit de développer $x.Q_m$ en séries de polynomes Q_m ; d'après ce que nous avons vu, ce développement ne contient que les trois termes suivants :

$$(24)\qquad xQ_m=A_{m-1}Q_{m-1}+A_mQ_m+A_{m+1}Q_{m+1}.$$

Les autres termes étant nuls conformément à l'équation (4′) ; et l'on a :

$$A_{m-1}=\frac{1}{S_{m-1}}\sum_{x=a}^{b}x.Q_mQ_{m-1}h,$$

$$A_m=\frac{1}{S_m}\sum_{x=a}^{b}x(Q_m)^2h,$$

$$A_{m+1}=\frac{1}{S_{m+1}}\sum_{x=a}^{b}x.Q_mQ_{m+1}h.$$

La détermination des coefficients A_{m-1} et A_{m+1} ne présente pas de difficultés, en répétant sur (24) la sommation par parties, on est conduit à la grandeur $\Sigma(x+mh-a)_m(x+mh-b)_m$ que l'on peut évaluer à l'aide d'une formule analogue à celle donnée par Cauchy pour le développement de $(x+y)_n$ en factorielles de x et y. On trouve enfin :

$$A_{m+1} = \frac{m+1}{2m+1}, \qquad A_{m-1} = \frac{m}{2m+1} \, \frac{(b-a)^2-m^2h^2}{4}.$$

La détermination du troisième coefficient, par la même méthode, conduirait à des difficultés. Elle nécessiterait l'évaluation de

$$\Sigma\,(x+hm+h-a)_m\,(x+hm+h-b)_m$$

ce qui est difficile, par contre on arrive directement au résultat en remarquant que l'équation (24) doit avoir lieu pour toutes les valeurs de x donc aussi pour $x = b$, mais de (9) il résulte que

$$Q_m(b) = (\tfrac{1}{2})^m\,(b-a+mh)_m.$$

En remplaçant dans (24) A_{m-1}, A_{m+1}, $Q_m(b)$, $Q_{m-1}(b)$, et $Q_{m+1}(b)$ par les valeurs correspondantes, on peut déterminer la seule inconnue A_m,

$$A_m = \tfrac{1}{2}\,(a+b-h).$$

Finalement on a

(25)

$$4(m+1)Q_{m+1}-2(2m+1)(2x-a-b+h)Q_m+m[(b-a)^2-m^2h^2]Q_{m-1} = 0.$$

En posant $a = -1$ et $b = 1$ nous obtenons l'équation de Tchebichef mentionnée au no. 1 ; si en outre nous posons $h = 0$, l'équation (25) coïncide avec l'équation bien connue vérifiée par les polynomes de Legendre,

$$(m+1)\,P_{m+1}-(2m+1)\,x.P_m+m.P_{m-1} = 0.$$

14. La méthode des fonctions génératrices de Laplace appliquée à (25) conduit à une équation différentielle dont la solution est la fonction génératrice des polynomes Q_m.

En désignant par $G[\psi(m)]$ la fonction génératrice de $\psi(m)$, on a :

$$G[\psi(m)] = \sum_{m=0}^{\infty} \psi(m)\,t^m.$$

Posons $$G(Q_m) = \phi \quad \text{et} \quad \frac{d\phi}{dt} = \phi',$$

nous aurons :

$$G[(m+2)Q_{m+2}] = \frac{1}{t}(\phi'-Q_1), \quad G[Q_{m+1}] = \frac{1}{t}(\phi-Q_0),$$

$$G[(m+1)Q_{m+1}] = \phi', \qquad G[mQ_m] = t\phi',$$

$$G[m^2Q_m] = t^2\phi''+t\phi', \qquad G[m^3Q_m] = t^3\phi'''+3t^2\phi''+t.$$

En écrivant dans (25) $m+1$ au lieu de m et x_1 au lieu de $x-\frac{1}{2}(a+b-h)$, on obtient à l'aide des quantités précédentes la fonction génératrice du premier membre de l'équation (25) ; en égalant cette fonction génératrice à zéro, nous obtenons une équation différentielle linéaire du troisième ordre dont la solution est la fonction génératrice cherchée :

$$(26) \quad \phi'''t^4h^2+6\phi''t^3h^2+\phi'[(7h^2-b^2+2ab-a^2)t^2+8x_1t-4] \\ +\phi[(h^2-b^2+2ab-a^2)t+4x_1] = Q_1-x_1Q_0.$$

Remarquons que dans le cas particulier des polynomes Q_ν, cette équation se simplifie, car $Q_0 = 1$ et $Q_1 = x_1Q_0$, donc le second membre de l'équation (26) est nul.

Si nous posons dans (26) $h=0$, $a=-1$, $b=1$, l'équation se transforme en une équation différentielle admettant comme solution la fonction génératrice des polynomes de Legendre,

$$\phi'(t^2-2x_1t+1)+\phi(t-x_1) = 0.$$

On en tire : $$\phi = (t^2-2x_1t+1)^{-\frac{1}{2}}.$$

Dans notre cas, il est possible de donner une forme plus simple à l'équation (26) en posant

$$\phi = t^{-2}\psi$$

et $$\frac{1}{h^2t^4}\{[h^2-(b-a)^2]t^2+8x_1t-4\} = R.$$

Il vient

$$(27) \qquad \psi'''+\psi'R+\tfrac{1}{2}\psi\frac{dR}{dt} = 0.$$

La résolution de cette équation donnerait la fonction génératrice des polynomes Q_m.

NOTES.

Ce mémoire ayant été communiqué à Mr. L. Fejér, il a démontré les propositions suivantes :

1. Étant donnés n points dont les abscisses sont par ordre de grandeur $x_0, x_1, x_2, \ldots, x_{n-1}$, parmi tous les polynomes $g_m(x)$ de degré m, dans lesquels le coefficient du terme en x^m est l'unité, *le polynome $G_m(x)$ qui rend minimum l'expression*

$$\sum_{i=0}^{n} [g_m(x_i)]^2 \quad (m < n) \tag{28}$$

est proportionnel au polynome ψ_m de Tchebichef, mentionné au no. 1.

Si les différences $x_i - x_{i-1}$ sont constantes, $G_m(x)$ est proportionnel à notre polynome $Q_m(x)$.

Cette proposition présente une analogie avec la suivante : Parmi les polynomes considérés précédemment, celui qui rend minimum l'intégrale

$$\int_{-1}^{1} [g_m(x)]^2 \, dx$$

est proportionnel aux polynome P_m de Legendre.*

2. Les racines de l'équation $G_m(x) = 0$ et par suite aussi des équations $\psi_m = 0$ et $Q_m = 0$ sont toutes réelles et comprises dans l'intervalle (x_0, x_{n-1}).

3. L'équation $G_m(x) = 0$ n'a pas de racines multiples et dans tout intervalle (x_i, x_{i+1}) il y a au plus une racine de cette équation.

Pour prouver ces propositions, nous allons poser avec Mr. Fejér :

$$g_m(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_{m-1} x^{m-1} + x^m.$$

Proposition I.—En vue de rendre minimum l'expression (28) égalons à zéro les dérivées de cette dernière par rapport aux coefficients c_ν,

$$\sum_{i=0}^{n} x_i^\nu \, g_m(x_i) = 0 \quad (0 \leqslant \nu < m). \tag{29}$$

Ces conditions sont identiques à nos conditions (4) qui déterminent les polynomes ψ_m et Q_m, on peut donc définir ces dernières comme rendant minimum l'expression (28).

Proposition II.—Cette proposition est la conséquence immédiate d'un théorème plus général dû à Mr. Fejér.

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* Voir Runge, *Praxis der Reihen*, p. 112.

Étant donnés n points $z_0, z_1, \ldots, z_{n-1}$ dans le plan de la variable complexe z, soit $G_m(z)$ le polynome de degré m ($m < n$), dans lequel le coefficient de z^m est égal à l'unité, et qui rende minimum l'expression suivante :

$$\sum_{i=0}^{n} |g_m(z_i)|^2. \tag{30}$$

Théorème : Si $a, b, c, \ldots$, sont les racines de $G_m(z) = 0$, aucune de ces racines ne peut représenter un point extérieur au plus petit polygone convex contenant les points $z_0, z_1, \ldots, z_{n-1}$.

Démonstration : Supposons que l'une de ces racines, p. ex. $z = a$ corresponde à un point extérieur au polygone mentionné. S'il est possible de déplacer le point a en a_1 de manière que toutes les distances $|z_i - a|$ diminuent, ($i = 0, 1, \ldots, n-1$), c.-à-d. que

$$|z_i - a_1| < |z_i - a|$$

pour toutes les valeurs de i, nous obtiendrons un polynome

$$H_m(z) = (z - a_1)(z - b)(z - c) \ldots$$

tel que

$$|H_m(z_i)| < |G(z_i)|$$

pour toutes les valeurs de i pour lesquelles $G(z_i) \neq 0$; aux autres valeurs $H_m(z_i) = G(z_i)$, par conséquent on aurait

$$\Sigma |H_m(z_i)|^2 < \Sigma |G_m(z_i)|^2$$

ce qui serait contraire à la supposition que $G_m(z)$ rend minimum l'expression (30).

Il reste encore à montrer que, le point a étant un point extérieur au polygone mentionné ci-dessus, il est effectivement possible de déplacer ce point de manière à diminuer toutes les distances $|z_i - a|$. En effet, si le point a est un point extérieur, il est toujours possible de mener une droite D de manière que le polygone soit situé d'un côté de la droite, et le point a de l'autre. Menons par le point a une perpendiculaire à D, si le point a se déplace sur cette perpendiculaire vers la droite D, on peut voir aisément que toutes les distances $|z_i - a|$ diminuent ; on en conclut qu'aucune des racines de $G_m(x) = 0$ ne peut être située en dehors du polygone considéré ; la démonstration du théorème est donc complète.

Dans le cas particulier où les n points donnés sont tous situés sur l'axe réel, le polygone est réduit au segment de droite (x_0, x_{n-1}), par suite d'après le théorème que l'on vient de démontrer, les racines de $G_m(x)$ sont toutes situées entre ces deux points, donc elles sont toutes réelles.

Proposition III.—Nous allons démontrer qu'entre deux points consécutifs quelconques x_i et x_{i+1} le polynome $Y = G_m(x)$ ne peut changer de signe plus d'une fois.

D'abord dans la suite $Y_0, Y_1, \ldots, Y_{n-1}$ correspondant à $x_0, x_1, \ldots, x_{n-1}$ la quantité Y_i change de signe m fois.

En effet $G_m(x)$ étant de degré m, il est évident que la série ci-dessus ne peut présenter plus de m changements de signe. Supposons, qu'il y ait moins, p. ex. μ ($\mu < m$), dans ce cas il serait possible de mener une courbe de degré μ soit $y = f_\mu(x)$ telle que pour toutes les valeurs de i les grandeurs $y_i = f_\mu(x_i)$ et Y_i aient le même signe, lorsque y_i et Y_i sont différentes de zéro.

Supprimons les points x_i correspondant aux valeurs nulles de Y_i; supposons qu'en suite les changements de signes de Y_i ont lieu entre les points x_k et x_p, entre x_q et x_r, etc., alors la courbe suivante de degré μ,

$$y = f_\mu(x) = (-1)^\mu\, Y_k(x - \tfrac{1}{2}x_k - \tfrac{1}{2}x_p)(x - \tfrac{1}{2}x_q - \tfrac{1}{2}x_r) \ldots \tag{31}$$

change de signe au milieu des mêmes intervalles que Y, et l'on voit facilement que le signe de $f_\mu(x_i)$ et Y_i est le même pour toutes les valeurs de i pour lesquelles y_i et Y_i sont différentes de zéro. Or il y a de telles valeurs, car d'après notre supposition Y_k est différente de zéro et à cause de (31) y_k l'est aussi. On en conclut que

$$\sum_{i=0}^{n} f_\mu(x_i)\, G_m(x_i) > 0,$$

ce qui contredit notre condition (29). Le polynome $G_m(x)$ ne rendrait pas l'expression (30) minimum, donc en supposant qu'il y a moins de m changements de signe dans la suite $Y_0, Y_1, \ldots, Y_{n-1}$ on arrive à une contradiction.

Ainsi nous avons démontré que le polynome $G_m(x)$ change de signe m fois entre x_0 et x_{n-1}, comme il est de degré m toutes ces racines sont simples. La première partie du théorème est donc démontrée. Pour montrer qu'entre deux racines consécutives de $G_m(x) = 0$ se trouve placé au moins un des points x_i, il suffit de remarquer que dans le cas contraire la suite $Y_0, Y_1, \ldots, Y_{n-1}$ présenterait nécessairement moins de m changements de signe, ainsi dans l'intervalle $x_i \leqslant x \leqslant x_{i+1}$ il y a au plus une racine de $G_m(x) = 0$.

En résumé : étant donnés n points, dont les abscisses sont par ordre de grandeur : $x_0, x_1, \ldots, x_{n-1}$, parmi tous les polynomes $g_m(x)$ de degré m, dans lesquels le coefficient de x^m est l'unité, soit $G_m(x)$ le polynome qui

rend minimum l'expression :

$$\sum_{i=0}^{n} [g_m(x_i)]^2 \quad (m < n).$$

Le polynome $G_m(x)$ est proportionnel au polynome ψ_m de Tchebichef, de plus toutes les racines de ce polynome sont réelles et comprises entre x_0 et x_{n-1} ; en outre, cette équation n'a pas de racine multiple et parmi les $n-1$ intervalles il y a m intervalles (x_i, x_{i+1}) renfermant une racine de $G_m(x) = 0$.

Dans le cas particulier où $m = n-1$, chacun des intervalles mentionnés contient une racine.

I. TABLE DES VALEURS DE $q_1(n, \xi)$.

n, ξ	0	1	2	3	4	5	6	7	8	9	10	11
2	−0·5	0·5										
3	−1	0	1									
4	−1·5	−0·5	0·5	1·5								
5	−2	−1	0	1	2							
6	−2·5	−1·5	−0·5	0·5	1·5	2·5						
7	−3	−2	−1	0	1	2	3					
8	−3·5	−2·5	−1·5	−0·5	0·5	1·5	2·5	3·5				
9	−4	−3	−2	−1	0	1	2	3	4			
10	−4·5	−3·5	−2·5	−1·5	−0·5	0·5	1·5	2·5	3·5	4·5		
11	−5	−4	−3	−2	−1	0	1	2	3	4	5	
12	−5·5	−4·5	−3·5	−2·5	−1·5	−0·5	0·5	1·5	2·5	3·5	4·5	5·5
13	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5
14	−6·5	−5·5	−4·5	−3·5	−2·5	−1·5	−0·5	0·5	1·5	2·5	3·5	4·5
15	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4
16	−7·5	−6·5	−5·5	−4·5	−3·5	−2·5	−1·5	−0·5	0·5	1·5	2·5	3·5
17	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3
18	−8·5	−7·5	−6·5	−5·5	−4·5	−3·5	−2·5	−1·5	−0·5	0·5	1·5	2·5
19	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2
20	−9·5	−8·5	−7·5	−6·5	−5·5	−4·5	−3·5	−2·5	−1·5	−0·5	0·5	1·5

Remarques: $q_1(n, n-1-\xi) = -q_1(n, \xi)$.

La table a été calculée à l'aide de la formule :

$$q_1 = \xi - \tfrac{1}{2}(n-1).$$

II. TABLE DES VALEURS DE $q_2(n, \xi)$.

n, ξ	0	1	2	3	4	5	6	7	8	9	10	11
3	0·5	−1	0·5									
4	1·5	**−1·5**	**−1·5**	1·5								
5	3	−1·5	**−3**	−1·5	3							
6	5	−1	**−4**	**−4**	−1	5						
7	7·5	0	−4·5	**−6**	−4·5	0	7·5					
8	10·5	1·5	−4·5	**−7·5**	**−7·5**	−4·5	1·5	10·5				
9	14	3·5	−4	−8·5	**−10**	−8·5	−4	3·5	14			
10	18	6	−3	−9	**−12**	**−12**	−9	−3	6	18		
11	22·5	9	−1·5	−9	−13·5	**−15**	−13·5	−9	−1·5	9	22·5	
12	27·5	12·5	0·5	−8·5	−14·5	**−17·5**	**−17·5**	−14·5	−8·5	0·5	12·5	27·5
13	33	16·5	3	−7·5	−15	−19·5	**−21**	−19·5	−15	−7·5	3	16·5
14	39	21	6	−6	−15	−21	**−24**	**−24**	−21	−15	−6	6
15	45·5	26	9·5	−4	−14·5	−22	−26·5	**−28**	−26·5	−22	−14·5	−4
16	52·5	31·5	13·5	−1·5	−13·5	−22·5	−28·5	**−31·5**	**−31·5**	−28·5	−22·5	−13·5
17	60	37·5	18	1·5	−12	−22·5	−30	−34·5	**−36**	−34·5	−30	−22·5
18	68	44	23	5	−10	−22	−31	−37	**−40**	**−40**	−37	−31
19	76·5	51	28·5	9	−7·5	−21	−31·5	−39	−43·5	**−45**	−43·5	−39
20	85·5	58·5	34·5	13·5	−4·5	−19·5	−31·5	−40·5	−46·5	**−49·5**	**−49·5**	−46·5

Remarque : $q_2(n, n-1-\xi) = q_2(n, \xi)$.

La table a été calculée à l'aide de la formule :

$$q_2 = \tfrac{1}{4}(2-n)_2 + \tfrac{3}{2}(2-n)\,\xi + \tfrac{3}{2}(\xi)_2.$$

III. TABLE DES VALEURS DE $q_3(n, \xi)$.

n, ξ	0	1	2	3	4	5	6	7	8	9	10
4	− 0·75	**2·25**	**− 2·25**	0·75							
5	− 3	6	**− 0**	− 6	3						
6	− 7·5	10·5	**6**	**− 6**	−10·5	7·5					
7	− 15	15	15	**0**	−15	−15	15				
8	− 26·25	18·75	26·25	**11·25**	**−11·25**	−26·25	−18·75	26·25			
9	− 42	21	39	27	**0**	−27	−39	−21	42		
10	− 63	21	52·5	46·5	**18**	**−18**	−46·5	−52·5	− 21	63	
11	− 90	18	66	69	42	**0**	−42	−69	− 66	− 18	90
12	−123·75	11·25	78·75	93·75	71·75	**26·25**	**−26·25**	−71·25	− 93·75	− 78·75	− 11·25
13	−165	0	90	120	105	60	**0**	−60	−105	−120	− 90
14	−214·5	− 16·5	99	147	142·5	100·5	**36**	**−36**	−100·5	−142·5	−147
15	−273	− 39	105	174	183	147	81	**0**	− 81	−147	−183
16	−341·25	− 68·25	107·25	200·25	225·75	198·75	134·25	**47·25**	**− 47·25**	−134·25	−198·75
17	−420	−105	105	225	270	255	195	105	**0**	−105	−195
18	−510	−150	97·5	247·5	315	315	262·5	172·5	**60**	**− 60**	−172·5
19	−612	−204	84	267	360	378	336	249	132	**0**	−132
20	−726·75	−267·75	63·75	282·75	404·25	443·25	414·75	333·75	215·25	**74·25**	**− 74·25**

Remarque : $q_3(n, n-1-\xi) = -q(n, \xi)$.

La table a été calculée par la formule :

$$q_3 = \tfrac{1}{8}(3-n)_3 + \tfrac{3}{2}(3-n)_3\,\xi + \tfrac{15}{4}(3-n)(\xi) + \tfrac{5}{2}(\xi)_3.$$

IV. TABLE DES VALEURS DE $q_4(n, \xi)$.

n, ξ	0	1	2	3	4	5	6	7	8	9	10
5	1·5	− 6	**9**	− 6	1·5						
6	7·5	− 22·5	**15**	**15**	− 22·5	7·5					
7	22·5	− 52·5	7·5	**45**	7·5	− 52·5	22·5				
8	52·5	− 97·5	− 22·5	**67·5**	**67·5**	− 22·5	− 97·5	52·5			
9	105	−157·5	− 82·5	67·5	**185**	675	− 82·5	− 157·5	105		
10	189	−231	− 178·5	31·5	**189**	**189**	31·5	− 178·5	− 231	189	
11	315	−315	− 315	− 52·5	210	**315**	210	− 52·5	− 315	− 315	315
12	495	−405	− 495	− 195	180	**420**	**420**	180	− 195	− 495	− 405
13	742·5	−495	− 720	− 405	82·5	480	**630**	480	82·5	− 405	− 720
14	1072·5	−577·5	− 990	− 690	− 97·5	472·5	**810**	**810**	472·5	− 97·5	− 690
15	1501·5	−643·5	−1303·5	−1056	−373·5	376·5	931·5	**1184**	931·5	376·5	− 373·5
16	2047·5	−682·5	−1657·5	−1507·5	−757·5	172·5	967·5	**1417·5**	**1417·5**	967·5	172·5
17	2730	−682·5	−2047·5	−2047·5	−1260	−157·5	892·5	1627·5	**1890**	1627·5	892·5
18	3570	−630	−2467·5	−2677·5	−1890	−630	682·5	1732·5	**2810**	**2810**	1732·5
19	4590	− 510	−2910	−3397·5	−2655	−1260	315	1702·5	2640	**2970**	2640
20	5814	−306	−3366	−4206	−3561	−2061	−231	1509	2844	**3564**	**3564**

Remarque : $q(n, \xi) = q(n, n-1-\xi)$.

La table a été calculée à l'aide de la formule :

$$q_4(\xi) = \tfrac{1}{16}\left[(4-n)_4 + 20\,(4-n)_3\,\xi + 90\,(4-n)_2\,(\xi)_2 + 140\,(4-n)(\xi)_3 + 70\,(\xi)_4\right].$$

V. TABLE DES VALEURS DE $q_5(n, \xi)$.

n, ξ	0	1	2	3	4	5	6	7	8	9	10
6	− 3·75	18·75	− **37·5**	**37·5**	− 18·75	3·75					
7	− 22·5	90	−112·5	**0**	112·5	− 90	22·5				
8	− 78·5	258·75	−191·25	−**168·75**	**168·75**	191·25	− 258·75	78·75			
9	− 210	577·5	−210	−472·5	**0**	472·5	210	− 577·5	210		
10	− 472·5	1102·5	− 78·75	−866·25	− **472·5**	**472·5**	866·25	78·75	−1102·5	472·5	
11	− 945	1890	315	−1260	−1260	**0**	1260	1260	− 315	−1890	945
12	−1732·5	2992·5	1102·5	−1522·5	−2310	− **1050**	**1050**	2310	1522·5	−1102·5	−2992·5
13	−2970	4455	2430	−1485	−3510	− 2700	**0**	2700	3510	1485	−2430
14	−4826·25	6311·25	4455	− 945	−4691·25	− 4893·25	− **2025**	**2025**	4893·25	4691·25	945
15	−7507·5	8580	7342·5	330	−5632·5	− 7500	− 5062·5	**0**	5062·5	7500	5632·5
16	−11261·25	11261·25	11261·25	2598·75	−6063·75	−10316·25	− 9056·25	− **8543·75**	**8543·75**	9056·25	10316·25
17	−16380	14332·5	16380	6142·5	−5670	−13072·5	−13860	− 8662·5	**0**	8662·5	13860
18	−23205	17745	22863·75	11261·25	−4095	−15435	−19241·25	−15303·75	−**5775**	**5775**	15303·75
19	−32130	21420	30870	18270	− 945	−17010	−24885	−23310	−13860	**0**	13860
20	−43605	25245	40545	27495	4207·5	−17347·5	−30397·5	−32422·5	−24210	−**8910**	**8910**

Remarques : On a $q_5(n, \xi) = -q_5(n, n-1-\xi)$.

Les valeurs ci-dessus ont été calculées à l'aide de la formule

$$q_5(n, \xi) = \tfrac{1}{32}(5-n)_5 + \tfrac{15}{16}(5-n)_4\,\xi + \tfrac{105}{16}(5-n)_3\,(\xi)_2 + \tfrac{35}{2}(5-n)_2\,(\xi)_3 + \tfrac{315}{16}(5-n)(\xi)_4 + \tfrac{63}{8}(\xi)_5$$

VI. TABLE DES VALEURS DE $\sum_{\xi=0}^{n}[q_m(n,\xi)]^2$.

n, m	1	2	3	4	5
2	0·5				
3	2	1·5			
4	5	9	11·25		
5	10	31·5	90	157·5	
6	17·5	84	405	1575	3543·75
7	28	189	1350	8662·5	42525
8	42	378	3712·5	34650	276412·5
9	60	693	8910	112612·5	1289925
10	82·5	1188	19305	315315	4837218·75
11	110	1930·5	38610	788287·5	15479100
12	143	3003	72393·75	1801800	43857450
13	182	4504·5	128700	3828825	112736300
14	227·5	6552	218790	7657650	267843637·5
15	280	9282	358020	14549535	595208250
16	340	12852	566865	26453700	1249937325
17	408	17442	872100	46293975	2499874650
18	484·5	23256	1308150	78343650	4791426412·5
19	570	30523·5	1918620	128707425	8845710300
20	665	39501	2758016·25	205931880	15795911250

Remarque : La table a été calculée à l'aide de la formule suivante :

$$\Sigma(q_m)^2 = \frac{n}{(2m+1)\,2^{2m}} \prod_{s=1}^{m+1}(n^2-s^2).$$

AN EXTENSION OF TWO THEOREMS ON JACOBIANS

By C. W. GILHAM.

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1. Two well known theorems on the reduction of Jacobians are here extended to forms involving any number of variables.

The theorems are:—

I. The Jacobian of a Jacobian is reducible.

II. The product of two Jacobians can be expressed as the sum of three-term products.

The extensions depend on the reduction of a certain covariant of weight 2.

2. *Reduction of* $(a_1 a_2 a_3 \ldots a_q)(a_1 b_2 b_3 \ldots b_q)$.

Let $$f_r \equiv a_{r x}^{p_r} \equiv (a_{r1}x_1 + a_{r2}x_2 + a_{r3}x_3 + \ldots + a_{rq}x_q)^{p_r} \quad [r = 1, 2, \ldots, q]$$

and $$\phi_r \equiv b_{r x}^{p'_r} \equiv (b_{r1}x_1 + b_{r2}x_2 + b_{r3}x_3 + \ldots + b_{rq}x_q)^{p'_r} \quad [r = 1, 2, \ldots, q],$$

be $2q$ quantics in the q variables $x_1 x_2 \ldots x_q$.

The fundamental identity for q-ary quantics is

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} & \ldots & a_{q1} & b_{21} \\ a_{12} & a_{22} & a_{32} & \ldots & a_{q2} & b_{22} \\ a_{13} & a_{23} & a_{33} & \ldots & a_{q3} & b_{23} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{1q} & a_{2q} & a_{3q} & \ldots & a_{qq} & b_{2q} \\ a_{1y} & a_{2y} & a_{3y} & \ldots & a_{qy} & b_{2y} \end{vmatrix} = 0,$$

i.e., $(a_1 a_2 \dots a_q)\, b_{2y} = (b_2 a_2 a_3 \dots a_q)\, a_{1y} + (a_1 b_2 a_3 \dots a_q)\, a_{2y}$

$$+(a_1 a_2 b_2 a_4 \dots a_q)\, a_{3y} + \dots + (a_1 a_2 \dots a_{q-1} b_2)\, a_{qy}.$$

Let $b_{2y} = (a_1 b_2 b_3 \dots b_q)$. Then

$a_{1y} = 0,\ a_{2y} = (a_1 a_2 b_3 \dots b_q),\ a_{3y} = (a_1 a_3 b_3 \dots b_q),\ \dots,\ a_{qy} = (a_1 a_q b_3 \dots b_q).$

Hence we get the following identity between covariants (as usual, only writing the bracket factors)

$$\begin{aligned}(a_1 a_2 a_3 \dots a_q)(a_1 b_2 b_3 \dots b_q) &= (a_1 b_2 a_3 \dots a_q)(a_1 a_2 b_3 \dots b_q) + (a_1 a_2 b_2 a_4 \dots a_q)(a_1 a_3 b_3 \dots b_q) + \dots \\ &\quad + (a_1 a_2 \dots a_{q-1} b_2)(a_1 a_q b_3 \dots b_q). \qquad \text{(I)}\end{aligned}$$

But $\quad (a_1 b_2 a_3 \dots a_q)(a_1 a_2 b_3 \dots b_q) \equiv (a_2 b_2 a_3 \dots a_q)(a_1 a_2 b_3 \dots b_q)$

$$\equiv (a_2 b_2 a_3 \dots a_q)(a_1 b_2 b_3 \dots b_q) \equiv -(a_1 a_2 a_3 \dots a_q)(a_1 b_2 b_3 \dots b_q)$$

reducible terms being omitted. Similarly,

$$(a_1 a_2 b_2 a_4 \dots a_q)(a_1 a_3 b_3 \dots b_q) \equiv -(a_1 a_2 a_3 \dots a_q)(a_1 b_2 b_3 \dots b_q), \text{ etc.}$$

Hence, substituting in (I), we have

$$(a_1 a_2 a_3 \dots a_q)(a_1 b_2 b_3 \dots b_q) \equiv 0,$$

i.e. it is reducible.

3. Let $\quad F = \dfrac{1}{p_1 p_2 \dots p_q} \dfrac{\partial (f_1 f_2 \dots f_q)}{\partial (x_1 x_2 \dots x_q)} = (a_1 a_2 \dots a_q).$

Then

$$\begin{aligned}\frac{1}{\lambda} \frac{\partial (F \phi_2 \phi_3 \dots \phi_q)}{\partial (x_1 x_2 \dots x_q)} &= (p_1 - 1)(a_1 a_2 \dots a_q)(a_1 b_2 b_3 \dots b_q) \\ &\quad + (p_2 - 1)(a_1 a_2 \dots a_q)(a_2 b_2 b_3 \dots b_q) + \dots \\ &\quad + (p_q - 1)(a_1 a_2 \dots a_q)(a_q b_2 b_3 \dots b_q), \qquad \text{(II)}\end{aligned}$$

where $\quad \lambda = p'_2 p'_3 \dots p'_q.$

But each term of (II) is reducible by § 2. Hence *the Jacobian of a Jacobian is reducible.*

4. Let $\quad F_1 = \dfrac{1}{p_1 p_2 \dots p_q} \dfrac{\partial (f_1 f_2 \dots f_q)}{\partial (x_1 x_2 \dots x_q)} = (a_1 a_2 \dots a_q)$

and $\quad F_2 = \dfrac{1}{p'_1 p'_2 \dots p'_q} \dfrac{\partial (\phi_1 \phi_2 \dots \phi_q)}{\partial (x_1 x_2 \dots x_q)} = (b_1 b_2 \dots b_q).$

Then $F_1F_2 = (a_1a_2 \dots a_q)(b_1b_2 \dots b_q)$

$$= (a_1a_2 \dots a_q)(a_1b_2 \dots b_q) - (a_1a_2 \dots a_q)(a_1b_1b_3 \dots b_q)$$
$$+ (a_1a_2 \dots a_q)(a_1b_1b_2b_4 \dots b_q) - \dots$$
$$+ (-1)^{q-1}(a_1a_2 \dots a_q)(a_1b_1b_2 \dots b_{q-1}).$$

By § 2, each of the terms of this expression can be expressed as the sum of terms, each of which only contains $(2q-2)$ of the symbols in its bracket factors; *i.e.* as the sum of terms each of which is the product of three covariants. Hence the product of two Jacobians can be expressed as the sum of three-term products.

It is assumed in the above work that each quantic is of order 2 at least.

5. In general, the evaluation of the reduced forms leads to somewhat complicated expressions. In particular cases, however, the results may be more compact, as, *e.g.* for the square of a Jacobian. Thus, if

$$f_1 = a_x^l = a_x'^l, \quad f_2 = b_x^m = b_x'^m, \quad f_3 = c_x^n = c_x'^n,$$

are three ternary quantics, I find, with the customary notation, the following result in terms of the fundamental covariants :—

If $$J = \frac{1}{lmn}\frac{\partial(f_1 f_2 f_3)}{\partial(x_1x_2x_3)} = (abc),$$

$$J^2 = (abc)^2 + (b_\alpha c_\alpha + c_\beta a_\beta + a_\gamma b_\gamma) - \tfrac{1}{2}(a_\beta^2 + a_\gamma^2 + b_\gamma^2 + b_\alpha^2 + c_\alpha^2 + c_\beta^2)$$
$$+ \tfrac{1}{4}\{(\beta\gamma x)^2 + (\gamma\alpha x)^2 + (\alpha\beta x)^2\}.$$

THE GROUP OF THE LINEAR CONTINUUM

By NORBERT WIENER.

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1. The linear continuum has already received a complete characterization in terms of order* and of limit.† Now, the author has shown that over a wide range of cases the notion of limit may be defined in terms of that of bicontinuous biunivocal transformation.‡ It is the purpose of this paper to develop a categorical theory of the structure of the line in terms of bicontinuous, biunivocal transformations, or, in other words, to give a complete postulational characterization of the analysis situs group of the line.

The set of postulates will be so framed that only one will have any direct effect on the dimensionality. All the other postulates together determine an analysis situs property of space which is shared by a large number of systems of a finite or infinite dimension number. A number of necessary conditions and a sufficient condition for a system to possess this property will be formulated.

INDEFINABLES.

2. Our indefinables are two in number—a set K of elements and a set Σ of one-one transformations of the whole of K into itself.

DEFINITIONS.

3. A sub-set E of K is said to have a *limit-element* A if A is invariant under every transformation belonging to Σ that leaves invariant every member of E except possibly A.

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* Cf. E. V. Huntington, "A Set of Postulates for Real Algebra," *Trans. Am. Math. Soc.* (1905); O. Veblen, "Definition in Terms of Order alone in the Linear Continuum," *ibid.*

† R. L. Moore, "The Linear Continuum in Terms of Point and Limit," *Annals of Mathematics* (1914–15).

‡ N. Wiener, "Limit in Terms of Continuous Transformation," *Bull. Soc. Math. de France* (1921–22).

A set E is *closed* if it contains all its limit-elements.

A set E is *connected* if, whenever it is divided into the two non-null sets, F and G, either F has a limit-element in G or G has a limit-element in F.

$\bar{E}$ is the set of all elements in K but not in E.

An *interior* element of E is one that is not a limit-element of $\bar{E}$.

An element A is *exterior* to E if it is interior to $\bar{E}$.

An element A is a *boundary-element** of E if it is at once a limit-element of E and of $\bar{E}$.

A *segment* is a closed, connected set with at least two boundary elements.

A *component*† of a set E is a greatest connected sub-set of E.

The transformation $\breve{R}$ is the inverse of R. $R|S$ is the transformation which consists in performing first S and then R.

POSTULATES.

4. I. K contains at least three distinct elements.

II. If R is a biunivocal transformation of the whole of K into itself that turns all closed sets into all closed sets and only into closed sets, then R belongs to Σ.

III. If R and S belong to Σ, so does $R|S$.

IV. If R belongs to Σ, so does $\breve{R}$.

V. If there is a transformation from Σ changing A and leaving every member of E invariant, while there is a transformation from Σ changing A and leaving every member of F invariant, then there is a transformation from Σ changing A and leaving every member of $E+F$ invariant.

VI. If A, B, C, and D belong to K, and $A \neq C$, $B \neq D$, then there is a transformation from Σ changing A to B and C to D.

VII. If E is any sub-set of K and A is an element of K not a limit-element of E, then there is a segment of which A is an interior element and which contains no element of E.

VIII. There is an at most denumerable sub-set K' of K such that no member of Σ except possibly the identity transformation leaves every member of K' invariant.

* Cf. F. Hausdorff, *Grundzüge der Mengenlehre*, p. 214. The notions of boundary element and *Randpunkt* are not identical.

† *Ibid.*, p. 245.

IX. If E and F are two connected sets, and two boundary elements of E are boundary elements of F, then every other element of E is an element of F.

Definitions of Systems.

5. A system satisfying postulates I–IX inclusive will be called a system (Li). A system satisfying postulates I–VII inclusive will be called a system (Sp).

A system (T_1) will be defined as in the author's previous paper in the *Bull. Soc. Math. de France*, as a system satisfying Postulates II–IV. A system (R) will be defined as by Fréchet,* as a system satisfying the conditions of F. Riesz.

1. Every limit-element of a set E is a limit-element of every set containing E.

2. Every limit-element of the sum of two sets E and F is a limit-element of at least one of the two sets.

3. A set containing a single element has no limit-element.

4. If A is a limit-element of a set E and B is distinct from A, there is always at least one set which has A for a limit-element without having B for a limit-element.

It has been proved by the author† that in the case of a (T_1), the necessary and sufficient condition that the system should also be an (R) is that it should satisfy the following three conditions:—

2′. This is verbally identical with V.

3′. Given any two elements, A and B, there is a transformation from Σ changing A but leaving B invariant.

4′. If there is a set E not containing the element A, but such that every transformation from Σ that leaves all the elements of E invariant leaves A also invariant, then, given any element B distinct from A, there is a set F not containing A such that there is no transformation from Σ changing A but leaving each member of F invariant, while there is a transformation from Σ changing B but leaving F invariant.

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* "Sur la notion de voisinage dans les ensembles abstraits," *Bulletin des Sciences Mathématiques*, May 1918.

† *Loc. cit.*

A system (H) is one in which neighbourhoods are so defined as to satisfy Hausdorff's "Umgebungsaxiome":*

(A) Given any point x, there is at least one neighbourhood U_x, of which x is a member.

(B) If U_x and V_x are two neighbourhoods of x, then there is a neighbourhood W_x contained in both.

(C) If y belongs to U_x, there is a neighbourhood of y, U_y, contained in U_x.

(D) If x and y are two points, then neighbourboods U_x and Y_x can be so chosen as not to overlap.

In a system (H) a set E is said to have a limit-point A if every neighbourhood U_A of A contains an infinity of points of E.†

A *vector-system*, or system (Ve), is defined as in my previous paper‡ as a system K of elements (represented by capitals), associated with entities called vectors (represented by Greek letters), real numbers (represented by lower case letters), and the operations $\odot$, $\oplus$, and $\| \;\|$ by the following laws:—

(1) If ξ and η are vectors, $\xi \oplus \eta$ is a vector.

(2) If ξ is a vector and $n \geqslant 0$, $n \odot \xi$ is a vector.

(3) If ξ is a vector, $\| \xi \|$ is a non-negative real number.

(4) $n \odot (\xi \oplus \eta) = (n \odot \xi) \oplus (n \odot \eta)$.

(5) $m \odot (n \odot \xi) = mn \odot \xi$.

(6) $(m \odot \xi) \oplus (n \odot \xi) = (m+n) \odot \xi$.

(7) $\| m \odot \xi \| = m \| \xi \|$.

(8) $\| \xi \oplus \eta \| \leqslant \| \xi \| + \| \eta \|$.

(9) If A and B belong to K, there is associated with them a unique vector AB.

(10) $\| AB \| = \| BA \|$.

(11) Given an element A of K and a vector ξ, there is an element B of K such that $AB = \xi$.

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* *Loc. cit.*, p. 213.

† *Ibid.*, p. 219, definition of *β-Punkt*.

‡ *Loc. cit.*

(12) $AC = AB \oplus BC$.

(13) $\| AB \| = 0$ when and only when $A = B$.

(14) If $AB = CD$, $DC = BA$.

A system (Vr), or a *restricted vector system* is a vector system of at least two elements in which the sum of two vectors is independent of their order, and in which, if A, B, and C are any three distinct elements such that $\| AB \| = \| AC \|$, then there is a finite set $B_1, B_2, \ldots, B_n$ of elements such that

(1) $B_1 = B$, $B_n = C$.

(2) For all k, $\| AB_k \| = \| AB \|$.

(3) For all k, $\| B_k B_{k+1} \| < \| AB \|$.

We shall say that a set E has A for a limit-element if, for all the B's that belong to E, the lower bound of $\| AB \|$ is zero.

Relations of Systems.

6. We shall say that a system of one of our classes belongs to another of our classes if a translation into the language of the second class is always possible in such a manner as to keep limit properties invariant. We have already seen that every (Sp) or (Li) is a (T_1), and every (Li) is clearly an (Sp); we shall prove the further relations:

(1) Every (Sp) is an (R).

(2) Every (Sp) is an (H).

(3) Every (Vr) is an (Sp).

Proof of (1).

All that we need to prove is contained in propositions 3′ and 4′ of § 5. If there are at least three elements, 3′ is a consequence of VI. Now, there are at least three elements, by I.

As to 4′ it is enough to show that, given any two elements A and B, there is a set E having A but not B as a limit-element. It follows from VII, I, and 3′, that there is at least one set F_1 which has limit-elements without having the whole of K for the class of its limit-elements.

Let A_1 be a limit-element of this set, and B_1 an element not a limit-element of the set. By VI, there is a transformation from Σ changing A_1 to A and B_1 to B. Let this transformation change F_1 to F. Then, as a result of III and IV, F will have A for a limit-element, but not B.

Proof of (2).

Let a neighbourhood U_A consist of all the interior elements of some set E of which A is an interior element. By I, 3', and VII at least one element has a neighbourhood, and by the use of VI, III, and IV, as above, every element will have at least one neighbourhood. Indeed, it may be shown by I, 3', and VII that there is at least one set with both interior and exterior elements, so that this same argument may be used to show that any two elements will have two mutually exclusive neighbourhoods, thus proving that Hausdorff's conditions (A) and (D) are satisfied. (C) is an obvious result of the definition of neighbourhood, for a neighbourhood is a neighbourhood of any of its elements. As to (B), the interior elements of a set E that are also interior to a set F are interior to the common part of E and F; this follows from condition 2 that our set be a set (R).

It remains to show that limit in a system (H) corresponds to limit in a system (Sp). It is a result of our definition of neighbourhood that if E is a set having A as a limit-element, every neighbourhood of A contains some element of E other than A. It results from Riesz's condition 2 that every neighbourhood of A contains a set of elements of E having A as a limit-element. From 2 and 3 together it follows that every such set is infinite. Hence every (Sp)-limit is an (H)-limit. The converse relation follows from VII.

Proof of (3).

Let Σ consist of all biunivocal, bicontinuous transformations in our system (Vr). That this will give the same notion of limit as that defined in a system (Sp) I have proved in my previous paper. Postulates I, II, III, IV, and V demand no discussion. VII will be obvious if we consider that a "sphere" with its boundary-elements will answer to our definition of a segment, for it is closed, has at least two boundary-elements, and is connected, for any point is connected with the centre by a radius. Moreover, the centre is an interior point. VII will then follow from our definition of limit.

There remains only condition VI. It is clear that any element A of K can be changed to any other member B of K by a transformation from Σ, for it will follow from II and the various properties of vectors that the transformation which turns C into the element D such that $CD = AB$ belongs to Σ. In a similar way, it may be shown that the transformation which consists in holding an element A fast and "multiplying" all the vectors AB by the same numerical factor also belongs to Σ. We shall

establish our theorem, then, if we show that if AB and AC are two vectors such that $\| AB \| = \| AC \|$, there is a transformation belonging to Σ holding A fixed and changing B into C, for every transformation of a point-pair into another may be reduced, as in ordinary geometry, into a "translation," an "expansion," and a "rotation." Our special hypothesis for a (Vr) enables us, moreover, without essentially limiting our problem, to suppose $\| BC \| < \| AB \|$.

Let us consider the vector transformation which turns ξ into

$$\xi \oplus \left\{ \frac{\| \xi \|}{\| AB \|} \odot BC \right\}.$$

This transformation is clearly univocal; it is, moreover, biunivocal. To prove this, let us make use of the fact that it results from our assumptions that if $\xi \oplus \eta = \vartheta$, η is uniquely determined by ϑ and ξ, and may be written $\vartheta \ominus \xi$. Now suppose that

$$\xi \oplus \left\{ \frac{\| \xi \|}{\| AB \|} \odot BC \right\} = \eta \oplus \left\{ \frac{\| \eta \|}{\| AB \|} \odot BC \right\}.$$

It results that
$$\xi \ominus \eta = \frac{\| \xi \| - \| \eta \|}{\| AB \|} \odot BC,$$

or
$$\| \xi \ominus \eta \| = \{ \| \xi \| - \| \eta \| \} \frac{\| BC \|}{\| AC \|}.$$

Now, by our hypothesis, $\| BC \| / \| AC \| < 1$. Hence either

$$\| \xi \ominus \eta \| = 0, \quad \text{or} \quad \| \xi \ominus \eta \| < \| \xi \| - \| \eta \|.$$

If we write this latter proposition in the form

$$\| \xi \ominus \eta \| + \| \eta \| < \| (\xi \ominus \eta) \oplus \eta \|,$$

it will be seen to contradict (8) in the definition of a (Ve). Hence

$$\| \xi \ominus \eta \| = 0,$$

or what results from (13), $\xi = \eta$.

Let us consider the point-transformation which retains A invariant and changes every other element P into the element P' such that

$$AP' = AP \oplus \left\{ \frac{\| AP \|}{\| AB \|} \odot BC \right\}.$$

It results from what has been said and the properties of vectors that this is biunivocal; let us consider how it affects the magnitude of vectors. If P is transformed into P' and Q into Q' by our transformation, we wish to determine a relation between PQ and $P'Q'$.

Now, as an immediate consequence of the commutative law and the definition of our transformation,

$$P'Q' = PQ \oplus \left\{ \frac{\| AQ \| - \| AP \|}{\| AB \|} \odot BC \right\}.^*$$

As a consequence,

$$\| P'Q' \| \leqslant \| PQ \| + \frac{\| BC \|}{\| AB \|} \left| \| AQ \| - \| AP \| \right|$$
$$\leqslant 2 \| PQ \|.$$

On the other hand, it may readily be proved that

$$\| P'Q' \| \geqslant \left| \| PQ \| - \frac{\| BC \|}{\| AB \|} \left| \| AQ \| - \| AP \| \right| \right|$$
$$\geqslant \| PQ \| \left\{ 1 - \frac{\| BC \|}{\| AB \|} \right\}.$$

It follows from these inequalities that, to put it roughly, $P'Q'$ is small when and only when PQ is small, and that a set of elements approaching indefinitely close to a given element is transformed into a set approaching indefinitely close to the transform of the given element, and *vice versa.* In other words, our transformation leaves limit-properties invariant in both directions, and so belongs to Σ. Moreover, our transformation leaves A invariant and changes B into the element D such that

$$AD = AB \oplus \left\{ \frac{\| AB \|}{\| AB \|} \odot BC \right\} = AB \oplus BC = AC,$$

or, in other words, into C. We thus have completed our proof of the equivalence of point-pairs by the consideration of "rotations."

EXAMPLES OF SETS (Vr).

7. (1) The system consists of all n-partite numbers $(x_1, x_2, \ldots, x_n)$. If $A = (x_1, x_2, \ldots, x_n)$ and $B = (y_1, y_2, \ldots, y_n)$, AB shall be the n-partite number $(x_1 - y_1, y_2 - y_2, \ldots, x_n - y_n)$, and every n-partite number shall be a vector. If

$$\xi = (u_1, u_2, \ldots, u_n) \quad \text{and} \quad \eta = (v_1, v_2, \ldots, v_n),$$

$$\| \xi \| = \sqrt{(u_1^2 + u_2^2 + \ldots + u_n^2)}, \quad k \odot \xi = (ku_1, ku_2, \ldots, ku_n),$$

and
$$\xi \oplus \eta = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n).$$

* $(-n) \odot UV$ is to be understood as $n \odot VU$.

The independence of addition on order is immediately obvious. The other specifically (Vr) property results from the fact that any arc of a circle can be traversed with a finite number of chords each less in length than ϵ, for any given ϵ.

(2) The system of elements and that of vectors alike consist in all ∞-partite numbers $(x_1, x_2, \ldots, x_k, \ldots)$ such that there is a finite X such that for all k, $|x_k| \leqq X$. If

$$A = (x_1, x_2, \ldots, x_k, \ldots) \quad \text{and} \quad B = (y_1, y_2, \ldots, y_k, \ldots),$$

$$AB = (x_1 - y_1, x_2 - y_2, \ldots, x_k - y_k, \ldots).$$

If $$\xi = (u_1, u_2, \ldots, u_k, \ldots) \quad \text{and} \quad \eta = (v_1, v_2, \ldots, v_k, \ldots),$$

$$\| \xi \| = \text{least upper bound } |u_k|,$$

$$m \odot \xi = (mu_1, mu_2, \ldots, mu_k, \ldots),$$

and $$\xi \oplus \eta = (u_1 + v_1, u_2 + v_2, \ldots, u_k + v_k, \ldots).$$

The commutative law is obvious; the other condition for a (Vr) can be demonstrated if we show that given ξ and η such that $\|\xi\| = \|\eta\| \neq 0$, there is a chain of vectors, $\xi_1 = \xi, \xi_2, \ldots, \xi_n = \eta$, such that for all j,

$$\| \xi_j \| = \| \xi \| \quad \text{and} \quad \| \xi_{j+1} \ominus \xi_j \| < \| \xi \|.$$

Such a chain may be constructed as follows; let ζ be the vector $(z_1, z_2, \ldots, z_k, \ldots)$, such that for all k, z_k is the larger of the two quantities u_k and v_k if they differ, and their common value, if they agree. Then

$$\| \zeta \| = \| \xi \|.$$

Let $$\frac{\| \zeta \ominus \xi \|}{\| \xi \|} = p, \quad \text{and} \quad \frac{\| \zeta \ominus \eta \|}{\| \xi \|} = q.$$

Let r be any integer larger than both p and q. Then the sequence of vectors

$$\xi,\ \xi \oplus \left\{ \frac{1}{r} \odot (\zeta \ominus \xi) \right\},\ \ldots,\ \xi \oplus \left\{ \frac{h}{r} \odot (\zeta \oplus \xi) \right\},\ \ldots,$$

$$\zeta,\ \zeta \oplus \left\{ \frac{1}{r} \odot (\eta \ominus \zeta) \right\},\ \ldots,\ \zeta \oplus \left\{ \frac{h}{r} \odot (\eta \ominus \zeta) \right\},\ \ldots,\ \eta,$$

may readily be shown to satisfy the conditions for a chain $\{\xi_j\}$.

(3) The system of all points and the system of all vectors consist alike

in all ∞-partite numbers $(x_1, x_2, \ldots, x_k, \ldots)$ such that the series

$$x_1^2+x_2^2+\ldots+x_n^2+\ldots$$

converges. AB, $m \odot \xi$, and $\xi \oplus \eta$ are defined as in (2). If

$$\xi = (u_1, u_2, \ldots, u_k, \ldots),$$

$$\| \xi \| = \sqrt{(u_1^2+u_2^2+\ldots+u_k^2+\ldots)}.$$

To show that our system is a (Vr), let us introduce a few considerations from the trigonometry of infinitely many dimensions. If

$$\xi = (u_1, u_2, \ldots, u_k, \ldots) \quad \text{and} \quad \eta = (v_1, v_2, \ldots, v_k, \ldots),$$

let us define $<\xi\eta$ as

$$\cos^{-1} \frac{u_1 v_1+u_2 v_2+\ldots+u_n v_n+\ldots}{\| \xi \| \cdot \| \eta \|}.$$

The first question to arise is under what circumstances $<\xi\eta$ will exist. It may easily be shown that if Σu_n^2 and Σv_n^2 converge, $\Sigma(u_n+v_n)^2$ and $\Sigma(u_n-v_n)^2$ will converge.* It results that $\Sigma \frac{1}{4} \{(u_n+v_n)^2-(u_n-v_n)^2\}$ will converge, or that $\Sigma u_n v_n$ will converge. Furthermore, it is obvious that to multiply ξ or η by a positive constant will not affect the magnitude or existence of $<\xi\eta$. We may thus assume $\| \xi \| = \| \eta \|$, which gives us

$$<\xi\eta = \cos^{-1} \frac{u_1 v_1+u_2 v_2+\ldots+u_n v_n+\ldots}{u_1^2+u_2^2+\ldots+u_n^2}.$$

Now, consider the inequality $\Sigma(u_n-v_n)^2 \geqslant 0$. We may write this

$$\Sigma u_n^2-2\Sigma u_n v_n+\Sigma v_n^2 \geqslant 0.$$

Making use of the fact that $\Sigma u_n^2 = \Sigma v_n^2$, this becomes

$$2\Sigma u_n v_n \leqslant 2\Sigma u_n^2.$$

It may be proved in precisely the same manner that

$$-2\Sigma u_n v_n \leqslant 2\Sigma u_n^2.$$

Hence $<\xi\eta$ is the anticosine of a number not greater in absolute value than 1, and consequently exists.

As in ordinary geometry,

$$\| \xi \ominus \eta \|^2 = \| \xi \|^2+\| \eta \|^2-2 \| \xi \| \cdot \| \eta \| \cos <\xi\eta.$$

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* Cf. Hausdorff, *loc. cit.*, p. 287.

This may be proved by writing the formula out at length, when it will reduce to an identity. All the series involved will be absolutely convergent, so there is no difficulty about changing the order of terms.

Let us suppose, as above, that $\|\xi\| = \|\eta\|$, and let us consider $\cos < \xi \{\xi \oplus \eta\}$. This will be

$$\frac{u_1(u_1+v_1)+u_2(u_2+v_2)+\ldots+u_n(u_n+v_n)+\ldots}{\sqrt{(u_1^2+u_2^2+\ldots+u_n^2+\ldots)}\sqrt{\{(u_1+v_1)^2+(u_2+v_2)^2+\ldots+(u_n+v_n)^2+\ldots\}}}.$$

By our previous remarks this is an essentially positive quantity. We shall moreover get the identity

$$\begin{aligned}
\cos 2 &< \xi \{\xi \oplus \eta\} \\
&= 2\cos^2 < \xi \{\xi \oplus \eta\} - 1 \\
&= \frac{2\{\Sigma(u_n^2+u_nv_n)\}^2 - [\Sigma u_n^2][\Sigma(u_n+v_n)^2]}{[\Sigma u_n^2][\Sigma(u_m+v_m)^2]} \\
&= \frac{\begin{array}{r}2\Sigma u_n^2 \Sigma u_m^2 + 4\Sigma u_n^2 \Sigma u_m v_m + 2\Sigma u_n v_n \Sigma u_m v_m - \Sigma u_n^2 \Sigma u_m^2 \\ -2\Sigma u_n^2 \Sigma u_m v_m - \Sigma u_n^2 \Sigma v_m^2\end{array}}{[\Sigma u_n^2][\Sigma(u_n+v_m)^2]} \\
&= \frac{2\Sigma u_n^2 \Sigma u_m v_m + 2\Sigma u_n v_n \Sigma u_m v_m}{[\Sigma u_n^2][\Sigma(u_m+v_m)^2]} \\
&= \frac{[\Sigma u_m v_m]\{\Sigma u_n^2 + 2\Sigma u_n v_n + \Sigma v_n^2\}}{[\Sigma u_n^2][\Sigma(u_m+v_m)^2]} \\
&= \frac{\Sigma u_m v_m}{\Sigma u_m^2} = \cos \xi\eta.
\end{aligned}$$

It results from this that $< \xi (\xi \oplus \eta)$ is the half of $< \xi\eta$ in the first or fourth quadrant.

Now, let ξ and η be any two vectors of equal magnitude, provided only that neither is made up entirely of 0's. Form the vector ξ_3, which shall be a positive multiple of $\xi \oplus \eta$ with the same magnitude as ξ. In a similar manner, interpolate ξ_2 between ξ and ξ_3, and ξ_4 between ξ_3 and η, and let us know ξ and η as ξ_1 and ξ_5, respectively. We have

$$\cos < \xi\xi_3 = \cos < \xi_3\eta = \sqrt{\{\tfrac{1}{2}(1+\cos < \xi\eta)\}} \geqslant 0.$$

Hence

$$\begin{aligned}
\cos < \xi_1\xi_2 = \cos < \xi_2\xi_3 = \cos < \xi_3\xi_4 = \cos < \xi_4\xi_5 \\
= \sqrt{\{\tfrac{1}{2}(1+\cos < \xi\xi_3)\}} \geqslant \tfrac{1}{2}\sqrt{2}.
\end{aligned}$$

It follows from the law of cosines that

$$\| \xi_h - \xi_{h+1} \| = \sqrt{(\| \xi_h \|^2 + \| \xi_{h+1} \|^2 - 2 \| \xi_h \| . \| \xi_{h+1} \| \cos \angle \xi_h \xi_{h+1})}$$
$$\leqq \| \xi \| \sqrt{(2-\sqrt{2})}$$
$$< \| \xi \| .$$

(4) The system of all elements and the system of all vectors both consist of all continuous functions of a real variable defined over a given closed interval. The vector fg is the function $f(x)-g(x)$. If $\xi = f(x)$ and $\eta = g(x)$, $\| \xi \| = \max |f(x)|$, $k \odot \xi = kf(x)$, and $\xi \oplus \eta = f(x)+g(x)$. The proof that this system is a (Vr) proceeds as in (2).

It may be noted that systems (1), (3), and (4) satisfy VIII.*

CONSISTENCY OF POSTULATES I–IX.

8. The following system satisfies Postulates I–IX: K consists of all the points on a line, and Σ consists of all bicontinuous, biunivocal transformations of the whole line into itself.

DEDUCTIONS FROM POSTULATES I–IX.

9. THEOREM I.—*If A and B are any two distinct members of K, there is a unique closed set (A, B), completely characterized by the facts that it is connected and that A and B are boundary elements of it.*

Proof.

It follows from Postulates I, VI, and VII that there is at least one set with at least two boundary elements. By VI, these can be transformed by a transformation from Σ into A and B, and by III and IV, this transformation will leave every connected set connected. By IX, this set is uniquely determined except as to whether it contains A and B. Adjoin to it its limit-elements, and it will clearly remain connected, while it will contain A and B.

THEOREM II.—*A and B are the only boundary-elements of (A, B).*

Proof.

Let D be any element not in (A, B). Consider the component† E of

* Hausdorff, *loc. cit.*, pp. 288, 289.

† Since we have proved that our system satisfies Hausdorff's axioms, we may take advantage of his proof of the existence of components.

(A, B) to which D belongs. This must have a limit-element P in (A, B), for otherwise the segment (D, A), which exists, by Theorem I, would not be connected. P is then a boundary-element of (A, B) which is the limit of the connected set E in (A, B).

Now, let C be any boundary-element of (A, B) other than A and B. If C were the limit of a connected set F in (A, B), then F would either have A for a limit-element, or B for a limit-element, or neither A nor B. In the first two cases it results from IX that F must coincide with (A, B), which is impossible. In the third case, it follows from V that A and B are boundary-elements of the connected set $(A, B)+F$, which hence must coincide with (A, B), by IX. This is again impossible. It follows that there is no such set as F.

Let Q be any boundary-element of (A, B) other than C and P. By IX, we may write (A, B) as (Q, C) or as (Q, P). Now, by VI, there is a transformation from Σ leaving Q invariant and changing P into C. By III and IV, this changes (Q, P) into (Q, C), and changes every connected set in (Q, P) having P as a limit-element into a connected set in (Q, C) having C as a limit-element. As the existence of sets of the latter sort has been disproved, while the existence of sets of the former sort has been proved, it follows that either our assumption of the existence of C or our assumption of the existence of P is inadmissible. If either assumption is incorrect, (A, B) has only two boundary-elements, which must be A and B.

THEOREM III.—*If* (A, B) *and* (A, C) *have an element in common other than* A, *either* (A, B) *contains* (A, C) *or vice versa.*

Proof.

Let E consist of all elements in (A, B) but not in (A, C), and let F be the component of E containing B. As (A, C) is connected, F has some limit-element D in (A, C). If A is the only limit-element of F in (A, C), $A+F$ is a connected set containing the boundary-elements A and B, and hence coincides with (A, B), which hence, contrary to assumptions, contains no other term than A in common with (A, C). By Theorem II, the only other value which D can have is C. Now, consider the set $F+(A, C)$. It is connected, and, by V, has A as a boundary element. By V, either B is a boundary-element or B belongs to (A, C). If B belongs to (A, C), then every element of (A, B) does likewise, for otherwise, as (A, C) has only two boundary-elements, E can have only A and C as limit-elements in (A, C). If B differs from C, this is clearly impossible, while if B coincides with C, $(A, B) = (A, C)$.

The only other possibility is that E contains no elements. In this case, (A, C) is contained in (A, B).

THEOREM IV.—*If C is interior to (A, B), $(A, B) = (A, C) + (C, B)$, and (A, C) shares with (C, B) no other element than C.*

Proof.

By Theorem III, (A, B) contains (A, C) and (C, B). If B belonged to (A, C), by Theorem III, (A, C) would contain, and hence coincide with (A, B). This contradicts our assumption. Hence, by Postulate V, B is a boundary-element of $(A, C) + (C, B)$. The same argument applies to A. Moreover, being the sum of two overlapping, closed, connected sets, by V, $(A, C) + (C, B)$ is closed and connected. Hence, by Theorem II,

$$(A, C) + (C, B) = (A, B).$$

If (A, C) and (C, B) had in common any other element than C, then, by Theorem III, either (A, C) would contain (C, B), or *vice versa*. In this case, either (A, C) or (C, B) would contain (A, B). Hence, by Theorem II, C would coincide with either A or B, and would not be an interior element of (A, B).

Definition.—If C is interior to (A, B), we shall write ACB. It is obvious that if ABC, A, B, and C are all different, and it is also obvious that ABC and CBA are equivalent. Furthermore, by Theorem III, ABC and ACB are incompatible.

THEOREM V.—*If ABC and ACD, then BCD.*

Proof.—By Theorem IV, ABD. Hence, by Theorem IV, either ACB or BCD. ACB, however is incompatible with ABC, by Theorem III.

THEOREM VI.—*ABC, ABD, and CBD are incompatible.*

Proof.—By Theorem IV, either ACB, $B = C$, or BCD. As Theorem III excludes the first two suppositions, which are incompatible with ABC, there remains only the last possibility, which, by III, is incompatible with BCD.

THEOREM VI.—*Either ABC, BAC, or ACB, if A, B and C are distinct.*

Proof.—Suppose the first two alternatives are not fulfilled. Then, by Theorem III, (A, C) and (B, C) have only C in common, $(A, C) + (B, C)$ is connected, and by Postulate V, has A and B as boundary-elements. Hence $(A, C) + (C, B) = (A, B)$, or, in other words, ACB.

THEOREM VII.—*If* ABC *and* BCD, *then* ACD.

Proof.—By Theorem VI, we have DAC, ADC, or ACD. If DAC and ABC, then by Theorem IV, DBC, which, by Theorem III, contradicts BCD. If ADC, then, by Theorem IV, ABD or DBC. DBC, by Theorem III, contradicts BCD. If ABD and BCD, then, by Theorem IV, ACD.

Definition.—$AB \mid CD$ shall mean any one of the following sets of relations:

(1) ACD, ABD.

(2) ACD, $B = D$.

(3) ACD, ADB.

(4) $A = C$, ABD.

(5) $A = C$, $B = D$, $A \neq B$.

(6) $A = C$, ADB.

(7) CAD, CAB.

(8) $A = D$, CAB.

(9) CDA, CAB.

THEOREM VIII.—*If* $AB \mid CD$ *and* $BP \mid CD$, *then* $AP \mid CD$.

Proof.—This involves merely the tabulation of the 81 possible cases and the application of Theorems III–VII in the instances in which they are appropriate.

THEOREM IX.—*If* $AB \mid CD$, $A \neq B$ *and* $C \neq D$.

Proof.—This follows from the fact that if ABC, $A \neq B \neq C$, and the definition of $AB \mid CD$.

THEOREM X.—*If* $A \neq B$, $C \neq D$, *then either* $AB \mid CD$ *or* $BA \mid CD$.

Proof.—This follows from Theorems VI, IV, V, and VII, as may be shown by tabulating the relations between A, B, C, and D, which are possible on the basis of Theorem VI.

THEOREM XI.—*If* $AB \mid CD$ *and* APB, *then* $AP \mid CD$ *and* $PB \mid CD$.

Proof.—As above, by tabulating the possible cases, and making use of Theorems IV–VII.

THEOREM XII.—*If* $AP \mid CD$ *and* $PB \mid CD$, *then* APB.

Proof.—As above, by tabulation.

THEOREM XIII.—*If M and N are two classes of elements exhausting K, and such that there are two fixed elements C and D such that if A belongs to M and B belongs to N, $AB|CD$, then there is an element P such that if Q belongs to M and R belongs to N and $Q \neq P \neq R$, QPR.*

Proof.

Suppose that X and Y belong to M, and that XZY. Either $XY|CD$ or $YX|CD$, by Theorem X. Similarly, either $XZ|CD$ or $ZX|CD$, and either $YZ|CD$ or $ZY|CD$. Making use of Theorems XII and VI, it turns out that the only admissible combinations of hypotheses are $XZ|CD$, $ZY|CD$, $XY|CD$ and $YZ|CD$, $ZX|CD$, $YX|CD$. Since we have $XB|CD$ and $YB|CD$ for all B in N, we have, by Theorem VIII, $ZB|CD$ in both cases. It follows then from Theorems VIII and IX that Z does not belong to N, so that it must belong to M. In other words, if M contains X and Y, it contains every element in (X, Y), so that M is connected. Likewise, N is connected.

It follows from Postulate IX and Theorem I that there is just one element P which is a limit-element of M belonging to N or a limit-element of N belonging to M. Let Q belong to M and R to N. As (Q, R) is connected, it must contain P.

THEOREM XIV.—*There is a denumerable set K' of elements such that if A and B are any two elements, there is an element C from K' such that ACB.*

Proof.

Let K' be the set to which reference is made in Postulate VIII. Then every element is a limit-element of K'. It follows from the fact that a single element has no limit-element and Postulate V that a segment has interior elements. Hence every segment contains at least one element of K'.

THEOREM XV.—*There is no element A such that for all $B \neq A$, $AB|CD$, and there is no element A such that for all $B \neq A$, $BA|CD$.*

Proof.—This follows directly from Postulate VI.

THEOREM XVI.—*K can be put into (1, 1)-correspondence with the set of all real numbers, in such a way that two elements C and D can be selected such that $AB|CD$ when and only when the correspondent of A is larger than the correspondent of B.*

Proof.—By Theorems VIII, IX, and X, order as defined by $AB \mid CD$ is serial. By Theorems XI, XII, and XIII, it is what Russell calls "Dedekindian." By XI, XII, and XIV, it contains a denumerable "median class." Hence, by a well known theorem,* it is ordinally similar to the series of reals.

THEOREM XVII.—*In the correspondence of Theorem XVI, Σ goes over into the set of all bicontinuous biunivocal transformations of the series of reals.*

Proof.—In the transformation of Theorem XVI, a segment goes over into a segment (Theorems XI, XII). Now, by Postulate VII, and Theorem I, the limit of a set E consists of all those elements A such that every segment (C, D) of which A is an element other than C and D contains a member of E. Hence limit goes over into limit, and in virtue of Postulates II, III, and IV, a transformation from Σ, which is precisely a transformation keeping limit-properties invariant, goes over into a bicontinuous, biunivocal transformation of the number-line, and every bicontinuous, biunivocal transformation of the number-line may be thus obtained.

Theorem XVII is equivalent to the statement that our set of postulates is a categorical set of postulates for the analysis-situs group of the line.

CONSIDERATIONS OF INDEPENDENCE.

10. Up to the present, the author has been unable to solve the question of the independence of Postulates IV, V, VII, and VIII. Each of the other postulates is independent of all the rest. The examples given below satisfy all the postulates except the one whose number they are given.

I. K consists of one element; Σ contains only the identity transformation.

II. K consists of all points on a line; Σ consists of all biunivocal, bicontinuous transformations that preserve direction.

III. K consists of all points on a line; Σ consists of all biunivocal, bicontinuous transformations, together with the transformations that displace all points with rational coordinates a rational distance in one direction, and all points with irrational coordinates a rational distance in the other.

* Whitehead and Russell, *Principia Mathematica*, Vol. 3, ✱ 275.

VI. K consists of all points on two mutually exclusive lines; Σ consists of all biunivocal, bicontinuous transformations of K.

IX. K consists of all points on a circle; Σ consists of all biunivocal, bicontinuous transformations of K.

It may be said that the independence of VIII would be proved if we could produce a closed homogeneous* series with a number of terms greater than $2^{\aleph_0}$. Homogeneous series with more than $2^{\aleph_0}$ terms are known, but they are not closed.

* Hausdorff, *loc. cit.*, p. 173.

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ON THE DISTRIBUTION OF ENERGY IN AIR SURROUNDING A VIBRATING BODY

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1. If a fluid be subject to a periodic disturbance, it is known that within regions whose maximum distance from the source of disturbance is small compared with the wave length, the fluid may be treated as if it were incompressible. This principle has been of great service in the approximate treatment of many acoustical problems, which do not lend themselves to rigorous solution. We infer that within such regions the potential energy may be neglected compared with the kinetic energy.*

In particular, if we consider the waves in an infinite medium, due to the vibration of a body, we deduce that there is a certain region in the neighbourhood of the body within which the energy is mainly kinetic. On the other hand, at large distances from the body, the waves tend to become plane, and, in a system of plane progressive waves, the potential energy is equal to the kinetic.† In one case the ratio of the potential to the kinetic is a minute fraction, in the other case it is unity. The question is, what is the law of distribution by which this ratio changes from its lower to its upper limit?

The case which is here considered in detail is that of a sphere, vibrating in an infinite medium. In the first place, the calculations have been carried out for the simplest type of vibration, viz. that of a pulsating sphere where each surface element vibrates radially in the same phase and with the same amplitude. A similar analysis is next carried out for a sphere vibrating in a straight line in the manner of a pendulum. Lastly, the problem is treated for the general vibration represented by a surface harmonic of order n, of which the two cases mentioned are, of course, particular examples. In particular, the case of $n = 2$ is examined.

* See Rayleigh, *Scientific Papers*, Vol. 4, No. 230; *Theory of Sound*, Vol. 2 (1878), p. 158, *et seq.*

† That is, if the mean be taken with respect to time.

2. *Summary of Results.*

Denoting the mean potential energy of the fluid enclosed between the sphere and any concentric spherical surface by V, and the mean kinetic by T, it is found that the ratio V/T depends on

(1) The ratio of the dimensions of the sphere to the wave length of the disturbance (ka).*

(2) The extent of the region considered (r/a).

(3) The type of vibration n.

A table has been drawn up in order to illustrate the variation in V/T when any one of the above factors is varied.†

Two facts emerge from an examination of the table. In the first place, when ka is small, there is a finite extent of the medium in which the ratio V/T is very small. This is the region referred to in the opening paragraph. Some idea, too, is gleaned of the relation between the extent of this region and the type of vibration. It is clear from the table, for example, that the simpler the type of vibration (*i.e.* the lower the value of n) the less extensive is the region. The question as to how the dimensions of the region change with the type of vibration is investigated and a formula is evolved giving the relation between the two.‡

The second fact is this: the ratio of V/T tends to the limit unity when the extent of the region is made infinitely great. In the case of plane waves, the value of V/T is always equal to unity, *i.e.* provided that the region so considered includes an integral number of wave lengths. It has been pointed out that, in the case of a system of divergent spherical waves, the *total* kinetic energy (*i.e.* reckoned throughout infinite space) is, under certain conditions, equal to the *total* potential energy.§ The condition mentioned is that $r.\phi^2$ shall vanish over the inner and outer boundaries of the system (ϕ denoting as usual the velocity potential). In the system of waves considered in this paper this condition does not hold, but a theorem is developed which gives the relation between V and T: and, in particular, it is found that the *limit* of V/T tends to unity when infinite space is considered.‖ It is interesting to note the change of V/T from a very minute fraction to its limiting value unity.

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* a denotes radius of the sphere, and $k = 2\pi/\lambda$, where λ is the wave length.

† § 11.

‡ § 10.

§ Lamb, *Proc. London Math. Soc.*, Ser. 1, Vol. xxxv, p. 160 (1902); *Hydrodynamics*, p. 484.

‖ *Appendix*, p. 361.

3. *Pulsating Sphere.*

If the centre be taken as origin, the velocity potential is given by

$$\phi = \frac{A \, . \, e^{-ik(r-ct)}}{r}, \tag{1}$$

where A is a complex constant to be determined, k stands for $2\pi/\lambda$, where λ is the wave length, and c is the velocity of wave propagation.

To determine the coefficient in (1) let the motion of the surface be given by

$$r = a + \alpha \, . \, e^{ikct}, \tag{2}$$

where α, the amplitude of vibration, is small. At the surface of the sphere, we have

$$-\partial\phi/\partial r = \dot{r},$$

whence

$$A \left\{ \frac{1+ika}{a^2} \right\} e^{-ika} = ikc\alpha = \beta \text{ (say)}. \tag{3}$$

Hence

$$\phi = \frac{a^2 \beta}{1+ika} \, \frac{e^{ik(ct-r+a)}}{r}, \tag{4}$$

and taking the real part, we have

$$\phi = \frac{a^2\beta}{(1+k^2a^2)^{\frac{1}{2}}} \, \frac{\cos k(ct-r+a+\epsilon)}{r}. \tag{5}$$

4. *The Mean Potential and Kinetic Energies.*

If we denote the elasticity of volume of the fluid by κ, and the condensation by s, the potential energy of unit volume is $\frac{1}{2}\kappa s^2$.

Remembering that $c^2 = \kappa/\rho_0$, where ρ_0 is the density in the undisturbed state, and that the dynamical equation of sound waves is

$$c^2 s = \partial\phi/\partial t,$$

we can write

$$\text{Potential Energy per unit volume} = \frac{1}{2} \, \frac{\rho_0}{c^2} \left(\frac{\partial\phi}{\partial t} \right)^2. \tag{6}$$

In the present case the mean value of this with respect to the time is

$$\tfrac{1}{4}\rho \, \frac{a^4\beta^2}{1+k^2a^2} \, \frac{k^2}{r^2}.$$

Hence the mean potential energy in the region between the sphere and a concentric spherical surface of radius r is

$$V = \int_a^r \tfrac{1}{4}\rho \, \frac{a^4\beta^2}{1+k^2a^2} \, \frac{k^2}{r^2} \, 4\pi r^2 dr = \frac{\pi\rho a^3\beta^2}{1+k^2a^2} \, k^2 a^2 \left(\frac{r}{a} - 1 \right). \tag{7}$$

The Kinetic Energy per unit volume at a distance r from the centre of the sphere is

$$\tfrac{1}{2}\rho\left(\frac{\partial\phi}{\partial r}\right)^2 = \tfrac{1}{2}\rho\,\frac{a^4\beta^2}{1+k^2a^2}\left\{\frac{\cos^2\omega - 2kr\sin\omega\cos\omega + k^2r^2\sin^2\omega}{r^4}\right\}, \tag{8}$$

where $$\omega = k(ct-r+a+\epsilon).$$

The mean value of this is

$$\tfrac{1}{4}\rho\,\frac{a^4\beta^2}{1+k^2a^2}\,\frac{1+k^2r^2}{r^4}.$$

The mean kinetic energy in the region considered in equation (7) is

$$T = \frac{\pi\rho a^3\beta^2}{1+k^2a^2}\left\{k^2a^2\left(\frac{r}{a}-1\right)+\left(1-\frac{a}{r}\right)\right\}. \tag{9}$$

5. *Special Cases.*

(1) When the radius of the sphere is small compared with the wave length, *i.e.* ka is small, the expression for V and T, in equations (7) and (9) reduce to

$$V = \pi\rho a^2\beta^2k^2a^2(r-a), \tag{10}$$

$$T = \pi\rho a^2\beta^2\,\frac{a}{r}(r-a). \tag{11}$$

Hence in the immediate neighbourhood of the sphere, the Potential Energy is very small compared with the Kinetic, and the motion is practically the same as if the fluid were incompressible.

(2) Consider next the case when r and a are both large compared with the wave length, *i.e.* ka and kr are large but $r-a$ is finite.

Then, from (7) and (9), we have

$$\begin{aligned} V &= \pi\rho a^2\beta^2(r-a) \\ &= 4\pi a^2(r-a)\times\tfrac{1}{4}\rho\beta^2 \\ &= T, \end{aligned} \tag{12}$$

approximately. The volume under consideration may be taken to be equal to $4\pi a^2(r-a)$. Under these conditions the Potential and Kinetic Energies are equal, and the mean value of each per unit volume is $\frac{1}{4}\rho\beta^2$, as in the case of plane waves.

(3) Whatever be the value of ka, we find from equations (7) and (9) that when $r\to\infty$, the ratio of V/T tends to unity. This may be proved

independently.* Thus, we have

$$r^2\left(\frac{\partial\phi}{\partial r}\right)^2 = \left\{\frac{\partial(r\phi)}{\partial r}\right\}^2 - \frac{\partial}{\partial r}(r\phi^2), \tag{13}$$

and since, from equation (1),

$$r\phi = A\cos k(ct-r+a+\epsilon),$$

we get
$$\left\{\frac{\partial(r\phi)}{\partial r}\right\}^2 = c^2r^2s^2,$$

since
$$c^2s = \frac{\partial\phi}{\partial t}.$$

Hence
$$\int_a^\infty \tfrac{1}{2}\rho\left(\frac{\partial\phi}{\partial r}\right)^2 4\pi r^2 dr = \int_a^\infty \tfrac{1}{2}\rho c^2s^2\, 4\pi r^2 dr - 2\pi\rho\Big[r\phi^2\Big]_a^\infty. \tag{14}$$

Since the last term on the right-hand side is finite at the surface of the sphere, and zero at infinity, it follows that $\underset{r\to\infty}{\mathrm{Lt}}\ V/T$ tends to unity.

The values of the expression V/T for various values of ka and r/a, are given in the column $n=0$ in the table.† For instance, when $ka = \cdot 01$, the value of r/a is 1000 before V is comparable with T. On the other hand, when $ka = 10$, the ratio V/T is nearly equal to unity when $r/a = 2$. The results are further discussed in § 9.

6. *The Pendulum.*

The velocity potential due to a swinging pendulum, when the amplitude is small, is that due to a *double* source.

Omitting the time factor, we can put

$$\phi = A\frac{\partial}{\partial x}\left(\frac{e^{-ikr}}{r}\right) = A\frac{\partial}{\partial r}\left(\frac{e^{-ikr}}{r}\right)\cos\theta, \tag{15}$$

if the axis of x is the line of motion of the sphere, and θ denotes the angle between r and x. If U be the velocity of the sphere at any instant, we have the following equation to determine A, viz.,

$$-\frac{\partial\phi}{\partial r} = U\cos\theta, \tag{16}$$

for $r = a$. If
$$U = \beta . e^{ikct}, \tag{17}$$

* *Proc. London Math. Soc.* (*l. c.*).

† P. 361.

we get
$$\phi = \beta a^3 \frac{2-k^2a^2-2ika}{4+k^4a^4} \frac{1+ikr}{r^2} e^{ik(ct-r+a)} \cos\theta, \tag{18}$$

and taking the real part of the expression

$$\phi = \frac{R}{r^2} \{\cos\omega - kr \sin\omega\} \cos\theta, \tag{19}$$

where
$$\omega = k(ct-r+a+\epsilon),$$

and
$$R = \frac{\beta a^3}{(4+k^4a^4)^{\frac{1}{2}}}.$$

The Potential Energy per unit volume is given by

$$\tfrac{1}{2} \frac{\rho_0}{c^2} \dot{\phi}^2 = \tfrac{1}{2} \frac{\rho_0}{c^2} k^2c^2 \frac{R^2}{r^4} \{\sin^2\omega + 2kr \sin\omega\cos\omega + k^2r^2\cos^2\omega\} \cos^2\theta. \tag{20}$$

Taking the mean of this expression over a long period of time and then the mean value over a spherical surface of radius r, we get the value of the potential energy in a spherical stratum of the medium of radius r and of thickness δr. Thus

$$\delta V = \tfrac{1}{3}\pi\rho_0 R^2k^2 \frac{1+k^2r^2}{r^2} \delta r.$$

Integrating from a to r, we get the expression corresponding to equation (7) for the pulsating sphere, viz.,

$$V = \tfrac{1}{3}\pi\rho_0\beta^2a^3 \frac{k^2a^2}{4+k^4a^4} \left\{ k^2a^2\left(\frac{r}{a}-1\right) + \left(1-\frac{a}{r}\right)\right\}. \tag{21}$$

The Kinetic Energy of the same volume of the medium is equal to the mean value with respect to time of the expression

$$\tfrac{1}{2}\rho_0 \iiint \left\{ \left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2 \right\} r^2\,dr\,d\varpi.$$

We find

$$T = \tfrac{2}{3}\pi\rho_0 \frac{\beta^2a^3}{4+k^4a^4} \left\{ \frac{k^4a^4}{2}\left(\frac{r}{a}-1\right) + k^2a^2\left(1-\frac{a}{r}\right) + \left(1-\frac{a^3}{r^3}\right)\right\}. \tag{22}$$

An analysis of these results for the special cases considered in § 4 leads to similar conclusions, but may be omitted in view of the general treatment in § 9.

7. *The General Vibration of a Sphere.*

The general equation of sound waves in the case of simple harmonic motion is

$$(\nabla^2+k^2)\phi = 0. \tag{23}$$

The solution of equation (23) when the waves are those produced by the vibration of a spherical surface is well known, having been given by Stokes in his classical paper "On the Communication of Vibrations from a Vibrating Body to a Surrounding Gas".* The most general type of vibration can be represented by terms of the type

$$\dot{r} = S_n e^{i\sigma t}. \tag{24}$$

The appropriate solution of $(\nabla^2+k^2)\phi = 0$ is then given by

$$\phi = C_n f_n(kr) r^n S_n e^{ikct}, \tag{25}$$

where C_n is a constant determined by the boundary conditions, S_n is a spherical surface harmonic of n-th order, and

$$f_n(kr) = \frac{i^n e^{-ikr}}{(kr)^{n+1}} \left\{ 1 + \frac{n\,.\,n+1}{2ikr} + \frac{n-1\,.\,n\,.\,n+1\,.\,n+2}{2\,.\,4(ikr)^2} + \dots \right.$$
$$\left. + \frac{1\,.\,2\,.\,3\,\dots\,2n}{2\,.\,4\,.\,6\,\dots\,2n(ikr)^n} \right\}. \tag{26}$$

Since $-\partial\phi/\partial r = S_n e^{ikct}$ at the surface $r = a$, we get

$$C_n = -\frac{1}{\{kaf_n'(ka) + nf_n(ka)\}\, a^{n-1}}. \tag{27}$$

The expression $f_n(kr)$ can be written

$$f_n(kr) = \frac{i^n\,.\,e^{-ikr}}{(kr)^{n+1}} \{g_n - ih_n\}, \tag{28}$$

which will be found convenient when the real part of ϕ is required. We have†

$$g_n = 1 - \frac{n-1\,.\,n\,.\,n+1\,.\,n+2}{2\,.\,4(kr)^2} + \dots + (-)^{\frac{1}{2}n} \frac{1\,.\,2\,\dots\,2n}{2\,.\,4\,\dots\,2n} \frac{1}{(kr)^n}, \tag{29}$$

$$h_n = \frac{n\,.\,n+1}{2\,.\,kr} - \frac{n-2\,.\,n-1\,.\,n\,.\,n+1\,.\,n+2\,.\,n+3}{2\,.\,4\,.\,6\,(kr)^3} + \dots$$
$$+ (-)^{\frac{1}{2}(n-2)} \frac{2\,.\,3\,.\,4\,\dots\,2n-1}{2\,.\,4\,.\,6\,\dots\,2n-2} \frac{1}{(kr)^{n-1}}. \tag{30}$$

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* *Phil. Trans.* (1868), *Papers*, Vol. 4, p. 299; Lamb, *Hydrodynamics* (1916), p. 502; Rayleigh, *Theory of Sound*, Vol. 2, pp. 205 *et seq.*

† The series for g_n and h_n will vary according as n is odd or even. In the above n is taken as even. A similar analysis for n odd leads to exactly the same results, but the series defining g_n and h_n are slightly different. For example, the last term in g_n becomes

$$(-)^{\frac{1}{2}(n-1)} \frac{2\,.\,3\,.\,4\,\dots\,2n-1}{2\,.\,4\,.\,6\,\dots\,2n-2} \frac{1}{(kr)^{n-1}}.$$

The real part of ϕ can then be written

$$\phi = (-)^{\frac{1}{2}n} | C_n | \frac{g_n \cos \omega + h_n \sin \omega}{(kr)^{n+1}} r^n . S_n, \tag{31}$$

where $\omega = k(ct - r + \epsilon)$, and $k\epsilon$ is the argument of C_n.

8. *The Potential and Kinetic Energies.*

Using the value of ϕ given by (25) and taking the mean value over a long period of time, we find that

$$V_1 = \frac{1}{2} \frac{\rho_0}{c^2} | C_n |^2 \sigma^2 \frac{g_n^2 + h_n^2}{2(kr)^{2n+2}} r^{2n} S_n^2,$$

where V_1 denotes potential energy per unit volume, at a distance r from the centre of the sphere.

Integrating over a spherical surface of radius r, we are led to the following expression for the mean potential energy of a spherical shell of radius r and thickness δr,

$$\delta V = \frac{1}{4} \rho_0 | C_n |^2 \frac{g_n^2 + h_n^2}{k^{2n}} \iint S_n^2 \, d\varpi \, dr. \tag{32}$$

The Kinetic Energy per unit volume of the medium is given by

$$T_1 = \frac{1}{2} \rho_0 \left\{ \left(\frac{\partial \phi}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \phi}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial \phi}{\partial \omega} \right)^2 \right\}.$$

We note that
$$\frac{\partial \phi}{\partial r} = C_n \{ kr f_n'(kr) + n f_n(kr) \} r^{n-1} S_n e^{i\sigma t}$$
$$= C_n \{ f_{n-1}(kr) - (n+1) f_n(kr) \} r^{n-1} S_n e^{i\sigma t}, \tag{33}$$

since
$$kr f_n'(kr) + (2n+1) f_n(kr) = f_{n-1}(kr). \tag{34}$$ *

The real part of $f_n(kr) e^{i\sigma t}$ is

$$(-)^{\frac{1}{2}n} \frac{g_n \cos(\sigma t - kr) + h_n \sin(\sigma t - kr)}{(kr)^{n+1}}, \tag{35}$$

and that of $f_{n-1}(kr) e^{i\sigma t}$ is

$$(-)^{\frac{1}{2}(n-1)} \frac{h_{n-1} \cos(\sigma t - kr) - g_{n-1} \sin(\sigma t - kr)}{(kr)^n}. \tag{36}$$

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* Lamb (*loc. cit.*), p. 499.

Hence the mean value of $(\partial\phi/\partial r)^2$ over a long period of time is

$$\tfrac{1}{2}\,|C_n|^2[\{(n+1)g_n+kr.h_{n-1}\}^2+\{(n+1)\,h_n-kr.g_{n-1}\}^2]\,\frac{r^{2n-2}}{(kr)^{2n+2}}S_n^2. \quad (37)$$

Similarly the mean value of $\dfrac{1}{r^2}\left(\dfrac{\partial\phi}{\partial\theta}\right)^2+\dfrac{1}{r^2\sin^2\theta}\left(\dfrac{\partial\phi}{\partial\omega}\right)^2$ is

$$\tfrac{1}{2}\,|C_n|^2\,(g_n^2+h_n^2)\,\frac{r^{2n-2}}{(kr)^{2n+2}}\left\{\left(\frac{\partial S_n}{\partial\theta}\right)^2+\frac{1}{\sin^2\theta}\left(\frac{\partial S_n}{\partial\omega}\right)^2\right\}. \quad (38)$$

Integrating the expressions (37) and (38) over the surface of a sphere of radius r will give us the kinetic energy of the medium of a spherical shell of thickness δr and radius r. Remembering that

$$\int_0^{2\pi}\int_0^{\pi}\left\{\left(\frac{\partial S_n}{\partial\theta}\right)^2+\left(\frac{\partial S_n}{\sin\theta\,\partial\omega}\right)^2\right\}\sin\theta\,d\theta\,d\omega = n(n+1)\int_0^{2\pi}\int_0^{\pi}S_n^2\sin\theta\,d\theta\,d\omega, \quad (39)$$

we get

$$\delta T=\frac{\frac{1}{4}\rho_0\,|C_n|^2}{k^{2n}k^2r^2}[\{(n+1)g_n+kr.h_{n-1}\}^2+\{(n+1)h_n-kr.g_{n-1}\}^2$$
$$+n(n+1)(g_n^2+h_n^2)]\,\delta r\int_0^{2\pi}\int_0^{\pi}S_n^2\sin\theta\,d\theta\,d\omega. \quad (40)$$

The expressions for V and T previously obtained for a pulsating and vibrating sphere are easily obtained by putting $n=0$ and $n=1$ respectively in equations (32) and (40).

The case of $n=2$ is here calculated by this method for purposes of illustration. In this case, we have

$$g_n = 1-\frac{3}{(kr)^2},\quad h_n=\frac{3}{kr},\quad g_{n-1}=1,\quad h_{n-1}=\frac{1}{kr} \quad (41)$$

Assuming that the vibration is symmetrical about an axis so that S_2 may be replaced by a zonal harmonic P_2, we have, using equation (32),

$$\delta V=\tfrac{1}{4}\rho\,\frac{|C_2|^2}{k^6}\left\{\left(1-\frac{3}{k^2r^2}\right)^2+\frac{9}{k^2r^2}\right\}\tfrac{4}{5}\pi.\delta r,$$

since
$$\iint P_n^2\,d\varpi=\frac{4\pi}{2n+1}.$$

Hence

$$V=\frac{\pi}{5}\rho_0\,|C_2|^2\frac{a}{k^4}\left\{\left(\frac{r}{a}-1\right)+\frac{3}{k^2a^2}\left(1-\frac{a}{r}\right)+\frac{3}{k^4a^4}\left(1-\frac{a^3}{r^3}\right)\right\}. \quad (42)$$

Similarly,

$$T = \frac{\pi}{5}\rho_0 \,|\, C_2 \,|^2 \frac{a}{k^4} \left\{ \left(\frac{r}{a} - 1 \right) + \frac{4}{k^2 a^2} \left(1 - \frac{a}{r} \right) + \frac{9}{k^4 a^4} \left(1 - \frac{a^3}{r^3} \right) \right.$$
$$\left. + \frac{27}{k^6 a^6} \left(1 - \frac{a^5}{r^5} \right) \right\}. \tag{43}$$

9. *Special Cases.*

A consideration of the special cases mentioned in § 4 leads to some interesting results.

CASE (i).—*Let the radius of the sphere be small compared with the wave length; i.e.* suppose *ka small.* If at the same time kr is small, we get

$$g_n = (-)^{\frac{1}{2}n} \frac{1.2.3 \ldots 2n}{2.4.6 \ldots 2n} \frac{1}{(kr)^n},$$

whilst h_n, g_{n-1}, and h_{n-1} are very small compared with g_n. Hence

$$V = \tfrac{1}{4}\rho_0 \,|\, C_n \,|^2 \frac{1}{k^{2n}} \int_a^r g_n^2 \, dr \iint S_n^2 \, d\varpi,$$

$$T = \tfrac{1}{4}\rho_0 \,|\, C_n \,|^2 \frac{1}{k^{2n+2}} \int_a^r (n+1)(2n+1) \frac{g_n^2}{r^2} \, dr \iint S_n^2 \, d\varpi;$$

and therefore
$$\frac{V}{T} = \frac{k^2 a^2}{(n+1)(2n-1)} \frac{1-(a/r)^{2n-1}}{1-(a/r)^{2n+1}}, \tag{44}$$

which is of the second order of small quantities.

We therefore arrive at the same conclusion as before (§ 4), viz., *that under these conditions the energy is mainly kinetic.*

CASE (ii).—*Suppose ka to be large and r comparable with a.* In this case we have

$$g_n = 1, \quad h_n = \frac{n(n+1)}{2kr}, \quad g_{n-1} = 1, \quad h_{n-1} = \frac{(n-1)n}{2kr},$$

approximately, and

$$\frac{V}{T} = \frac{k^2 \int_a^r g_n^2 \, dr}{\int_a^r \frac{k^2 r^2}{r^2} g_{n-1}^2 \, dr} = 1, \tag{45}$$

since the remaining terms are small. We therefore have

$$V = T = \tfrac{1}{4}\rho_0 \, |C_n|^2 \frac{r-a}{k^{2n}} \iint S_n^2 \, d\varpi.$$

Now $$C_n = \frac{1}{\{(n+1) f_n(ka) - f_{n-1}(ka)\} \, a^{n-1}},$$

from equations (27) and (34).

When ka is large we get, on reduction,

$$|C_n| = k^n . a.$$

Hence $$V = T = \tfrac{1}{4}\rho_0 a^2 (r-a) \iint S_n^2 \, d\varpi,$$

and since the volume of the medium under consideration can be regarded as equal to $4\pi a^2 (r-a)$, we have

$$V = T = \tfrac{1}{4}(\text{vol}) \, \rho_0 \frac{\beta^2}{4\pi} \iint S_n^2 \, d\varpi, \tag{46}$$

where β is the maximum velocity, previously omitted.

It is interesting to compare this with the result for plane waves of sound, viz.,

$$V = T = \tfrac{1}{4} (\text{vol}) \, \rho_0 \beta^2.$$

The formula involves the mean value of S_n^2 taken over the surface of a sphere. In the case of symmetry about an axis, we replace S_n by P_n and we find that the energy per unit volume in this region is $1/(2n+1)$ of that due to plane waves of the same amplitude.

CASE (iii).—*Any value for* ka, $r \to \infty$. The complete expression for the ratio of the total potential to the total kinetic energy is

$$\frac{V}{T} = \frac{k^2 \int_a^r (g_n^2 + h_n^2) \, dr}{\int_a^r [\{(n+1) g_n + kr . h_{n-1}\}^2 + \{(n+1) h_n - kr . g_{n-1}\}^2 + n(n+1)(g_n^2 + h_n^2)] \frac{dr}{r^2}}. \tag{47}$$

Now $$g_n^2 + h^2 = 1 + \frac{A_2}{r^2} + \frac{A_4}{r^4} + \ldots + \frac{A_{2n}}{r^{2n}},$$

and the integrand in the denominator

$$= k^2 + \frac{B_2}{r^2} + \frac{B_4}{r^4} + \ldots + \frac{B_{2n+2}}{r^{2n+2}}, \tag{48}$$

where A_2, B_2, etc. are constants. Hence

$$\underset{r\to\infty}{\mathrm{Lt}} \frac{V}{T} = \underset{r\to\infty}{\mathrm{Lt}} \frac{k^2 \left\{ (r-a) + A_2 \left(\frac{1}{a} - \frac{1}{r} \right) + \text{etc.} \ldots \right\}}{k^2 (r-a) + B_2 \left(\frac{1}{a} - \frac{1}{r} \right) + \text{etc.}} = 1. \tag{49}$$

This result may be arrived at by a more direct analysis, but, for convenience, this has been put separately as an appendix.*

10. *The extent of the region in which the Potential Energy may be neglected.*

When the radius of the sphere was supposed small in comparison with the wave length, we found that the energy stored in the fluid owing to the alternate condensations and rarefactions was very small compared with the energy due to the actual motion of the medium. We proceed to examine the extent of the region throughout which these conditions hold good.

In the analysis referred to above [§ 9, Case (i)] it was assumed that the ratio r/a was never very large. If we abandon this restriction as to the value of r, we note that, in the expression for V/T in equation (47) above we may neglect h_n and h_{n-1} in comparison with g_n and $kr \,.\, g_{n-1}$, and write

$$g_n = 1 + (-)^{\frac{1}{2}n} \frac{1.2.3 \ldots 2n}{2.4.6 \ldots 2n} \frac{1}{(kr)^n}, \tag{50}$$

$$g_{n-1} = 1 + (-)^{\frac{1}{2}(n-1)} \frac{3.4 \ldots 2n-2}{2.4 \ldots 2n-4}. \tag{50.1}$$

For, we find that the terms h_n and h_{n-1} appear in the expression for V/T only in the form $\int h_n^2 \, dr$, which is equivalent to a series

$$\frac{B_1}{ka}\left(1 - \frac{a}{r}\right) + \frac{B_3}{(ka)^3}\left(1 - \frac{a^3}{r^3}\right) + \ldots + \frac{B_{2n-3}}{(ka)^{2n-3}}\left(1 - \frac{a^{2n-3}}{r^{2n-3}}\right).$$

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Since ka is small, it follows that the largest term, viz., the last, is small compared with the corresponding term in $\int g_n^2\, dr$.

Substituting for g_n and g_{n-1} the values given above in equations (50) and (50 . 1), we have

$$\frac{V}{T} = \frac{k^2 \int_a^r g_n^2\, dr}{\int_a^r \{k^2 r^2 g_{n-1}^2 + (n+1)(2n+1) g_n^2\} \frac{dr}{r^2}}$$

$$= \frac{ka\left(\frac{r}{a} - 1\right) + \frac{A_n^2}{(2n-1)(ka)^{2n-1}}\left(1 - \frac{a^{2n-1}}{r^{2n-1}}\right)}{ka\left(\frac{r}{a} - 1\right) + \frac{(n+1)A_n^2}{(ka)^{2n+1}}\left(1 - \frac{a^{2n+1}}{r^{2n+1}}\right)}, \tag{51}$$

where
$$A_n = \frac{1.2.3 \ldots 2n}{2.4.6 \ldots 2n}.$$

It is now clear that retaining more terms in the expression for g_n and g_{n-1} would only introduce terms of the type

$$\frac{A_m^2}{(2m-1)(ka)^{2m-1}}\left\{1 - \frac{a^{2m-1}}{r^{2m-1}}\right\},$$

where $m < n$, and such terms are small compared with the factor of A_n^2.

When r/a is not large, the first term in both numerator and denominator of (51) is small compared with the other, and the expression reduces to that obtained before in § 9. But values of r/a may evidently be chosen which will make the first term, viz. kr, the predominant term in the numerator but yet not of more importance than the second term in the denominator. For, obviously, when r/a is large, the expression for V/T may be written

$$\frac{V}{T} = \frac{kr(ka)^{2n+1} + \frac{1}{2n-1} A_n^2 (ka)^2}{kr(ka)^{2n+1} + (n+1) A_n^2},$$

and there will be a range of values for r for which we may write

$$\frac{V}{T} = \frac{kr(ka)^{2n+1}}{kr(ka)^{2n+1} + (n+1) A_n^2}. \tag{52}$$

From this expression we may deduce the range of values of r for which the quantity V/T has any assigned small value. Thus we observe:—

(1) If V/T is to be small, then $kr(ka)^{2n+1}$ is to be small compared with the absolute term $(n+1)A_n^2$.

(2) The order of the expression V/T is determined by the order of $kr(ka)^{2n+1}$, and hence *vice versa*, if V/T is to be of order 10^{-c}, then $kr(ka)^{2n+1}$ is to be of order $10^{-c}(n+1)A_n^2$.

(3) Assuming that ka (or $2\pi a/\lambda$) is to be of order 10^{-b} we deduce that kr must be of order $10^{-c}(n+1)A_n^2 \div 10^{-(2n+1)b}$. Hence kr is to be of order $10^{(2n+1)b-c}A_n^2(n+1)$, or, alternatively, r/a must not be of higher order than $10^{2(n+1)b-c}A_n^2(n+1)$.

A numerical example will help to make the matter clear. Thus, supposing that the waves propagated are such that the ratio of the circumference of the sphere to the wave length is 10^{-2}, the extent of the region corresponding to $V/T = 10^{-3}$ is given by $r/a = 10^{4n+1}A_n^2(n+1)$, since $b = 2$, $c = 3$.

In the three cases $n = 0$, 1, and 2, we get respectively $r/a = 10$, 2×10^5, and 27×10^9, in agreement with the values in the Table.

11. A table has been drawn up showing the value of the expression V/T for the special cases of $n = 0$, 1, and 2. The case $n = 0$, of course, represents the case of a pulsating sphere vibrating radially, $n = 1$ that of a sphere moving in simple harmonic motion. The formulæ for V/T in the three cases are :—

$$n=0,\ \frac{V}{T}=\frac{k^2a^2\left(\frac{r}{a}-1\right)}{k^2a^2\left(\frac{r}{a}-1\right)+\left(1-\frac{a}{r}\right)},$$

$$n=1,\ \frac{V}{T}=\frac{k^4a^4\left(\frac{r}{a}-1\right)+k^2a^2\left(1-\frac{a}{r}\right)}{k^4a^4\left(\frac{r}{a}-1\right)+2k^2a^2\left(1-\frac{a}{r}\right)+2\left(1-\frac{a^3}{r^3}\right)},$$

$$n=2,\ \frac{V}{T}=\frac{k^6a^6\left(\frac{r}{a}-1\right)+3k^4a^4\left(1-\frac{a}{r}\right)+3k^2a^2\left(1-\frac{a^3}{r^3}\right)}{k^6a^6\left(\frac{r}{a}-1\right)+4k^4a^4\left(1-\frac{a}{r}\right)+9k^2a^2\left(1-\frac{a^3}{r^3}\right)+27\left(1-\frac{a^5}{r^5}\right)}.$$

In conclusion the writer would like to express his best thanks to Prof. H. Lamb whose unremitting kindness and helpful criticism have been most valuable during the writing of this paper.

TABLE.

ka	r/a	V/T		
		$n = 0$	$n = 1$	$n = 2$
·01	2	·00020	·00003	·00001
	10	·00099	·00004	·00001
	10^2	·0098	·00005	·00001
	10^3	·0909	·00005	·00001
	10^4	·5000	·00010	·00001
	10^5	·9090	·00055	·00001
	10^6	·9900	·0050	·00001
	10^7	·9990	·0476	·00001
	10^8	·9999	·3334	·00001
	10^9	1·0000	·8333	·00005
	10^{10}	1·0000	·9804	·00036
	10^{11}	1·0000	·9980	·0035
	10^{12}	1·0000	·9998	·0357
	10^{13}	1·0000	1·0000	·2703
	10^{14}	1·0000	1·0000	·7884
	10^{15}	1·0000	1·0000	·9737
·1	2	·0196	·0029	·0009
	10	·082	·0049	·0011
	10^2	·500	·0097	·0011
	10^4	·999	·3344	·0015
	10^6	1·000	·9804	·0368
	10^8	1·000	1·0000	·7874
	10^{10}	1·000	1·0000	·9973
1	2	·666	·310	·135
	10	·909	·790	·302
	10^2	·990	·971	·755
	10^3	·999	·997	·967
10	2	·995	·9949	·9944

APPENDIX.

*On the Relation between the Mean Kinetic and Potential Energies of the Wave-System due to a Vibrating Body.**

1. It is a well known property of plane waves of sound that the mean

* In the first draft of this paper, an analytical proof of the theorem which follows was given applicable to a spherical surface only. That the theorem was of more general application was pointed out by Prof. H. Lamb, to whom the main steps of the following proof are due.

kinetic and potential energies are equal, whatever be the extent of the medium considered. In the foregoing paper an analogous property has been found to hold for the system of waves due to the vibration of a sphere, viz. that the *ratio* of the potential to the kinetic energy *tends to the limit unity* when infinite space is considered. It will be shown presently that the same conclusion holds independently of the form of the vibrating body.

This does not imply that the amounts of the two kinds of energy are actually equal. In fact, the analysis brings to light the fact that *although the potential and kinetic energies both tend to become infinite, yet there is always a finite difference between them.*

2. The expression for the kinetic energy of the fluid contained in any given region is given by

$$2T = \rho \iiint \left\{ \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 \right\} dx\,dy\,dz. \tag{1}$$

By Green's theorem we may write this in the form

$$2T = -\rho \iint \phi \frac{\partial\phi}{\partial n}\, dS - \rho \iiint \phi \nabla^2 \phi\, dx\,dy\,dz, \tag{2}$$

where the triple integral is taken throughout the region and the surface integral over its boundary. The gradient $\partial\phi/\partial n$ is towards the interior of the region.

The general equation of sound waves of simple harmonic type is

$$(\nabla^2 + k^2)\,\phi = 0,$$

where $k = \sigma/c$, σ being the frequency, and c, as usual, the velocity of propagation. Hence

$$2T = -\rho \iint \phi \frac{\partial\phi}{\partial n}\, dS + \rho k^2 \iiint \phi^2\, dx\,dy\,dz. \tag{3}$$

The expression for the Potential Energy is

$$2V = \iiint \kappa s^2\, dx\,dy\,dz = \frac{\rho}{c^2} \iiint \left(\frac{\partial\phi}{\partial t}\right)^2 dx\,dy\,dz. \tag{4}$$

Now when the waves are due to a surface vibrating in simple harmonic motion, we can write

$$\phi = P \cos \sigma t + Q \sin \sigma t, \tag{5}$$

where P and Q are functions of position only.

Hence the value of ϕ^2 at any point in space taken over a long period of time is equal to $\frac{1}{2}(P^2+Q^2)$. Similarly the mean value of $(\partial\phi/\partial t)^2$ is $\frac{1}{2}\sigma^2(P^2+Q^2)$. The mean values of T and V are therefore

$$2T = -\tfrac{1}{2}\rho \iint \left(P\frac{\partial P}{\partial n}+Q\frac{\partial Q}{\partial n}\right) dS + \tfrac{1}{2}\rho k^2 \iiint (P^2+Q^2)\, dx\, dy\, dz,$$

$$2V = \tfrac{1}{2}\rho k^2 \iiint (P^2+Q^2)\, dx\, dy\, dz,$$

so that

$$4(T-V) = -\iint \left(P\frac{\partial P}{\partial n}+Q\frac{\partial Q}{\partial n}\right) dS. \tag{6}$$

This expression is quite general no matter what part of the fluid be considered. If we apply it to the infinite region surrounding the vibrating body, the surface integral is to be taken over the surface of the body and over a sphere of infinite radius.

Now the velocity potential at infinity is of the order e^{-ikr}/r at most. Hence at infinity

$$\phi\frac{\partial\phi}{\partial n} = \frac{C^2\{\cos^2 k(r-ct)+kr\cos k(r-ct)\sin k(r-ct)\}}{r^3},$$

the mean value of which is $C^2/2r^3$. The part of the surface integral due to the infinitely distant spherical boundary therefore vanishes.

On the other hand, the volume integral tends to become infinite, for P^2+Q^2 is of order $1/r^2$ and $dx\,dy\,dz$ is equal to $r^2 d\omega\, dr$. Hence, whereas T and V both become infinite for infinite space, $T-V$ approaches a finite lower limit.

3. In the case of the vibrating sphere, equation (6) can be written

$$T-V = -\frac{\rho_0}{8}\iint \frac{\partial}{\partial n}|F_n|^2 S_n^2 r^2 \sin\theta\, d\theta\, d\omega$$

$$= \frac{\rho_0}{8}\left[r^2\frac{\partial}{\partial r}|F_n|^2\right]_a^\infty \iint S_n^2 \sin\theta\, d\theta\, d\omega,$$

since, on the outer surface, *i.e.* the infinite one

$$\frac{\partial F_n}{\partial n} = -\frac{\partial F_n}{\partial r}.$$

In the notation of § 7, we have

$$F_n = \frac{i^n e^{-ikr}}{k^n(kr)}\{g_n - ih_n\},$$

and the mean value with respect to time of

$$|F_n e^{\iota\sigma t}|^2 = \tfrac{1}{2}|C_n|^2 \frac{g_n^2+h_n^2}{k^{2n}(kr)^2}.$$

Hence in the particular cases, corresponding to $n = 0$, 1, and 2, we get

$$n=0, \qquad T-V = \frac{|C_0|^2\rho_0}{4k^2a}\iint S_0^2\,d\varpi,$$

$$n=1, \qquad T-V = \frac{|C_1|^2\rho_0}{4k^4a}\left(1+\frac{2}{k^2a^2}\right)\iint S_1^2\,d\varpi,$$

$$n=2, \qquad T-V = \frac{|C_2|^2\rho_0}{4k^6a}\left(1+\frac{6}{k^2a^2}+\frac{27}{k^4a^4}\right)\iint S_2^2\,d\varpi.$$

These values confirm those obtained in § 8.

ON A GENERALISATION OF LAGRANGE'S SERIES

By M. KÖSSLER.

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THE solution due to Lagrange of equation (2.1) of this paper gives only one root of the equation. By forming the slightly modified equations (2.3), (3.1), and (4.2), we get other roots, and, in some cases, *all* the roots of the original equation.

One of the consequences of this result is that we are thereby enabled to solve any given algebraic equation by means of series of polynomials; I therefore hope that the contents of this paper are of some interest. The methods are a novel application of the method of variable parameters, which has proved to be a powerful weapon in attacking the theory of integral equations.

2. Lagrange's solution of the equation

$$x-a-uf(x) = 0 \tag{2.1}$$

is given by the formula

$$x = a+\sum_{m=1}^{\infty} a_m u^m, \qquad a_m = \frac{1}{m!}\left[\frac{d^{m-1}}{dx^{m-1}}\{f(x)\}^m\right]_{x=a}, \tag{2.2}$$

when $f(x)$ is a function of x, which is analytic at the point $x = a$, such that $f(a) \neq 0$. The radius of convergence of the series may be determined without difficulty.

Two generalisations of this expansion are possible. In the case of the first, we take the equation to be

$$(x-a)^n-uf(x) = 0, \tag{2.3}$$

or

$$u = \frac{(x-a)^n}{f(x)}.$$

By writing $\qquad f(x) = f(a)+(x-a)f'(a)+\dots,$

we get
$$u = (x-a)^n \left[\frac{1}{f(a)} + \mathfrak{p}(x-a)\right],$$

where $\mathfrak{p}$ denotes a power series. When this expansion is reverted, $x-a$ is expressed as a function of u with a branch-point at $u=0$. We thus obtain n values of x, say $x_0, x_1, \ldots, x_{n-1}$, where

$$x_k - a = \sum_{m=1}^{\infty} a_m u_k^m \quad (k = 0, 1, 2, \ldots, n-1), \tag{2.4}$$

$$u_k = u^{1/n} e^{2k\pi i/n},$$

and
$$u^{1/n} = |u^{1/n}|\, e^{i\phi} \quad (0 < \phi < 2\pi/n).$$

To evaluate the coefficients a_m, we write equation (2 . 3) in the form

$$x_k - a - u_k f^{1/n}(x_k) = 0,$$

whence, by (2 . 2), we have

$$a_m = \frac{1}{m!}\left[\frac{d^{m-1}}{dx^{m-1}} \{f(x)\}^{m/n}\right]_{x=a}. \tag{2.5}$$

3. It is now possible to solve the equation

$$\phi(x) - uf(x) = 0, \tag{3.1}$$

where $\phi(x)$ and $f(x)$ are functions of x which are both analytic in a well-defined region of the x-plane, if the roots of the equation $\phi(x) = 0$ are supposed known. If these roots are $a_1, a_2, \ldots, a_n$, of multiplicities $r_1, r_2, \ldots, r_n$ respectively, and if the functions $\phi(x)$ and $f(x)$ have no common zeros, then we transform equation (3 . 1) into

$$(x-a_k)^{r_k} - u\frac{(x-a_k)^{r_k}}{\phi(x)} f(x) = 0.$$

In this equation, the coefficient of u is analytic at a_k, and it does not vanish at that point. Hence, by the formula of § 2, we obtain r_k roots of the equation, and then, by putting $k = 1, 2, \ldots, n$, we get n sets of roots of equation (3 . 1).

The radii of convergence of the series (2 . 2) and (2 . 4) are given by the distance of the point $u = 0$ from the nearest singularity of the functions inverse to

$$u = \frac{x-a}{f(x)}, \quad u = \frac{(x-a)^n}{f(x)},$$

respectively.

When, as is frequently the case, we can solve the equation

$$\frac{du}{dx} = 0,$$

the radius of convergence is obtained by taking the roots $\beta_1, \beta_2, \ldots$ of this equation and constructing the set of expressions

$$u_l = \frac{\beta_l - a}{f(\beta_l)}, \qquad u_l^{1/n} = \frac{\beta_l - a}{\{f(\beta_l)\}^{1/n}},$$

in the respective cases, and selecting that one which has the smallest modulus; the modulus in question is the radius of convergence.

We apply to the expansions now obtained the well known theorem, due to Mittag-Leffler,* by which the power series

$$F(u) = a_0 + a_1 u + a_2 u^2 + \ldots$$

is transformed into a series of polynomials

$$\text{(M)} \qquad F(u) = \sum_{k=1}^{\infty} P_k(u),$$

where the coefficients in the polynomials P_k are linear functions of the coefficients $a_0, a_1, a_2, \ldots$. This series is convergent throughout Mittag-Leffler's *star* (étoile).

The application of Mittag-Leffler's transformation to the generalisations of Lagrange's series leads directly to the solution of the algebraic equation.

4. Let $f(x, y)$ be an analytic function of both of the variables x, y, and suppose that there exists a constant a such that the roots of the equation in x,

$$f(x, a) = 0,$$

are known; let these roots be $a_1, a_2, \ldots, a_n$.

Suppose also that the roots of the equation in x,

$$(4.1) \qquad f(x, y) = 0,$$

are not independent of the variable y.

To solve the last equation we consider the modified equation

$$(4.2) \qquad f(x, a) - u[f(x, a) - f(x, y)] = 0,$$

* *Acta Mathematica*, Vol. 23 (1899), pp. 43 *et seq.*

which reduces to (4.1) when $u = 1$. By (2.2), a solution of the last equation, valid near $u = 0$, is

$$(4.3) \qquad x_k = a_k + \sum_{m=1}^{\infty} a_m u^m,$$

$$a_m = \frac{1}{m!}\left[\frac{d^{m-1}}{dx^{m-1}}\left(\frac{x-a_k}{f(x,\, a)}\right)^m \{f(x,\, a)-f(x,\, y)\}^m\right]_{x=a_k},$$

provided that a_k is a simple zero of $f(x, a)$; the modification to be made in the case of a multiple zero is evident.

The circle of convergence of (4.3) either does or does not contain the point $u = 1$. If it does, we may calculate n roots of the equation (4.1) by putting $u = 1$. If it does not, we must transform the power series by using the formula (M).

This transformation cannot fail by reason of the point $u = 1$ being a summit of the star of convergence, provided that equation (4.2) has no multiple roots in x, for the system

$$u \equiv \frac{f(x,\, a)}{f(x,\, a)-f(x,\, y)} = 1, \qquad \frac{du}{dx} = 0,$$

which forms the conditions that $u = 1$ should be a summit of the star, is equivalent to the system

$$f(x,\, y) = 0, \qquad f_x(x,\, y) = 0,$$

and this system is not satisfied if there is no multiple root.

5. Now take any trinomial equation

$$(5.1) \qquad x^n - u(ax+1) = 0,$$

in which n is a positive integer.

The formulæ (2.4), (2.5) give immediately all the roots of the equation in the form

$$(5.2) \qquad x_k = \frac{1}{a} \sum_{m=1}^{\infty} \frac{1}{m}\binom{m/n}{m-1} a^m u^{m/n} e^{2km\pi i/n},$$

where $\quad u^{1/n} = |u^{1/n}|\, e^{i\phi}, \quad 0 \leqslant \phi < 2\pi/n, \quad k = 1, 2, \ldots, n.$

The roots are algebraic functions of u whose only singularities are at the branch-points, which are given as the solutions of the system

$$u = \frac{x^n}{ax+1}, \qquad \frac{du}{dx} \equiv \frac{(n-1)\, ax^n + nx^{n-1}}{(ax+1)^2} = 0.$$

The only value of u besides zero which satisfies this system is

$$u = -\frac{(-n)^n}{a^n(n-1)^{n-1}}.$$

Hence the series (5 . 2) is convergent when

$$|u| < \frac{n^n}{|a|^n(n-1)^{n-1}} = \rho.$$

If u does *not* satisfy this inequality, we put

$$x = \frac{1}{y}, \quad u = \frac{1}{v},$$

so that (5 . 1) transforms into

$$y^{n-1}(y+a)-v = 0.$$

When $v = 0$, the roots of this equation are 0 and $-a$, the former having multiplicity $n-1$. Hence in the neighbourhood of $v = 0$ we obtain the solutions

$$(5\,.\,3) \qquad y_k = a\sum_{m=1}^{\infty}\binom{-m/(n-1)}{m-1}\frac{v^{m/(n-1)}}{m\,a^{mn/(n-1)}}e^{2km\pi i/(n-1)}$$

$$(k = 1, 2, \ldots, n-1),$$

$$y_n = -a+\sum_{m=1}^{\infty}\binom{-m(n-1)}{m-1}\frac{(-1)^{mn-1}v^m}{m\,a^{mn-1}}.$$

It is easy to verify that these series converge when

$$|v| < \frac{|a|^n(n-1)^{n-1}}{n^n} = \frac{1}{\rho},$$

i.e. when $|u| < \rho$.

We have thus obtained the fundamental theorem:

The roots of the trinomial equation (5 . 1) *are given by* (5 . 2) *when** $|u| \leqslant \rho$, *and they are given by* (5 . 3) *when* $|u| \geqslant \rho$, *if* $x_k = 1/y_k$.

The only case of exception occurs when

$$u = -\frac{(-n)^n}{a^n(n-1)^{n-1}},$$

* It has been proved by Riesz, *Palermo Rendiconti*, t. 30 (1910), pp. 339–345, that such a series is convergent on the circumference of the circle of convergence.

but, in this case, the equation has a repeated root

$$x = \frac{-n}{a(n-1)},$$

and the degree is reducible by elementary methods.

The convergence is sufficiently rapid for numerical applications whenever $|u|$ is appreciably less than or greater than ρ.

The special case in which $n = 5$, $u = -1$, gives the solution of the general quintic equation when reduced to the trinomial form by the method of Bring and Jerrard. The method just described is obviously simpler than Hermite's well known solution of the quintic equation.

6. Now take the general algebraic equation in the form

$$x^n - u(c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_n) = 0, \tag{6.1}$$

where n is a positive integer, and $c_1, c_2, \ldots, c_n$ are constants of which c_n is not zero. By formulæ (2 . 4) and (2 . 5), the solution is

$$x_k = \sum_{m=1}^{\infty} a_m u^{m/n} e^{2km\pi i/n} \quad (k = 1, 2, \ldots, n), \tag{6.2}$$

$$a_m = \frac{1}{m!}\left[\frac{d^{m-1}}{dx^{m-1}}(c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_n)^{m/n}\right]_{x=0}.$$

To determine the radius of convergence, we have to solve the equation

$$nf(x) - xf'(x) = 0, \tag{6.3}$$

where $$f(x) = c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_n,$$

and construct the set of expressions

$$u = x^n / f(x), \tag{6.4}$$

where x is given the values of these roots in turn; we then select that value of u which has the smallest modulus; and the modulus in question is the radius of convergence.

This procedure evidently involves the solution of an algebraic equation of degree $n-1$.

The values of u which are determined by (6 . 3) and (6 . 4) are the only singularities of the functions x_k defined by the series. It is therefore possible to construct the star for each of the functions x_k, and then transform the power series into the expansions of polynomials (M), which are convergent at all points of the star with the exception of points on the

boundary. But it has been shown by Painlevé* that it is possible to effect a transformation of the expansions (M), such that the transformed expansions converge at all points of the star, including points on the boundary, with the sole exception of the summits of the star.

For values of u which correspond to one of the summits, the equation (6.1) has a repeated factor, and it is consequently reducible.

Hence, for all values of u, *the equation* (6.1) *has been solved by an expression of the form*

$$x_k = \sum_{m=1}^{\infty} P_m(u^{1/n} e^{2k\pi i/n}) \quad (k = 1, 2, \ldots, n), \tag{6.5}$$

where the coefficients in the polynomials P_m *are linear functions of the coefficients* a_m *of equation* (6.2).

The formation of the expansion (6.5) does not depend upon the critical values of u. Hence, if the variable u is so chosen that equation (6.1) has no repeated roots, the form of the solution given by (6.5) is independent of the solution of an equation of lower degree. If the coefficients $c_1, c_2, \ldots, c_n$ in the equation are not constants, but functions of a variable, the same remark holds good.

It is evident that this solution of the general algebraic equation is complicated and it is not adapted for numerical applications, though it is simple and short in comparison with the solution due to F. Lindemann.†

7. The application of the general formulæ to equations involving integral transcendental functions leads to interesting results, but in this paper I shall confine myself to stating two formal examples.

(I) Let $f(x)$ be an integral function with simple zeros, none of which has any of the values $0, \pm 1, \pm 2, \ldots$. The equation

$$\sin \pi x - u[\sin \pi x - f(x)] = 0 \tag{7.1}$$

is of the form (2.1). We thus obtain the solution

$$x_k = k + (-1)^{k+1} \frac{f(k)}{\pi} u + \frac{(-1)^k \pi - f'(k)}{2\pi^2} u^2 + \ldots$$
$$(k = 0, \pm 1, \pm 2, \ldots).$$

If the radii of convergence of these power series are different from zero,‡

* Cf. Borel, *Leçons sur les fonctions de variables réelles* (1905), Note 1, pp. 140–145.

† *Nachrichten der k. Ges. der Wiss. Göttingen*, 1884, p. 245.

‡ This is by no means an essential restriction.

we can transform them into polynomial expansions (M) which are valid at the point $u = 1$; we have thus calculated an infinite set of zeros of the equation

$$f(x) = 0.$$

(II) Consider the equation

$$P(x) - ue^{Q(x)} = 0, \tag{7.2}$$

where

$$P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n,$$

$$Q(x) = b_0 x^p + b_1 x^{p-1} + \ldots + b_p.$$

When $u = 0$, this equation has n roots $a_1, a_2, \ldots, a_n$, and therefore, by (2.2),

$$x_k = a_k + \sum_{m=1}^{\infty} a_m^{(k)} u^m \quad (k = 1, 2, \ldots, n). \tag{7.3}$$

The radii of convergence and the Mittag-Leffler stars of these series can be constructed by solving the equation

$$\frac{du}{dx} = 0,$$

which, when written in the form

$$P'(x) - P(x)\,Q'(x) = 0,$$

is obviously algebraic, and substituting the roots in

$$u = \frac{P(x)}{e^{Q(x)}}.$$

But the n roots (7.3) are, of course, not all of the roots of the proposed equation. We therefore form from (7.2)

$$Q(x) = \log P(x) - \log u \pm 2k\pi i = \log P(x) - v,$$

where $\log P(x)$ denotes any definite branch of the multiform function.

We know n roots of the last equation when $v = 0$, and hence we can find a set of n roots for every value of k (by putting $v = \log u \pm 2k\pi i$) by using the expansion (2.2).

The equation

$$P(\sin x, \cos x) - ue^x = 0,$$

where P is a polynomial in both variables, may be treated in a similar

manner, and many similar equations which are soluble by these methods can be constructed without difficulty.

The solution of the last equation is of some theoretical interest, though it is of little use in numerical applications. But a slightly modified method is effective in the asymptotic calculation of zeros of functions of types discussed by G. H. Hardy.* I hope to return to this topic in a subsequent paper.

In conclusion I have to express my thanks to Prof. G. H. Hardy for his kind help, and to Prof. G. N. Watsou for the trouble he has taken by revising the equations and my imperfect English.

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* *Proceedings*, Ser. 2, Vol. 2 (1905), pp. 1–7, 401–431.

ON CERTAIN CLASSES OF MATHIEU FUNCTIONS

By E. G. C. Poole.

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1. *Mathieu's Equation.*

We shall be concerned with the well-known equations which arise when the equation of wave-motions in two dimensions is transformed to confocal coordinates. A full account of the equation and its history will be found in Whittaker and Watson's *Modern Analysis*, Ch. xix, so that the following brief recapitulation will be sufficient to remind the reader of the salient facts.

The equation
$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+k^2 V=0$$

is transformed by putting

$$x+iy = a\cosh(\xi+i\eta),$$

and gives
$$\frac{\partial^2 V}{\partial \xi^2}+\frac{\partial^2 V}{\partial \eta^2}+k^2a^2(\cosh^2\xi-\cos^2\eta)\,V=0.$$

Assuming a normal solution of type

$$V=F(\xi)\,G(\eta),$$

we see that the functions $F(\xi)$ and $G(\eta)$ must satisfy

$$\left.\begin{aligned}\frac{d^2F}{d\xi^2}+(k^2a^2\cosh^2\xi-p)\,F=0\\ \frac{d^2G}{d\eta^2}+(p-k^2a^2\cos^2\eta)\,G=0\end{aligned}\right\}. \tag{1.1}$$

These are reducible to the same form by interchanging the real and imaginary axes in one of them.

We may also put $\lambda = \cosh \xi, \quad \mu = \cos \eta,$

which gives two equations of identical form:

$$\left.\begin{aligned}(\lambda^2-1)\frac{d^2F}{d\lambda^2}+\lambda\frac{dF}{d\lambda}+(k^2a^2\lambda^2-p)F &= 0\\ (1-\mu^2)\frac{d^2G}{d\mu^2}-\mu\frac{dG}{d\mu}+(p-k^2a^2\mu^2)G &= 0\end{aligned}\right\}. \tag{1.2}$$

We can express the original x, y in terms of λ, μ by the formulæ

$$x = a\lambda\mu, \quad y = a\sqrt{\{(\lambda^2-1)(1-\mu^2)\}}.$$

A third form, employed by Lindemann, can be obtained by writing $\zeta = \lambda^2$ (or μ^2),

$$4\zeta(1-\zeta)\frac{d^2u}{d\zeta^2}+2(1-2\zeta)\frac{du}{d\zeta}+(p-k^2a^2\zeta)u = 0, \tag{1.3}$$

where u is written for F or G, as the case may be. Let us consider more particularly the form (1.2).

2. *Group of the Equation.*

Since the equations in λ, μ are identical, we may consider

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+(p-k^2a^2x^2)y = 0.$$

This has regular singularities at $x = \pm 1$, the exponents at these being equal by symmetry. These are found to be 0 and $\frac{1}{2}$. The point at infinity is an irregular singularity of rank unity. We shall write the two fundamental solutions at $x = 1$ in the form

$$\left.\begin{aligned}F_1(1-x) &= 1+a_1(1-x)+\ldots+a_n(1-x)^n+\ldots\\ F_2(1-x) &= \sqrt{(1-x)}\,[1+b_1(1-x)+\ldots+b_n(1-x)^n+\ldots]\end{aligned}\right\}. \tag{2.1}$$

These series are convergent within a circle with centre at $x = 1$, and whose circumference grazes the nearest singularity $x = -1$. The radical is taken positively to begin with, at points on the real axis lying between the singular points.

By symmetry, the solutions at $x = -1$ can be expressed in the form $F_1(1+x)$, $F_2(1+x)$, with a similar convention about the sign of $\sqrt{(1+x)}$.

Now at all points lying within the domain of convergency of *both* sets of solutions, we must be able to express these solutions in terms of two independent ones. Since F_1 and F_2 are independent, we have relations of the form

$$\left.\begin{aligned} F_1(1-x) &= \alpha F_1(1+x)+\beta F_2(1+x) \\ F_2(1-x) &= \gamma F_1(1+x)+\delta F_2(1+x) \end{aligned}\right\}, \tag{2.2}$$

provided that $\quad |1-x| < 2, \quad |1+x| < 2.$

Changing the sign of x and repeating the substitution, we must have identically

$$\left.\begin{aligned} F_1(1-x) &\equiv (\alpha^2+\beta\gamma)\, F_1(1-x)+\beta(\alpha+\delta)\, F_2(1-x) \\ F_2(1-x) &\equiv \gamma\, (\alpha+\delta)\, F_1(1-x)+(\beta\gamma+\delta^2)\, F_2(1-x) \end{aligned}\right\}. \tag{2.3}$$

Hence
$$\left.\begin{aligned} \alpha^2+\beta\gamma = \beta\gamma+\delta^2 &= 1 \\ \beta(\alpha+\delta) = \gamma(\alpha+\delta) &= 0 \end{aligned}\right\}. \tag{2.4}$$

We must therefore have either

$$\text{(i)} \quad \alpha = \delta, \qquad \beta = 0, \qquad \gamma = 0, \qquad \alpha^2 = 1,$$

or
$$\text{(ii)} \quad \alpha = -\delta, \qquad \beta\gamma = 1-\alpha^2.$$

We shall discuss these cases *seriatim*.

3. *Discussion of Particular Cases.*

(i*a*) $\alpha = \delta = +1, \ \beta = \gamma = 0.$

In this case
$$\left.\begin{aligned} F_1(1-x) &\equiv F_1(1+x) \\ F_2(1-x) &\equiv F_2(1+x) \end{aligned}\right\}, \tag{3.1}$$

provided $|1 \pm x| < 2$, which is certainly true if $|x| < 1$. Now the solutions F_1, F_2 being independent, these equations imply that there are two independent *even* solutions of the equation. The one, which is expressible in the form $F_1(1-x)$ near $x = 1$, and in the form $F_1(1+x)$ near $x = -1$, has no singularity in the finite part of the plane, and is therefore an even integral function. The other, by similar reasoning, takes the form

$$\sqrt{(1-x^2)} \times (\text{even integral function of } x).$$

Now, by considering the solutions near the origin, which is an ordinary

point, we see that there is only *one even* and *one odd* series in x satisfying the equation. Hence the present hypothesis, which requires the existence of *two independent even* solutions, must be rejected.

(i*b*) $\alpha = \delta = -1,\ \beta = \gamma = 0.$

This gives $F_1(1-x) = -F_1(1+x), \quad F_2(1-x) = -F_2(1+x),$

provided that $$|1 \pm x| < 2.$$

As above, we can show that this requires the existence of *two independent odd* solutions, which is again impossible.

Both sub-cases of (i) being excluded and no other sub-case being possible, we are left with case (ii), $\alpha+\delta = 0$, $\beta\gamma = 1-\alpha^2$. This admits an infinity of possible solutions, some of which will be discussed in the next section. But the following particularly simple cases will at once occur to us.

(ii*a*) $\alpha = -\delta = \pm 1,\ \beta = 0.$

This gives $$F_1(1-x) = \pm F_1(1+x), \tag{3.2}$$

provided $$|1 \pm x| < 2.$$

It follows that there is *one* solution, either even or odd, which has *no singularity in the finite part of the plane.* This solution is therefore an integral function in x. Putting $x = \cos\theta$, we can express it as a series of cosines of *either even or odd* integral multiples of θ, according as $\alpha = +1$ or $\alpha = -1$ respectively. The series will converge throughout the θ-plane except at infinity.

These functions form half the set discussed by Whittaker, and we may conveniently denote them by $C_{2n}(\theta)$, $C_{2n+1}(\theta)$, because they reduce to $\cos 2n\theta$, or $\cos(2n+1)\theta$, when we make $k^2a^2 \to 0$ in Mathieu's equation. The condition that $\beta = 0$ implies that the parameter p must satisfy a certain transcendental equation, and in the limiting case $k^2a^2 \to 0$, we have $p = m^2$, where m is an integer. A method of calculating p is given in Whittaker and Watson's treatise, and we shall return to this point below.

(ii*b*) $\alpha = \pm 1 = -\delta,\ \gamma = 0.$

Here $$F_2(1-x) \equiv \mp F_2(1+x),$$

provided $$|1 \pm x| < 2. \tag{3.3}$$

In this case the solution which changes sign on describing a small circuit about $x = 1$, will also change sign on describing a small circuit about $x = -1$. Hence there is a solution of the form $\sqrt{(1-x^2)} \times$ (uniform function of x). Since *neither* solution tends to infinity at $x = \pm 1$, the uniform function of x will be finite everywhere except at infinity, and is therefore an *integral* function. The latter will be *even* if $\delta = +1 = -\alpha$, and *odd* if $\delta = -1 = -\alpha$. Putting $x = \cos\theta$, $\sqrt{(1-x^2)} = \sin\theta$, and expanding the integral function as a cosine series of integral multiples of θ, all odd or all even, we see that the solution can be written as a *sine series* of integral multiples of θ, all even or all odd respectively. These solutions are the other half of the set found by Whittaker, and we shall denote them by $S_{2n}(\theta)$, $S_{2n+1}(\theta)$, to show that they reduce to $\sin 2n\theta$, or $\sin(2n+1)\theta$, when $k^2a^2 \to 0$. The values of p again tend to m^2 (m integer), but are not of the same form as those of the $C_m(\theta)$ functions.

Note.—If the two transcendental relations between p and k^2a^2, viz. $\beta = 0$, $\gamma = 0$, were simultaneously true, then Mathieu's equation would admit two periodic solutions with period 2π. This could only occur if the value of k^2a^2 were a zero of the p-eliminant of β and γ.

(ii*c*) $\alpha = \delta = 0$, $\beta\gamma = 1$.

This interesting case, which appears to have escaped attention, gives

$$\left.\begin{aligned} F_1(1-x) &= \beta\ F_1(1+x) \\ F_2(1-x) &= \frac{1}{\beta}\ F_1(1+x) \end{aligned}\right\}, \tag{3.4}$$

provided $$|1 \pm x| < 2.$$

These relations are deducible from one another by a change in the sign of x.

These relations imply that there is *one* solution which is uniform in the vicinity of $x = 1$, and changes sign when it describes a circuit about $x = -1$; and there is another solution which is uniform at $x = -1$, and changes sign when it describes a circuit about $x = \pm 1$. These two solutions are of the form

$$\sqrt{(1-x)}\,\phi(x) \quad \text{and} \quad \sqrt{(1+x)}\,\phi(-x), \tag{3.5}$$

where ϕ is an integral function of x. Putting $x = \cos\theta$, the solutions take the form

$$\sin\tfrac{1}{2}\theta\, f(\theta), \quad \cos\tfrac{1}{2}\theta\, f(\pi-\theta),$$

where $f(\theta)$ is a cosine series converging throughout the θ-plane except at infinity, and proceeding by integral multiples of θ.

On rearranging we find series of the form

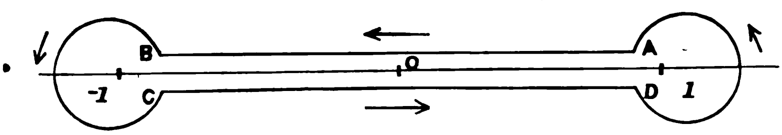
$$\sum_{(n)} a_n \cos(n+\tfrac{1}{2})\,\theta, \qquad \sum_{(n)} (-)^n a_n \sin(n+\tfrac{1}{2})\,\theta.$$

We shall denote the solutions of this type by the symbols $C_{n+\frac{1}{2}}(\theta)$, $S_{n+\frac{1}{2}}(\theta)$ to indicate that they reduce to $\cos(n+\frac{1}{2})\theta$ and $\sin(n+\frac{1}{2})\theta$, when $k^2a^2 \to 0$. The corresponding values of p tend to $(n+\frac{1}{2})^2$ when $k^2a^2 \to 0$. In this case, we also note that the two solutions are *coexistent* for the same value of p, and if one is $F(\theta)$, the other is $F(\pi-\theta)$.

They admit the period 4π.

4. *General Case of Periodic Solutions.*

Having found that there are solutions admitting the period 4π, we are led to inquire whether there may not also be solutions admitting the period $2s\pi$, where s is any integer. Now when the angle θ increases by 2π, the variation in $x = \cos\theta$ is represented by a single circuit enclosing the points $x = \pm 1$. We therefore consider the effect of such a circuit on the fundamental solutions; we shall premise that in $F_2(1 \pm x)$, the radicals $\sqrt{(1 \pm x)}$ shall be *positive* at points on the real axis where $-1 < x < 1$.



Consider the solutions u, v, which are defined at A on the upper edge of a cut along the real axis between $x = +1$ and $x = -1$ as

$$u_A = F_1(1-x), \qquad v_A = F_2(1-x). \tag{4.1}$$

Proceeding along the upper edge of the cut to B, we get

$$u_B = \alpha F_1(1+x)+\beta F_2(1+x), \qquad v_B = \gamma F_1(1+x)+\delta F_2(1+x). \tag{4.2}$$

Now $F_2(1+x)$ changes sign as x describes the path BC. Hence

$$u_C = \alpha F_1(1+x)-\beta F_2(1+x), \qquad v_C = \gamma F_1(1+x)-\delta F_2(1+x). \tag{4.3}$$

Returning to D along the lower edge of the cut

$$\left.\begin{aligned} u_D &= (\alpha^2-\beta\gamma)\, F_1(1-x)+\beta(\alpha-\delta)\, F_2(1-x) \\ v_D &= \gamma(\alpha-\delta)\, F_1(1-x)+(\beta\gamma-\delta^2)\, F_2(1-x) \end{aligned}\right\}. \tag{4.4}$$

Finally describing the circuit DA, we return to the starting point with the values

$$\left.\begin{aligned}\bar{u}_A &= (a^2-\beta\gamma)\,F_1(1-x)-\beta(a-\delta)\,F_2(1-x)\\ \bar{v}_A &= \gamma(a-\delta)\,F_1(1-x)+(\delta^2-\beta\gamma)\,F_2(1-x)\end{aligned}\right\}.$$

Now since $a=-\delta$, and $\beta\gamma=1-a^2$, we have

$$\left.\begin{aligned}\bar{u} &= (2a^2-1)\,u-\quad 2a\beta v\\ \bar{v} &= \quad 2a\gamma u\quad+(2a^2-1)\,v\end{aligned}\right\}, \tag{4.5}$$

$$\beta\gamma=(1-a^2).$$

Let this substitution be reduced to its canonical form, by assuming

$$(A\bar{u}+B\bar{v}) = K(Au+Bv).$$

We shall find, on eliminating $A : B$, the following equation for K

$$\begin{vmatrix}(2a^2-1-K), & 2a\gamma\\ -2a\beta, & (2a^2-1-K)\end{vmatrix}=0,$$

that is to say $\qquad (2a^2-1-K)^2+4a^2(1-a^2)=0,$

because $\beta\gamma=1-a^2$. This reduces to

$$K^2+2K\,(1-2a^2)+1=0.$$

It follows that the two values of K are *reciprocal*, and that their sum is

$$2\,(2a^2-1).$$

Now, if $2a^2-1>1$, *i.e.* $a^2>1$, both values of K are *real*. But if $a^2<1$, both are *complex* and of modulus unity.

If K is real, the two fundamental solutions of form $(Au+Bv)$, $(Cu+Dv)$ will become $K^n(Au+Bv)$ and $K^{-n}(Cu+Dv)$ after n circuits. Such solutions are *not periodic*.

But if K is complex and of modulus unity, it may happen that $K^n=1$, for an *integer* value of n. In that case, both solutions will regain their initial value after n circuits, or, in terms of θ, they will admit the period $2n\pi$.

Since $$K=e^{\pm 2r\pi i/n},$$

we must have $$(2a^2-1)=\cos\left(\frac{2r\pi}{n}\right). \tag{4.6}$$

This is the transcendental equation which p must satisfy, and when it is satisfied, we shall have solutions of type

$$e^{r\theta i/n} f(\theta), \qquad e^{-r\theta i/n} f(-\theta),$$

where f denotes a function of θ with the period 2π, which remains finite everywhere except at infinity.

If therefore
$$f(\theta) = \sum_{-\infty}^{\infty} c_m e^{m\theta i},$$

the coefficients c_m, c_{-m} must tend to zero like the coefficients of an integral function. This hypothesis will enable us to construct the corresponding solutions, and to obtain the transcendental equations for p.

5. *Construction of Solutions.*

We have
$$\frac{d^2y}{d\theta^2} + \left\{p - \frac{k^2a^2}{2}(1+\cos 2\theta)\right\} y = 0, \tag{5.1}$$

where the value of p is at first undetermined. We shall proceed to construct a solution of the form

$$y = e^{ir\theta/S} \sum_{-\infty}^{\infty} c_n e^{in\theta}, \tag{5.2}$$

on the hypothesis that (c_n) are the coefficients of an integral function, which is finite everywhere except at $\theta \to \infty$. On substituting the series (5.2) in the equation (5.1), the terms with even and odd suffixes fall into separate groups; since we may add or subtract s in assigning the value of r outside the sign of summation, there is no loss of generality in supposing all the n's even. On substituting, and picking out the terms multiplying the same power of $e^{i\theta}$, we find that the (c_n) are determined by

$$\frac{k^2a^2}{4}(C_{2n+2}+C_{2n-2}) = \left\{p - \frac{k^2a^2}{2} - \left(2n+\frac{r}{s}\right)^2\right\} C_{2n}.$$

Let us put
$$L_n \equiv \frac{4}{k^2a^2}\left\{\left(2n+\frac{r}{s}\right)^2 + \frac{k^2a^2}{2} - p\right\}. \tag{5.3}$$

Then
$$C_{2n+2} + L_n C_{2n} + C_{2n-2} = 0. \tag{5.4}$$

We shall solve on the hypothesis

$$\underset{n\to+\infty}{\mathrm{L}} \frac{C_{2n+2}}{C_{2n}} \to 0, \qquad \underset{n\to+\infty}{\mathrm{L}} \frac{C_{-2n-2}}{C_{-2n}} \to 0.$$

We shall follow a method employed by Whittaker in a more restricted

case (see Whittaker and Watson, *Modern Analysis*, 19.52), and which is perhaps most beautifully illustrated in Hough's theory of the tides on a rotating globe (*Phil. Trans.*, A, Vols. 189, 191). It is not necessary to justify the method from a theoretical point of view here. It depends on a theorem in the convergency of continued fractions given by Poincaré, *Les Nouvelles Méthodes de la Mécanique Céleste*, Vol. 2, p. 257.

We divide the equations (5.4) into two groups, for which n is positive or negative respectively, reserving the equation for which $n = 0$ for future use. If we divide by C_{2n}, and solve the one set on the hypothesis

$$\frac{C_{2n+2}}{C_{2n}} \to 0,$$

and the other on the hypothesis

$$\frac{C_{-2n-2}}{C_{-2n}} \to 0,$$

(n positive), we have the formulæ

$$\left.\begin{aligned} \frac{C_{2n-2}}{C_{2n}} &= -L_n + \frac{1}{L_{n+1}-}\,\frac{1}{L_{n+2}-}\,\ldots \text{ to } \infty \\ \frac{C_{-2n+2}}{C_{-2n}} &= -L_{-n} + \frac{1}{L_{-n-1}-}\,\frac{1}{L_{-n-2}-}\,\ldots \text{ to } \infty \end{aligned}\right\}. \qquad (5.5)$$

Since L_n is $O(n^2)$, these fractions will ultimately converge very rapidly. In the formulæ (5.5), we now put $n = 0$, and substitute in the equation

$$\frac{C_2}{C_0} + L_0 + \frac{C_{-2}}{C_0} = 0.$$

We now find that p must be so chosen that it verifies the relation

$$L_0 = \left\{\frac{1}{L_1-}\,\frac{1}{L_2-}\,\frac{1}{L_3-}\,\ldots\right\} + \left\{\frac{1}{L_{-1}-}\,\frac{1}{L_{-2}-}\,\frac{1}{L_{-2}-}\,\ldots\right\}, \qquad (5.6)$$

which is the same as the infinite determinant

$$\begin{vmatrix} 0 & 0 & 1 & L_{-2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & L_{-1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L_0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & L_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & L_2 & 1 \end{vmatrix} = 0.$$

This defines a series of values of p, for which there exists a solution admitting the multiplier $e^{2r\pi i/s}$ when θ increases by 2π. It can be proved by a discussion analogous to Hough's, or more directly by the "oscillation theorem," that the values of p will ultimately approximate to the roots of $L_n = 0$, as $n \to \infty$.

On separating real and imaginary parts, we have two solutions $C_{r/s+h}(\theta)$, $S_{r/s+h}(\theta)$, which reduce to $\cos(r/s+h)\theta$ and $\sin(r/s+h)\theta$ when $k^2a^2 \to 0$. The index h corresponds to the different values of p satisfying the transcendental equation.

It would be possible to develop a theory of the Mathieu functions admitting the period $2s\pi$, and to show that these form a "complete system" for the expansion of certain classes of functions. The functions of a system would admit an increasing number of nodes and loops in the interval $(0, 2s\pi)$ as p increased, and would possess "orthogonal" properties over this range. The values of p are necessarily real. The discussion and proof of these properties would however exceed our limits, and we shall conclude by giving certain integral equations, some of them already known and others new, satisfied by functions of the types n, $n+\frac{1}{2}$, where n is an integer.

6. *Integral Equations.*

6.1. In the harmonic analysis, we have to form potential functions and wave functions adapted to many types of curvilinear coordinates. If we expand a function of one type in terms of those of other types, we obtain certain "addition theorems" or expansion formulæ, such as those of the Legendre or Bessel functions.

Now it is possible to expand certain large classes of two-dimensional wave functions in terms of Mathieu functions, the typical term of the expansion being $F(\xi)\,G(\eta)$, where F, G satisfy the equations (1.1). If the given function is everywhere finite except at infinity, and admits the periods $2s\pi$ for η, and $2s\pi i$ for ξ, it will in general be expansible in Mathieu functions of order n/s. We shall not attempt a rigorous proof of this theorem, but we shall refer the reader to treatises such as Kneser's or Hilbert's on integral equations, where analogous problems are rigorously treated.

Suppose now that our wave function is a *uniform* integral function of x, y. It will be expansible, if at all, in terms of the simplest class of Mathieu functions, those with period 2π (or $2i\pi$ in the case of the second solution).

Now consider the solutions of the equation of wave motion

$$e^{ikx}, \quad e^{iky}, \quad ye^{ikx}, \quad xe^{iky}. \tag{6.11}$$

These are integral functions of x and y. Hence on putting

$$x+iy = a\cosh(\xi+i\eta),$$

we shall look for expansions of the form $\Sigma F_n(\xi)\,G_n(\eta)$, where

$$F_n(\xi) \equiv G_n(i\xi),$$

and the $G_n(\eta)$ are periodic with period 2π.

If we proceed to determine the coefficients of the expansion

$$f(\xi, \eta) = \Sigma A_n G_n(i\xi)\,G_n(\eta), \tag{6.12}$$

by means of the well known orthogonal property, we find

$$\int_0^{2\pi} f(\xi, \eta)\,G(\eta)\,d\eta = A_n G_n(i\xi)\int_0^{2\pi} G_n^2(\eta)\,d\eta,$$

or say

$$G_n(z) = a_n\int_0^{2\pi} f(iz, \eta)\,G_n(\eta)\,d\eta, \tag{6.13}$$

for $G_n(-z) = \pm G_n(z)$, being either an odd or an even integral function of z. This will be an integral equation for $G_n(z)$.

We cannot however *define* $G_n(z)$ as a system of solutions of (6.13), for the function $f(iz, y)$ need not necessarily require *all* the G_n's of the complete system in its expansion. With this proviso, we can however set up integral equations satisfied by *an infinite number* of the G_n's. Our method will be to separate the real and imaginary parts of (6.11) and to replace ξ by $i\xi'$ in the result, to obtain symmetry in the two variables. We thus obtain eight *kernels* $K(\xi', \eta)$. Four of these give integral equations already given by Whittaker, the others appear to be new. Whittaker's kernels are marked with an asterisk in the following table. By considering the nature of the function, whether even or odd and whether proceeding by even or odd multiples of the variables, we can specify the nature of the functions $F_n(\theta)$ satisfying the equation

$$F_n(\theta) = a_n\int_0^{2\pi} K(\theta, \phi)\,F_n(\phi)\,d\phi. \tag{6.14}$$

$K(\theta, \phi)$	$F_n(\theta)$
$\cos(ka\cos\theta\cos\phi)$* $\cosh(ka\sin\theta\sin\phi)$*	$C_{2n}(\theta)$
$\sin(ka\cos\theta\cos\phi)$* $\cos\theta\cos\phi\cosh(ka\sin\theta\sin\phi)$	$C_{2n+1}(\theta)$
$\sinh(ka\sin\theta\sin\phi)$* $\sin\theta\sin\phi\cos(ka\cos\theta\cos\phi)$	$S_{2n+1}(\theta)$
$\cos\theta\cos\phi\sinh(ka\sin\theta\sin\phi)$ $\sin\theta\sin\phi\sin(ka\cos\theta\cos\phi)$	$S_{2n}(\theta)$

(6.15)

We notice the appearance of hyperbolic functions in some of the kernels, as the result of putting $i\sin\phi$ for $\sinh\xi$.

6.2. We shall now attempt by similar methods to find an integral equation satisfied by the Mathieu functions with period 4π.

If we suppose p so chosen that the equation in η in (1.1) has a solution with period 4π, or the corresponding equation in μ in (1.2) has a solution of form $\sqrt{(1-\mu)}\,\phi(\mu)$, where ϕ is an integral function, let us consider the "element"

$$\sqrt{\{(\lambda-1)(1-\mu)\}}\,\phi(\lambda)\,\phi(\mu). \tag{6.21}$$

This is a wave function made up of two similar Mathieu functions, and by changing the sign of μ or λ, we can include the case where the second Mathieu function is taken instead of the first. Now in order that a function may be expansible in a convergent series of elements (6.21), we shall restrict ourselves to such functions as are of the form

$$\sqrt{\{(\lambda-1)(1-\mu)\}} \times (\text{symmetric integral function of } \lambda, \mu).$$

Now let us write

$$\xi = \sqrt{\{(\lambda+1)(1+\mu)\}}, \quad \eta = \sqrt{\{(\lambda-1)(1-\mu)\}}. \tag{6.22}$$

Then allowing for a change in sign of λ, μ, the functions required are

---

* Whittaker's types.

such as can be expanded as an integral function in ξ, η, *even in one variable and odd in the other.*

Now we have
$$\left.\begin{aligned} a(\xi^2-\eta^2) &= 2(x+a) \\ a\xi\eta &= y \end{aligned}\right\}, \tag{6.23}$$

so that
$$(x+iy) = \frac{a}{2}(\xi+i\eta)^2 - a. \tag{6.24}$$

This change of variable reduces the equation of wave motions to

$$\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} + k^2a^2(\xi^2+\eta^2)\,V = 0. \tag{6.25}$$

And the kernel we require must be an integral solution of this equation.

Now this equation is well known in the theory of the parabolic cylinder [Lamb, *Proceedings*, Vol. 4 (1907); Bateman, *Wave Motions*, p. 98; Whittaker and Watson, *Modern Analysis*, Ch. xvi]. We shall pick out the simplest solution, namely an *even* integral function of ξ, multiplying an *odd* integral function of η. If $V = L(\xi)\,M(\eta)$, we find

$$\frac{d^2L}{d\xi^2} + (k^2a^2\xi^2+p)L = 0, \qquad \frac{d^2M}{d\eta^2} + (k^2a^2\eta^2-p)M = 0. \tag{6.26}$$

The choice of p being at our disposal, we make $p = -ika$. Then a solution of the first equation is

$$L(\xi) = e^{\frac{1}{2}ika\xi^2}. \tag{6.27}$$

Now the *even* solution of the second equation is clearly $M = e^{-ika\eta^2}$. This enables us to find the *odd* solution required by elementary reasoning, and we get

$$M(\eta) = e^{-\frac{1}{2}ika\eta^2}\int_0^{\eta} e^{ikat^2}\,dt. \tag{6.28}$$

Hence the kernel takes the form

$$e^{ik(x+a)}\int_0^{\eta} e^{ikat^2}\,dt.$$

Or, introducing the elliptic coordinates, which we may call θ and $\phi' = i\phi$, to distinguish from our auxiliary variable ξ, η, we have

$$e^{ika\cosh\phi\cos\theta}\int_0^{2\sinh\frac{1}{2}\phi\sin\frac{1}{2}\theta} e^{ikat^2}\,dt = i\,e^{ika\cos\theta\cos\phi'}\int_0^{2\sin\frac{1}{2}\theta\sin\frac{1}{2}\phi'} e^{-ikat^2}\,dt.$$

Neglecting constant factors, we shall write

$$\left.\begin{aligned} K_1(u,\, v) &\equiv e^{ika\cos u\cos v}\int_0^{2\sin\frac{1}{2}u\sin\frac{1}{2}v} e^{-ikat^2}\,dt \\ K_2(u,\, v) &= K_1(u+\pi,\, v+\pi) = e^{ika\cos u\cos v}\int_0^{2\cos\frac{1}{2}u\cos\frac{1}{2}v} e^{-ikat^2}\,dt \end{aligned}\right\}. \quad (6.29)$$

The kernel K_1 will give functions of the type $S_{m+\frac{1}{2}}$, and K_2 will give functions of type $C_{m+\frac{1}{2}}$. In fact, on expanding the kernels in series of the type

$$\left.\begin{aligned} K_1(u,\, v) &= \Sigma A_m S_{m+\frac{1}{2}}(u)\, S_{m+\frac{1}{2}}(v) \\ K_2(u,\, v) &= \Sigma B_m C_{m+\frac{1}{2}}(u)\, C_{m+\frac{1}{2}}(v) \end{aligned}\right\}, \quad (6.291)$$

we have by the usual method of determining the coefficients

$$\left.\begin{aligned} S_{m+\frac{1}{2}}(u) &= a_m \int_0^{2\pi} K_1(u,\, v)\, S_{m+\frac{1}{2}}(v)\,dv \\ C_{m+\frac{1}{2}}(u) &= \beta_m \int_0^{2\pi} K_2(u,\, v)\, C_{m+\frac{1}{2}}(v)\,dv \end{aligned}\right\}, \quad (6.292)$$

these being deducible from one another, by changing u into $(\pi - u)$. We cannot, however, base the whole theory of the functions on the equations (6 . 292), because it has not been proved, and it appears improbable that it *can* be proved, that *every* function of the complete system is present in the expansion of these kernels.

To investigate the completeness of the system theoretically, we require to form the "Green's function" for the interval $(0,\, 2\pi)$ and to discuss the integral equation whose kernel is symmetric and continuous, but whose differential coefficient has a discontinuity at $u = v$. Into this question we cannot enter here.

7. *Conclusion.*

We may remark in conclusion that the methods of this paper can be applied with little more than verbal alterations to a large class of equations, which depend on the equation of wave motions in spheroidal coordinates.

In the case of the prolate spheroid, we put

$$\xi = a\lambda\mu, \qquad \rho = a\sqrt{\{(\lambda^2-1)(1-\mu^2)\}},$$

and retain the azimuth ϕ. A wave function of the form

$$V = e^{mi\phi} F(\lambda)\, G(\mu),$$

gives us two similar equations in λ, μ for F, G of the form

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+\left(p-\frac{m^2}{1-x^2}-k^2a^2x^2\right)y=0. \tag{7.1}$$

In the case of the oblate spheroid, we put

$$\xi = a\lambda\mu, \qquad \rho = a\sqrt{\{(\lambda^2+1)(1-\mu^2)\}},$$

and with similar assumptions, we find an equation in μ of the form

$$\frac{d}{dx}(1-x^2)\frac{dy}{dx}+\left(p-\frac{m^2}{1-x^2}+k^2a^2x^2\right)y=0, \tag{7.2}$$

only differing from the former in the sign of k^2a^2. The corresponding equation in λ is found by putting $\lambda = ix$.

If m is *not an integer*, the theory of these equations presents the closest analogy with that of Mathieu's. They are in fact reducible to the canonical form

$$(1-x^2)\frac{d^2y}{dx^2}-2(1\pm m)x\frac{dy}{dx}+(A+Bx^2)y=0,$$

of which Mathieu's equation is a special case when $\pm m = -\frac{1}{2}$. The cases where m is an integer require different treatment, owing to the singularities at $x=\pm 1$ becoming logarithmic.

ON THE TORSION OF A PRISM, ONE OF THE CROSS-SECTIONS OF WHICH REMAINS PLANE

By S. TIMOSCHENKO.

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1. In the theory of torsion due to Saint-Venant the twisting couple is supposed to be applied by means of tangential tractions exerted upon the terminal sections, and these tractions are supposed to be distributed over the sections according to determinate laws. Such a torsion is generally accompanied by the distortion of cross-sections of the twisted prism.

In some cases we have to solve the problem of the torsion of a prism, one of the cross-sections of which is maintained plane by suitable forces.

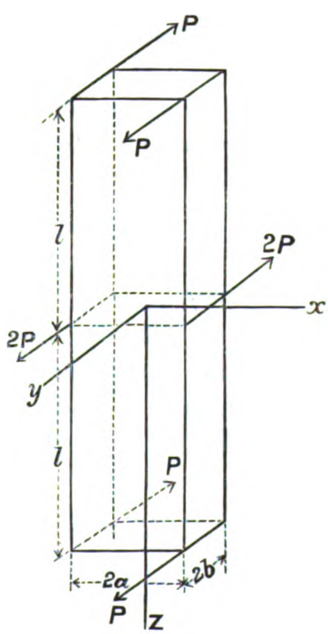

FIG. 1.

For instance, in the case illustrated in Fig. 1, it is a consequence of the

symmetry that the middle cross-section ($z = 0$) must remain plane. In such cases near the constrained cross-section there will appear "local irregularity." The influence of that on the magnitude of the angle of twisting can be neglected in cases where the linear dimensions of the cross-section are small in comparison with the length of the prism, but in some cases this influence can be of practical interest.

We have met with this problem when investigating the stability of the plane form of the I girder under bending loads, and have given an approximate solution of the question, by estimating the effect of the bending of the flanges, which accompanies the torsion.* In the recently published paper by A. Föppl† another method for the approximate solution of the same problem is given. A. Föppl works with expressions for the components of stress, which satisfy the differential equations of equilibrium and the boundary conditions. The constant quantities which enter into these expressions must be calculated so as to make the potential energy of the twisted prism a minimum.

By this method A. Föppl solves the problem in the case of an elliptical boundary, and uses the result for a very extended ellipse to estimate the influence of constraint in the case of a twisted prism of narrow rectangular cross-section.

The last problem can be of interest in connexion with the investigation of stability of a flat blade bent in its plane. For that reason we give here a more detailed solution of the question by using the method of A. Föppl and also another method, where we work with expressions for the displacements. The last method seems to be more appropriate in the case of a twisted flat blade.

2. The exact solution of Saint-Venant's problem in the case of a rectangular boundary involves infinite series. For our purpose it is more convenient to proceed with a simple approximate solution, which we can get by using the "membrane analogy." The stress-equations of equilibrium will be satisfied, if we take the components of stress on a cross-section as follows

$$Z_x = \frac{\partial \psi}{\partial y}, \quad Z_y = -\frac{\partial \psi}{\partial x}. \tag{1}$$

* S. Timoschenko, "Einige Stabilitätsprobleme der Elastizitätstheorie," *Zeitschrift f. Math. u. Phys.*, Bd. 58 (1910), S. 361.

† *Sitzungsberichte d. Bayerischen Akademie der Wissenschaften*, 1920, S. 261.

In case of an exact solution the stress function $\psi(x, y)$ satisfies the equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+2\mu\tau=0, \tag{2}$$

and remains constant on the boundary of the cross-section. Here μ denotes the rigidity and τ the twist of the prism.

Let a homogeneous membrane be stretched with uniform tension T and fixed at its edge, which is the same as the bounding curve of the cross-section of the twisted prism. When the membrane is subjected to uniform pressure of amount p per unit of area, it will undergo a small displacement z.

If we put
$$\frac{p}{T}=2\mu\tau, \tag{a}$$

the equation of equilibrium of the membrane coincides with (2) and the surface of the membrane is given by the equation

$$z=\psi(x, y).$$

In order to get this surface we use the variational method. In a small variation of the displacement $z=\psi(x, y)$, the uniform tension T will do work of amount
$$-\frac{T}{2}\,\delta\iint\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right]dx\,dy.$$

The corresponding work of the uniform pressure p will be

$$p\delta\iint\psi\,dx\,dy.$$

It follows from (a) that the condition of equilibrium of the membrane will be
$$\delta\left\{\tfrac{1}{2}\iint\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right]dx\,dy-2\mu\tau\iint\psi\,dx\,dy\right\}=0.$$

In this way the solution of the problem of torsion is reduced to seeking the minimum of the integral

$$S=\iint\left\{\tfrac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right]-2\mu\tau\psi\right\}dx\,dy, \tag{3}$$

where the function ψ must be constant at the boundary. This method is especially appropriate when approximate solutions are sought. In the case of a rectangular cross-section (Fig. 1) the general form of the stress-

function will be

$$\psi = \sum_{m=1,3,5,\ldots}^{\infty} \sum_{n=1,3,5,\ldots}^{\infty} A_{mn} \cos \frac{m\pi x}{2a} \cos \frac{n\pi y}{2b}.$$

The coefficients A_{mn} can be found from the equations

$$\frac{\partial S}{\partial A_{mn}} = 0,$$

and we can get Saint-Venant's solution in this way.

In the case of a very narrow rectangle* we can get the approximate solution by taking for the surface of the membrane the cylindrical surface

$$\psi = \mu\tau (b^2 - y^2).$$

Then we have

$$Z_x = -2\mu\tau y, \qquad Z_y = 0. \tag{4}$$

The corresponding displacements will be

$$u = -\tau z y, \qquad v = \tau z x, \qquad w = -\tau x y. \tag{5}$$

In order to get a more exact solution and obtain a correction, due to the influence of the edges at $x = \pm a$, we can take (for $x > 0$)

$$\psi = \mu\tau (b^2 - y^2)(1 - e^{-\kappa(a-x)}),$$

$$Z_x = -2\mu\tau y (1 - e^{-\kappa(a-x)}), \qquad Z_y = \mu\tau\kappa (b^2 - y^2) e^{-\kappa(a-x)}. \tag{6}$$

We choose the quantity κ in such a manner as to make the integral (3) a minimum and get in this way

$$\kappa = \frac{1}{b}\sqrt{\tfrac{5}{2}}. \tag{7}$$

This solution gives for the twisting couple the approximate formula

$$M = 4\int_0^a \int_{-b}^{+b} \psi \, dx \, dy = \tfrac{16}{3}\mu\tau a b^3 \left(1 - 0{\cdot}632 \frac{b}{a}\right), \tag{8}$$

or in the case of a very narrow rectangle

$$M = \tfrac{16}{3}\mu\tau a b^3. \tag{8$'$}$$

3. We shall now use these results in order to get an approximate solution in the case illustrated by Fig. 1. In consequence of symmetry the cross-section $z = 0$ remains plane and the corresponding displacement w will be equal to zero.

* We suppose that a is large in comparison with b.

In accordance with (5) we can suppose that the stress Z_z at this cross-section is as follows

$$Z_z = -E\gamma\tau e^{-\gamma z} xy, \tag{9}$$

where γ is a constant quantity to be determined.

We will suppose also that

$$X_x = Y_y = 0. \tag{10}$$

Then the stress-equations of equilibrium and the condition that the cylindrical bounding surface of the prism is free from traction are satisfied by taking

$$X_y = -\tfrac{1}{8}E\gamma^3\tau e^{-\gamma z}(a^2-x^2)(b^2-y^2), \tag{11}$$

$$X_z = \tfrac{1}{4}E\gamma^2\tau e^{-\gamma z}(a^2-x^2)\,y - 2\mu\tau y, \tag{12}$$

$$Y_z = \tfrac{1}{4}E\gamma^2\tau e^{-\gamma z}(b^2-y^2)\,x. \tag{13}$$

As z increases the expressions (9)–(13) approach the expressions (4) that were found previously. The quantity γ must be chosen in such a manner as to make the potential energy V of twisting a minimum.

If we calculate V from the formula

$$V = \frac{1}{2\mu}\int_0^l\int_{-a}^{+a}\int_{-b}^{+b}\left(X_y^2+X_z^2+Y_z^2+\frac{1}{2(1+\sigma)}Z_z^2\right)dx\,dy\,dz,$$

and assume that $$\int_0^l e^{-\gamma z}\,dz = \frac{1}{\gamma},$$

we get

$$V = \tfrac{1}{9}E\tau^2a^3b^3\left\{-3\gamma+(1+\sigma)\left[\tfrac{2}{25}a^2b^2\gamma^5+\tfrac{1}{5}(a^2+b^2)\gamma^3+\frac{12}{(1+\sigma)^2}\frac{l}{a^2}\right]\right\}. \tag{14}$$

The equation for γ will be

$$(1+\sigma)\left[\tfrac{2}{5}a^2b^2\gamma^4+\tfrac{3}{5}(a^2+b^2)\gamma^2\right] = 3. \tag{15}$$

In the case of a very narrow rectangle we get

$$a^2\gamma^2 = \frac{5}{1+\sigma}. \tag{15$'$}$$

In order to get the angle of twisting θ, we put the potential energy (14) equal to the work of twisting couple M. From (8)$'$, we get

$$\theta = \tau\left(l - \frac{\sqrt{[5(1+\sigma)]}}{6}\,a\right),$$

or, σ being taken to be 0·3,

$$\theta = \tau(l - 0{\cdot}425a). \tag{16}$$

The influence of "local irregularity," at $z = 0$, on the value of θ is the same as the influence of diminution of the length l by $0{\cdot}425a$. If we wish to take into account the boundaries $x = \pm a$, we must change the expressions for X_z, Y_z to the following:—

$$X_z = \tfrac{1}{4}E\gamma^2\tau e^{-\gamma z}(a^2 - x^2)\, y - 2\mu\tau y(1 - e^{-\kappa(a-x)}),$$

$$Y_z = \tfrac{1}{4}E\gamma^2\tau e^{-\gamma z}(b^2 - y^2)\, x + \mu\tau\kappa\,(b^2 - y^2)e^{-\kappa(a-x)}.$$

The values of $a^2\gamma^2$ and the diminutions δl of the length l, corresponding to the diminutions of the angle θ, are given in the following table:—

$a/b =$	∞	10	5
$a^2\gamma^2 =$	3·846	3·604	3·047
$\delta l =$	$0{\cdot}425a$	$0{\cdot}428a$	$0{\cdot}390a$

4. We can get the solution of our problem in another way by working with expressions for the displacements. These expressions must be chosen in such a manner as to satisfy the conditions at the plane cross-section $z = 0$. They will contain one or more constant quantities, which will represent the coordinates of the system. These quantities can be determined by the variational equations of equilibrium.

In the case of a narrow rectangular cross-section each half of the prism may be considered as a thin plate built in at the edge $z = 0$. In such a case we can assume that the displacement v in the direction of y is given by an equation of the form

$$v = \frac{Mx}{C}\left[z - \frac{1}{\alpha}(1 - e^{-\alpha z})\right], \tag{17}$$

where M denotes the twisting couple, α the constant quantity to be determined, and C the torsional rigidity of the prism.

We see that the conditions

$$(v)_{z=0} = 0, \quad \left(\frac{\partial v}{\partial z}\right)_{z=0} = 0,$$

are satisfied. Further, as z increases, the twist

$$\frac{\partial^2 v}{\partial x \partial z} = \frac{M}{C}(1-e^{-az})$$

approaches the value M/C. Neglecting the small quantity e^{-al}, we get for the angle of twisting

$$\theta_l = \left(\frac{\partial v}{\partial x}\right)_{z=l} = \frac{M}{C}\left(l - \frac{1}{a}\right). \tag{18}$$

We again find that the decrease of θ, resulting from the "local irregularity" is the same as that corresponding to the diminution of the length by the quantity $1/a$, which is independent of l. The equation for the determination of a is obtained by equating the potential energy of the bending of the plate to the work done by the twisting couple M. In this way we get

$$\frac{2aD}{C}\left[\frac{aa^2}{6} + 2(1-\sigma)\left(l - \frac{3}{2a}\right)\right] = l - \frac{1}{a}, \tag{b}$$

where D denotes the "flexural rigidity" of the plate. In the case of a narrow rectangular cross-section we have

$$\frac{2aD}{C} = \frac{1}{2(1-\sigma)},$$

and the equation (b) gives us

$$\frac{1}{a} = \frac{a}{\sqrt{[6(1-\sigma)]}} = 0{\cdot}488a. \tag{19}$$

In order to get a higher approximation, we must work with a more complicated expression for the displacement v, which must contain two or more constants to be determined.

The calculation made with the expression

$$v = \frac{Mx}{C}\left[z - \frac{1}{a}(1-e^{-az}) + \beta z^2 x^2 e^{-az}\right], \tag{20}$$

which contains two constants a and β, gives us a result differing from (19) in the last decimal only.

5. We will use the result (17) in order to estimate the stiffening effect of the "local irregularity" in relation to the question of the stability of a

flat blade bent in its plane* (Fig. 2). It is known that by increasing the

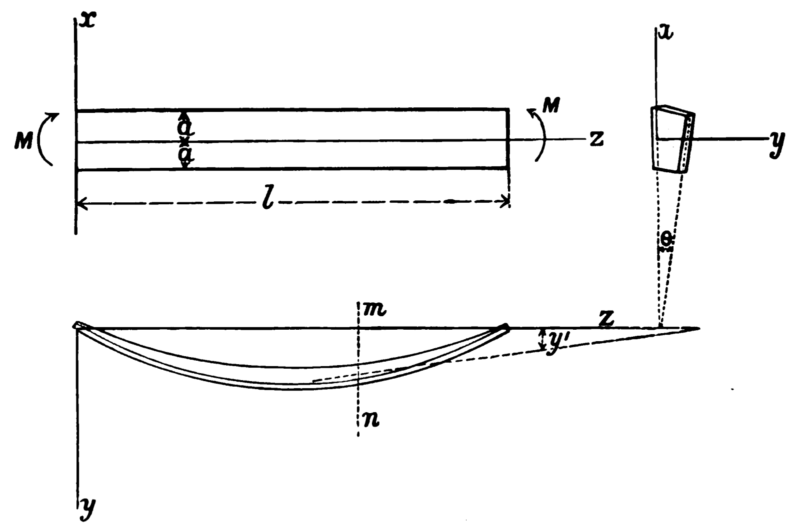

FIG. 2.

bending moment we can reach the state in which the plane form of bending becomes unstable and sidewise buckling occurs, as is illustrated in Fig. 2. Such buckling is accompanied by torsion.

In order to find the value M_{crt} of the bending moment, rendering possible this kind of instability, we take the differential equations of equilibrium.

From the assumption of small displacements, we can conclude that the twisting moment at the cross-section mn will be equal to $M\,dy/dz$, and in accordance with (17), we get the differential equation, corresponding to the torsion, in the form

$$M\frac{dy}{dz} = C\left(\frac{d\theta}{dz} - \frac{1}{a^2}\frac{d^3\theta}{dz^3}\right). \tag{21}$$

The bending moment in the plane of smallest flexural rigidity will be $M\theta$, and the corresponding equation of equilibrium is

$$M\theta = -B\frac{d^2y}{dz^2}, \tag{22}$$

where B is the flexural rigidity.

* A. G. Michell, *Phil. Mag.*, Ser. 5, Vol. 48 (1899); S. Timoschenko, *Isvestia Petrogradskago Polytechnicheskago Instituta* (1905).

From (21) and (22) we get

$$\frac{d^4\theta}{dz^4} - a^2\frac{d^2\theta}{dz^2} - \frac{M^2a^2}{BC}\theta = 0. \tag{23}$$

Integrating this equation and observing that θ and $d^2\theta/dz^2$ vanish at $x = 0$ and $x = l$, we get

$$\theta = A\sin\beta z,$$

where
$$\beta = \sqrt{\left[-\tfrac{1}{2}a^2 + \sqrt{\left(\frac{a^4}{4} + \frac{M^2a^2}{BC}\right)}\right]} = \frac{\pi}{l}. \tag{24}$$

From (24), we get
$$M_{crt} = \frac{\pi\sqrt{(BC)}}{l}\sqrt{\left(1 + \frac{1}{a^2}\,\frac{\pi^2}{l^2}\right)},$$

or, with the solution (18) for $1/a$,

$$M_{crt} = \frac{\pi\sqrt{(BC)}}{l}\left(1 + 1\cdot 18\,\frac{a^2}{l^2}\right). \tag{25}$$

The second term in the bracket gives us the stiffening effect of the "local irregularity."

We see that it is very small and cannot have so much practical importance as in the case of an I girder.

A MEMBRANE ANALOGY TO FLEXURE

By S. TIMOSCHENKO.

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THE membrane analogy, which is of great importance in the case of torsion, can be applied in some cases to the investigation of the bending of prisms. This analogy combined with the Rayleigh-Ritz method for determining the form of a stretched membrane, subjected to normal pressure, enables us in some cases to get an approximate solution of the flexure problem, when the exact solution is unknown, or is very complicated and inconvenient for numerical calculation.*

We take the central-line of the beam of length l to be horizontal, and one end of it to be fixed, and we suppose that forces are applied to the cross-section containing the other end in such a way as to be statically equivalent to a vertical load W acting downwards in a line through the centroid of the section. We take the origin at the fixed end, and the axis of z along the central-line, and we draw the axis of x vertically downwards. Further we suppose that the axes of x and y are the principal axes of inertia of the cross-section. In such a case we have, in accordance with Saint-Venant's solution,

$$X_x = Y_y = X_y = 0,$$

$$Z_z = -W(l-z)\,\frac{x}{I}. \tag{1}$$

The stress components X_z and Y_z will be the functions of x and y only. The stress-equation of equilibrium

$$\frac{\partial X_z}{\partial x} + \frac{\partial Y_z}{\partial y} + \frac{\partial Z_z}{\partial z} = 0,$$

* Applying this method to the case of a rectangular cross-section, we had occasion to observe some errors in the well known table, calculated for this cross-section by Saint-Venant. The corresponding corrections are given below.

will be satisfied, if we put

$$X_z = \frac{\partial\phi}{\partial y} - \frac{Wx^2}{2I} + f(y), \qquad Y_z = -\frac{\partial\phi}{\partial x}, \tag{2}$$

where ϕ denotes the "stress-function" and f an arbitrary function of y only.

The condition that the cylindrical bounding surface is free from traction is

$$X_z \cos(x\nu) + Y_z \cos(y\nu) = 0,$$

and can be written as follows

$$\frac{\partial\phi}{\partial s} = \left[\frac{Wx^2}{2I} - f(y)\right]\frac{\partial y}{\partial s}. \tag{3}$$

Substituting (2) in the equations of compatibility

$$\nabla^2 Y_z = 0, \qquad \nabla^2 X_z = -\frac{W}{(1+\sigma)I},$$

we get

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{\sigma}{1+\sigma}\,\frac{Wy}{I} - f'(y) + c. \tag{4}$$

In particular cases we must adjust the quantity c in such a way as to make the couple about the axis of z due to tractions on the cross-section vanish.

In cases where it is possible, by appropriate choice of $f(y)$, to make the right-hand member of (3) vanish, our problem will be the same as the problem of seeking the form of a uniformly stretched membrane, subjected to normal pressure. Provided that the edge of the membrane is the same as the bounding curve of the cross-section of the prism, the uniform tension of the membrane is equal to unity, and the intensity of normal pressure is represented by the right-hand member of (4) with negative sign, the equation of equilibrium of the membrane will be identical with (4).

The form of equilibrium can be found by using the variational method. If we give to the displacements of the membrane small variations the corresponding work due to the uniform tension is

$$-\delta\iint \tfrac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] dx\,dy,$$

and the work due to the normal pressure is

$$-\delta\iint \phi\left[\frac{\sigma}{1+\sigma}\,\frac{Wy}{I} - f'(y) + c\right] dx\,dy.$$

It follows that the function ϕ can be found from the condition that the integral

$$S = \iint \left\{ \tfrac{1}{2} \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 \right] + \phi \left(\frac{\sigma}{1+\sigma} \frac{Wy}{I} - f'(y) + c \right) \right\} dx\, dy \tag{5}$$

is a minimum.

Using the method of Rayleigh-Ritz, we put

$$\phi = a_0 \psi_0 + a_1 \psi_1 + a_2 \psi_2 + \ldots, \tag{6}$$

where ψ_0, ψ_1, ... denote functions which vanish at the boundary. The coefficients a_0, a_1, ... can be calculated from the minimum conditions of the form

$$\frac{\partial S}{\partial a_n} = 0. \tag{7}$$

The accuracy of our solution will depend on the number of terms in the expression (6).

If the boundary of the cross-section is given by the equation

$$F(x, y) = 0,$$

and the function F is different from zero within the cross-section, we can take the solution (6) in the following form

$$\phi = F(x, y) \sum_{m=0,\, n=0}^{m,\, n} a_{mn} x^m y^n. \tag{8}$$

We shall now show how to find the function ϕ when the boundary of the section of the beam has one or other of certain special forms.

(*a*) *The ellipse.*

The equation of the bounding curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The right-hand side of (3) will be equal to zero if we put

$$f(y) = \frac{Wa^2}{2I} \left(1 - \frac{y^2}{b^2}\right).$$

The differential equation (4) will be

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{Wy}{I} \left(\frac{\sigma}{1+\sigma} + \frac{a^2}{b^2}\right). \tag{a}$$

That is, the membrane is subjected to a linear distribution of normal

pressure vanishing at the x axis. We can conclude that ϕ is an even function of x and an uneven function of y. It is easy to see that in this case the term

$$a_{01}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)y$$

of the expression (8) gives us the exact solution of (a), if we take

$$a_{01}=\frac{Wa^2}{I}\,\frac{(1+\sigma)\,a^2+\sigma b^2}{2(1+\sigma)(3a^2+b^2)}.$$

With $a = b$, we get the solution for a circle.

(b) *The rectangle.*

In the case of a rectangle the boundaries are given by the equations $x = \pm a$, $y = \pm b$. The right-hand member of (3) will be equal to zero if we put

$$f=\frac{Wa^2}{2I}.$$

The corresponding equation for the membrane will be

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\frac{\sigma}{1+\sigma}\,\frac{Wy}{I}.$$

Also in this case ϕ is an even function of x and an uneven function of y. These conditions and the conditions at the boundary will be satisfied if we take the expression (6) in the form

$$\phi=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}a_{2m+1,\,n}\cos\frac{(2m+1)\pi x}{2a}\sin\frac{n\pi y}{b}. \qquad (9)$$

From the equations (7), we can get

$$a_{2m+1,\,n}=\frac{\sigma}{1+\sigma}\,\frac{W}{I}\,\frac{8b^3}{\pi^4}\,\frac{(-1)^{m+n-1}}{n(2m+1)\left[\frac{1}{4}\alpha^2(2m+1)^2+n^2\right]},$$

where

$$\alpha = b/a.$$

The shearing stresses (2) will be

$$X_z=\frac{\partial\phi}{\partial y}+\frac{W}{2I}(a^2-x^2),\qquad Y_z=-\frac{\partial\phi}{\partial x}. \qquad (10)$$

The second term in the expression for X_z gives us the shearing stresses, which are usually calculated in treatises on Applied Mechanics from the stress-equations of equilibrium, without reference to the conditions of compatibility. The calculation of corrections to this elementary solution is facilitated by the use of the function ϕ.

In the case of a very narrow rectangle we can at once reach some conclusions in regard to these corrections.

If a is large in comparison with b, we can assume that, at the points distant from the short sides of the rectangle, the surface of the membrane is effectively cylindrical. The corresponding differential equation will be

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\sigma}{1+\sigma}\frac{Wy}{I},$$

and we get
$$\phi = \frac{\sigma}{1+\sigma}\frac{W}{6I}(y^3-b^2y),$$

$$X_z = \frac{W}{2I}\left[a^2-x^2+\frac{\sigma}{1+\sigma}\left(y^2-\frac{b^2}{3}\right)\right]. \tag{11}$$

At the centre of the cross-section we have

$$(X_z)_{x=y=0} = \frac{Wa^2}{2I}\left(1-\frac{\sigma}{3(1+\sigma)}a^2\right).$$

If b is large in comparison with a, the displacement of the membrane at points distant from the short sides of the rectangle will be a linear function of y, and we can put

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\sigma}{1+\sigma}\frac{Wy}{I},$$

from which
$$\phi = \frac{\sigma}{1+\sigma}\frac{W}{2I}y(x^2-a^2),$$

$$X_z = \frac{1}{1+\sigma}\frac{W}{2I}(a^2-x^2), \qquad Y_z = -\frac{\sigma}{1+\sigma}\frac{W}{I}xy. \tag{12}$$

In comparison with the usual elementary solution the shearing stresses are reduced in the ratio $1 : 1+\sigma$.

It may be pointed out that the differential equation

$$dx/X_z = dy/Y_z$$

of the "lines of shearing stresses" in accordance with (12) gives

$$y = C(a^2-x^2)^\sigma. \tag{13}$$

The expressions (12) will constitute an exact solution of the flexural problem if the boundary is represented by (13), or, what is the same thing, by the equation

$$\left(\frac{y}{b}\right)^{1/\sigma} = 1-\frac{x^2}{a^2}.$$

In fact, the right-hand member of (3) will be equal to zero, if we put

$$f(y) = \frac{Wa^2}{2I}\left[1-\left(\frac{y}{b}\right)^{1/\sigma}\right].$$

The corresponding solution of (4) will be

$$\phi = \frac{\sigma}{1+\sigma}\,\frac{W}{2I}\left[y(x^2-a^2)+a^2b\left(\frac{y}{b}\right)^{1+1/\sigma}\right],$$

and the expressions (2) give the solution (12).

If a and b are of the same order of magnitude, we use the complete solution (9), and using the results

$$\sum_1^\infty \frac{1}{n^2} = \tfrac{1}{6}\pi^2, \qquad \sum_1^\infty \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12},$$

$$\sum_0^\infty \frac{(-1)^m}{(2m+1)[(2m+1)^2+\kappa^2]} = \frac{\pi^3}{32}\,\frac{\operatorname{sech}\frac{1}{2}\kappa\pi-1}{\frac{1}{2}(\frac{1}{2}\kappa\pi)^2},$$

we get

$$\left.\begin{aligned}(X_z)_{x=0,\,y=0} &= \frac{3}{8}\,\frac{W}{ab}\left[1-\frac{\sigma}{1+\sigma}a^2\left\{\frac{1}{3}+\frac{4}{\pi^2}\sum_1^\infty\frac{(-1)^n}{n^2\cosh n\pi/a}\right\}\right]\\ (X_z)_{x=0,\,y=b} &= \frac{3}{8}\,\frac{W}{ab}\left[1+\frac{\sigma}{1+\sigma}a^2\left\{\frac{2}{3}-\frac{4}{\pi^2}\sum_1^\infty\frac{1}{n^2\cosh n\pi/a}\right\}\right]\end{aligned}\right\}. \quad (14)$$

These formulæ coincide with the well known solution of Saint-Venant.

Using the Rayleigh-Ritz method, we can get the solution of the problem in another form more convenient for numerical calculation. Reducing the general expression (8) to two terms only, and putting

$$\phi = (x^2-a^2)(y^2-b^2)(Ay+By^3),$$

we get from the equations of the form (7)

$$A = -\frac{\sigma}{1+\sigma}\,\frac{W}{8Ib^2}\,\frac{\dfrac{1}{11}+\dfrac{8}{a^2}}{\left(\dfrac{1}{7}+\dfrac{3}{5}\,\dfrac{1}{a^2}\right)\left(\dfrac{1}{11}+\dfrac{8}{a^2}\right)+\dfrac{1}{21}+\dfrac{9}{35a^2}},$$

$$B = -\frac{\sigma}{1+\sigma}\,\frac{W}{8Ib^4}\,\frac{1}{\left(\dfrac{1}{7}+\dfrac{3}{5}\,\dfrac{1}{a^2}\right)\left(\dfrac{1}{11}+\dfrac{8}{a^2}\right)+\dfrac{1}{21}+\dfrac{9}{35a^2}}.$$

The corresponding shearing stress-components (10) will be

$$(X_z)_{x=0,\,y=0} = \frac{Wa^2}{2I}+Aa^2b^2, \qquad (X_z)_{x=0,\,y=b} = \frac{Wa^2}{2I}-2a^2b^2(A+Bb^2). \quad (15)$$

In order to estimate the accuracy of this approximate solution we have calculated the stresses at the centre of the rectangle ($x = 0$, $y = 0$) and at the point ($x = 0$, $y = b$), σ being taken to be $\frac{1}{4}$. The values of the expressions in square brackets of (14) and the corresponding values for the solution (15) are given in the following table.

	$b/a =$	0·5	1	2	4
Point	Exact Solution	0·983	0·940	0·856	0·805
$x = 0,\ y = 0$	Approx. Solution	0·981	0·936	0·856	0·826
Point	Exact Solution	1·033	1·126	1·396	1·988
$x = 0,\ y = b$	Approx. Solution	1·040	1·143	1·426	1·934

We see that, if a and b are of the same order of magnitude, the approximate solution (15) is sufficiently accurate. In case of necessity we can always increase the accuracy of the solution by increasing the number of terms in the general expression (8).

In order to get the approximate solution for the points near the short sides of a rectangle, when a is a large number,* we can take the stress-function in the following form (satisfying the conditions at the boundary), viz.

$$\phi = \frac{\sigma}{1+\sigma} \frac{W}{2I} (x^2 - a^2)\, y (1 - e^{-(b-y)\kappa}), \tag{16}$$

where κ is a constant to be determined from the equation (7). If we put

$$e^{-b\kappa} = 0,$$

this equation will be

$$0{\cdot}8a^2(a\kappa)^4 - 0{\cdot}8a(a\kappa)^3 + (0{\cdot}4 - 2a^2)(a\kappa)^2 + 6aa\kappa - 7 = 0, \tag{17}$$

If a is a very large number, we get

$$a\kappa = \tfrac{1}{2}\sqrt{10}.$$

If $a = 4$, we get from (17) $a\kappa = 1{\cdot}298$, and the formula (10) gives us

$$(X_x)_{x=0,\, y=b} = \frac{Wa^2}{2I}\, 2{\cdot}038.$$

* In such a case the exact solution (14) is not convenient for numerical calculation.

This result is only 2·5 per cent. less than the exact solution, given in the table above, and we can conclude that in the case of narrow rectangles the approximate solution (16) is sufficiently accurate.

By the method used for the rectangle, we can obtain an approximate solution in some other cases. For instance, if the equations of the boundaries are (Fig. *a*)

$$y = \pm b, \qquad x^2+y^2-r^2 = 0,$$

we put
$$f(y) = \frac{W}{2I}(r^2-y^2).$$

As an approximate expression for the function ϕ we can take

$$\phi = (y^2-b^2)(x^2+y^2-r^2)(Ay+By^3),$$

where A and B can be calculated from the equation (7). In the same way the problem can be solved in other cases when the cross-section is bounded by vertical lines $y = \pm b$, and by two curves symmetrically situated in relation to y axis.

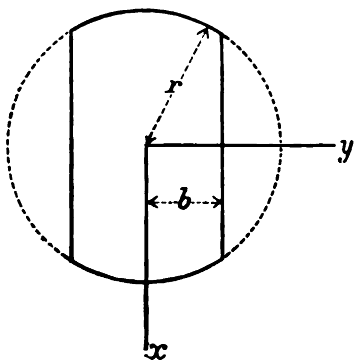

FIG. *a*.

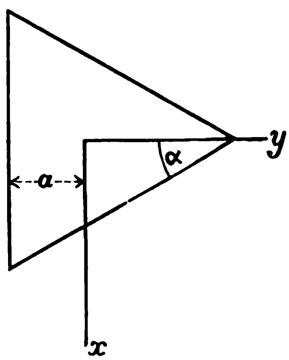

FIG. *b*.

(*c*) *The triangle.*

If the triangle is symmetrical with respect to the y axis (Fig. *b*), the equations of the boundaries will be

$$y+a = 0, \qquad x = \pm \tan \alpha(2a-y).$$

If we put
$$f(y) = \frac{W}{2I}\tan^2\alpha\,(2a-y)^2,$$

the right-hand member of (3) will be equal to zero, and we have to solve

the following equation of equilibrium of the membrane, fixed at its edge:—

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\frac{\sigma}{1+\sigma}\,\frac{Wy}{I}+\tan^2\alpha\,\frac{W}{I}(2a-y)+c. \tag{18}$$

We utilise the general solution (8) and adjust the constant c so as to make the twisting couple equal to zero.

If
$$\tan^2\alpha=\frac{\sigma}{1+\sigma},$$

the solution of the problem is particularly simple. We get it by superposing on the stresses

$$X'_z=\frac{W}{2I}\left[-x^2+\tfrac{1}{3}(2a-y)^2\right],\qquad Y'_z=0, \tag{19}$$

the torsional stresses calculated from the known stress-function

$$\phi=-\frac{\mu\tau}{2a}(y+a)\left[x^2-\tfrac{1}{3}(2a-y)^2\right].$$

These last stresses will be

$$X''_z=\frac{\partial\phi}{\partial y}=-\frac{\mu\tau}{2a}(x^2+2ay-y^2),\qquad Y''_z=-\frac{\partial\phi}{\partial x}=\frac{\mu\tau}{2a}2x(y+a). \tag{20}$$

We have only to adjust the value of $\mu\tau$.

The twisting couple, corresponding to (19), will be

$$-\iint X'_z\,y\,dx\,dy=\tfrac{2}{5}Wa.$$

To the tractions (20) corresponds the twisting couple

$$2\iint\phi\,dx\,dy=\frac{27}{5\sqrt{3}}\mu\tau a^4.$$

The condition that the twisting couple vanishes will be

$$\tfrac{2}{5}Wa+\frac{27}{5\sqrt{3}}\mu\tau a^4=0,$$

and we get
$$\mu\tau=-\frac{2\sqrt{3}}{27}\,\frac{W}{a^3}.$$

Substituting for $\mu\tau$ in (20) and combining (19) and (20), we get

$$X_z = X_z' + X_z'' = \frac{2\sqrt{3}}{27a^4} W[-x^2 + a(2a - y)],$$

$$Y_z = Y_z' + Y_z'' = \frac{2\sqrt{3}}{27a^4} Wx(y + a).$$

The stresses X_z at points of the y axis are represented by the linear function

$$(X_z)_{x=0} = \frac{2\sqrt{3}}{27a^3} W(2a - y).$$

The greatest value of this stress will be

$$(X_z)_{x=0,\ y=-a} = \frac{2\sqrt{3}}{9a^2} W.$$

NOTE ON CERTAIN MODULAR RELATIONS CONSIDERED BY MESSRS. RAMANUJAN, DARLING, AND ROGERS

By L. J. MORDELL.

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IN a recent number of these *Proceedings*,* Messrs. Darling and Rogers have dealt with a number of results enunciated by Ramanujan which may be stated as follows.

Put $$f(r) = r^{\frac{1}{5}} \frac{(1-r)(1-r^4)(1-r^6)(1-r^9)\dots(1-r^{5n\pm 1})\dots}{(1-r^2)(1-r^3)(1-r^7)(1-r^8)\dots(1-r^{5n\pm 2})\dots}$$

so that $$f(r) = r^{\frac{1}{5}}(1-r+r^2+r^3\dots),$$

and write for shortness f, f_1 instead of $f(r)$, $f(r^2)$. Then

(1) $$f^2-f_1+ff_1^2(f^2+f_1) = 0,$$

(2) $$f^{-5}-f^5-11 = \frac{1}{r}\left[\frac{(1-r)(1-r^2)(1-r^3)\dots}{(1-r^5)(1-r^{10})(1-r^{15})\dots}\right]^6,$$

a result which can also be written in the form

$$HG^{11}-r^2GH^{11} = 1+11rG^6H^6,$$

where $$G = 1/(1-r)(1-r^4)\dots(1-r^{5n\pm 1})\dots,$$

$$H = 1/(1-r^2)(1-r^3)\dots(1-r^{5n\pm 2})\dots .$$

(3) $$f^{-1}-f-1 = \frac{1}{r^{\frac{1}{5}}}\, \frac{(1-r^{\frac{1}{5}})(1-r^{\frac{2}{5}})(1-r^{\frac{3}{5}})\dots}{(1-r^5)(1-r^{10})(1-r^{15})\dots},$$

(4) $$\frac{df}{dr}\Big/f = \frac{1}{5r}\, \frac{[(1-r)(1-r^2)(1-r^3)\dots]^5}{(1-r^5)(1-r^{10})(1-r^{15})\dots},$$

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* Issued March 7th, 1921, Ser. 2, Vol. 19. H. B. C. Darling, "Proofs of certain Identities and Congruences enunciated by S. Ramanujan"; L. J. Rogers, "On a Type of Modular Relation."

(5) $$\sum_{0}^{\infty} T(5n+5)r^n = 4830[(1-r)(1-r^2)(1-r^3)\dots]^{24} - 5^{11}r^4[(1-r^5)(1-r^{10})(1-r^{15})\dots]^{24},$$

where $$r[(1-r)(1-r^2)(1-r^3)\dots]^{24} = \sum_{1}^{\infty} T(n)r^n,$$

(6) $$\sum_{0}^{\infty} p_{5n+4}r^n = 5\frac{[(1-r^5)(1-r^{10})(1-r^{15})\dots]^5}{[(1-r)(1-r^2)(1-r^3)\dots]^6},$$

where $$1/(1-r)(1-r^2)(1-r^3)\dots = \sum_{0}^{\infty} p_n r^n,$$

(7) $$\sum_{0}^{\infty} p_{7n+5}r^n = 7\frac{[(1-r^7)(1-r^{14})(1-r^{21})\dots]^3}{[(1-r)(1-r^2)(1-r^3)\dots]^4} + 49r\frac{[(1-r^7)(1-r^{14})(1-r^{21})\dots]^7}{[(1-r)(1-r^2)(1-r^3)\dots]^8}.$$

Mr. Darling gives very complicated proofs of the results (1)–(6), while Prof. Rogers proves (1) and (2) and finds also the formulæ corresponding to (1), when $f(r^2)$ is replaced by $f(r^3)$, $f(r^5)$, ...

Neither of them gives a proof of (7).* I wish to point out that all these formulæ are simple consequences of well known theorems on the Modular Functions. For put as usual

$$\theta_{11}(x, \omega) = 2r^{\frac{1}{8}} \sin \pi x \prod_{1}^{\infty} (1-r^n e^{2\pi i x})(1-r^n e^{-2\pi i x}),$$

where $$r = e^{2\pi i \omega}.$$

Then changing ω into 5ω, putting $x = \omega$, 2ω respectively and dividing, we have

$$\frac{\theta_{11}(\omega, 5\omega)}{\theta_{11}(2\omega, 5\omega)} = \frac{\sin \pi\omega}{\sin 2\pi\omega} \prod_{1}^{\infty} \left(\frac{1-r^{5n\pm 1}}{1-r^{5n\pm 2}}\right).$$

Writing now $\zeta(\omega)$ instead of $f(r)$, or simply ζ when more convenient, we have†

$$\zeta(\omega) = \frac{r^{\frac{1}{10}}\theta_{11}(\omega, 5\omega)}{r^{\frac{3}{10}}\theta_{11}(2\omega, 5\omega)} = r^{\frac{1}{5}}\frac{(1-r)(1-r^4)(1-r^6)(1-r^9)\dots}{(1-r^2)(1-r^3)(1-r^7)(1-r^8)\dots}.$$

* The results (6) and (7) are given by Ramanujan in his paper "Some Properties of $p(n)$, the Number of Partitions of n" in the *Proceedings of the Cambridge Philosophical Society*, Vol. 9.

† The factors $r^{\frac{1}{10}}$, $r^{\frac{3}{10}}$ are put in, because the expression $\zeta(\omega)$ is only a particular case in the study of the quantities $\theta_{11}(a\omega, p\omega)$, $a = 1, 2, \dots, p-1$.

See Klein-Fricke, *Vorlesungen über die Theorie der elliptischen Modulfunktionen*, Vol. 2, p. 383. This treatise will be referred to hereafter as K.F.

The non-homogeneous modular group is defined as the substitutions

$$\omega' = \frac{a\omega+b}{c\omega+d}, \tag{A}$$

where a, b, c, d are any integers satisfying the equation $ad-bc = 1$. Then the most important properties* of $\zeta(\omega)$ are—

(1) It is an invariant of the sub-group Γ_{60} defined by the congruences

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \pmod 5, \tag{B}$$

that is† $$\zeta(\omega') = \zeta(\omega).$$

(2) It has a simple pole and a simple zero in the fundamental polygon F_{60} associated with the sub-group Γ_{60}. The polygon F_{60} is of genus zero and is formed from 60 of the triangles occurring in the well known modular division of the plane. Also‡

$$\zeta(i\infty) = 0, \quad \zeta(2/5) = \infty, \quad \zeta(0) = e^{-2\pi i/5}+e^{2\pi i/5}.$$

(3) Any modular function which is an invariant of the sub-group Γ_{60}, can be expressed rationally in terms of $\zeta(\omega)$ which is called the "Haupt Modul" of the group Γ_{60}. In fact, $\zeta(\omega)$ plays practically the same part for the polygon F_{60} that an ordinary complex variable z does in the z plane.

An another important sub-group§ of the modular group is the group Γ_6 defined by the substitutions (A) wherein

$$c \equiv 0 \pmod 5.$$

The corresponding fundamental polygon F_6 is of genus zero, and is formed from six triangles of the modular division of the plane, namely, the well known fundamental triangle and the five triangles derived from it by the substitutions $-1/(\omega+\kappa)$, $\kappa = 0, 1, 2, 3, 4$.

We have again a "Haupt Modul," usually denoted by $\tau_5(\omega)$, with the

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* K.F., Vol. 2, p. 383.

† We are not considering in this paper the invariants $f(\omega)$ for which $f(\omega')/f(\omega)$ is a root of unity.

‡ K.F., Vol. 1, p. 613.

§ K.F., Vol. 1, p. 635.

same properties for the group Γ_6 that $\zeta(\omega)$ has for the group Γ_{60}. Also*

$$\tau_5(i\infty) = 0, \qquad \tau_5(0) = \infty,$$

while in the neighbourhood of $\omega = 0$, $\tau_5(\omega) = e^{2\pi i/5\omega}$.

Invariants for the groups Γ_6 and Γ_{60} can be found very simply by the use of the modular invariant defined by

$$\Delta(\omega_1, \omega_2) = (2\pi/\omega_2)^{12} r \prod_1^\infty (1-r^n)^{24}, \quad \omega = \omega_1/\omega_2,$$

which, as is well known, is unaltered by the substitutions of the linear homogeneous modular group

$$\omega_1' = a\omega_1 + b\omega_2,$$

$$\omega_2' = c\omega_1 + d\omega_2,$$

where a, b, c, d are any integers for which

$$ad - bc = 1.$$

Thus $\Delta(5\omega_1, \omega_2)/\Delta(\omega_1, \omega_2)$ is a modular function of ω which is invariant for the group Γ_6; for writing

$$5(a\omega_1 + b\omega_2) = a(5\omega_1) + 5b\omega_2,$$

$$c\omega_1 + d\omega_2 = (c/5)(5\omega_1) + d\omega_2,$$

we have at once

$$\Delta[5(a\omega_1 + b\omega_2),\ c\omega_1 + d\omega_2] = \Delta(5\omega_1, \omega_2),$$

since $c/5$ is an integer.

Its value† is given by (K.F., Vol. 2, p. 64)

$$\Delta(\omega_1, \omega_2/5) = \tau_5^4(\omega)\,\Delta(\omega_1, \omega_2).$$

Hence the result (2), which can be written as‡

$$\zeta^{-5} - \zeta^5 - 11 = [\Delta(\omega_1, \omega_2)/\Delta(5\omega_1, \omega_2)]^{\frac{1}{4}},$$

is equivalent to $$\zeta^{-5} - \zeta^5 - 11 = 125/\tau_5(\omega),$$

* K.F., Vol. 1, pp. 637, 638. The $\tau(\omega)$ of K.F. is equal to $-\tau_5(\omega)$.

† Since $\Delta(\omega_1, \omega_2)$ is homogeneous in ω_1, ω_2, $\Delta(\omega_1, \omega_2/5) = 5^{12}\Delta(5\omega_1, \omega_2)$.

‡ All the radicals in this paper are one-valued functions of ω.

a known relation [K.F., Vol. 1, p. 639, formula 4, since $\tau = -\tau_5(\omega)$]. As there remarked, it can also be easily proved by noting that $\tau_5(\omega)$, being also an invariant of the group Γ_{60}, can be expressed rationally in terms of ζ.

The result (3) can be written as

$$\zeta^{-1} - \zeta - 1 = \left[\frac{\Delta(\omega_1/5,\ \omega_2)}{\Delta(5\omega_1,\ \omega_2)}\right]^{\frac{1}{24}}. \tag{C}$$

The right-hand side is an invariant of the sub-group Γ_{60} as both

$$[\Delta(5\omega_1,\ \omega_2)/\Delta^5(\omega_1,\ \omega_2)]^{\frac{1}{24}} \quad \text{and} \quad [\Delta(\omega_1,\ 5\omega_2)/\Delta^5(\omega_1,\ \omega_2)]^{\frac{1}{24}}$$

are invariants of this sub-group [K.F., Vol. 2, p. 67, equation (7), and Vol. 1, p. 644], and hence it can be expressed as a rational function of ζ. As in the neighbourhood of $\omega = i\infty$, the expansion of the right hand-side starts with $r^{-\frac{1}{5}}$, this rational function must, except for a term ζ^{-1}, be a polynomial in ζ. Since a substitution in (A) wherein

$$b \equiv c \equiv 0 \pmod 5,$$

will leave the right-hand side of (C) unaltered and change ζ into $-1/\zeta$,* this polynomial must be $-\zeta$ and a constant, clearly $-\zeta - 1$, which proves the result (3).

The result (4) is an illustration of the method of deducing from the modular function $\zeta(\omega)$, modular forms ζ_1, ζ_2 homogeneous in ω_1, ω_2 and defined by (K.F., Vol. 1, p. 618)

$$\left.\begin{aligned} \zeta_1\sqrt{(20\pi)} &= (1+i)\,\omega_2\zeta\Big/\left(\frac{d\zeta}{d\omega}\right)^{\frac{1}{2}} \\ \zeta_2\sqrt{(20\pi)} &= (1+i)\,\omega_2\Big/\left(\frac{d\zeta}{d\omega}\right)^{\frac{1}{2}} \end{aligned}\right\}. \tag{D}$$

The result (4) is

$$\frac{1}{\zeta}\,\frac{d\zeta}{dr} = \left[\frac{\Delta(\omega_1,\ \omega_2)^5}{\Delta(5\omega_1,\ \omega_2)}\right]^{\frac{1}{24}} \frac{1}{5r}\left(\frac{\omega_2}{2\pi}\right)^2,$$

and since

$$dr = 2\pi i r\,d\omega,$$

this becomes

$$\frac{1}{\zeta}\,\frac{d\zeta}{d\omega} = \frac{2\pi i}{5}\left(\frac{\omega_2}{2\pi}\right)^2 \left[\frac{\Delta(\omega_1,\ \omega_2)^5}{\Delta(5\omega_1,\ \omega_2)}\right]^{\frac{1}{24}},$$

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* K.F., Vol. 1, p. 639.

or, from equations (D),

$$\frac{(1+i)^2\,\omega_2^2}{\zeta_1\zeta_2(20\pi)} = \frac{2\pi i}{5}\left(\frac{\omega_2}{2\pi}\right)^2 \left[\frac{\Delta(\omega_1,\ \omega_2)^5}{\Delta(5\omega_1,\ \omega_2)}\right]^{\frac{1}{24}},$$

or

$$\zeta_1\zeta_2 = \left[\frac{\Delta(\omega_1,\ \omega_2)^5}{\Delta(5\omega_1,\ \omega_2)}\right]^{-\frac{1}{24}} = 5^{-\frac{1}{2}}\left(\frac{\tau_5(\omega)}{\Delta(\omega_1,\ \omega_2)}\right)^{\frac{1}{4}},$$

by using the value of $\tau_5(\omega)$. The identity then becomes a known result, K.F., Vol. 1, p. 640, equation (5).

The result (5) is an expression of the fact* that

$$\Delta(5\omega_1,\ \omega_2) + \sum_{\kappa=0}^{4} \Delta(\omega_1+\kappa\omega_2,\ 5\omega_2) = C\Delta(\omega_1,\ \omega_2),$$

where C is independent of ω_1, ω_2. The left-hand side is an invariant of the homogeneous modular group, its six terms being permuted by these substitutions, and its expansion in powers of r starts off with a term $(2\pi/\omega_2)^{12}\,r$ except for a numerical factor.

Putting now $$\Delta(\omega_1,\ \omega_2) = \left(\frac{2\pi}{\omega_2}\right)^{12} \sum_{1}^{\infty} T(n)\,r^n,$$

we have at once (5), the constant C being found by equating terms in r.

By equating coefficients, we have, as on p. 119 of my paper just cited,

$$T(5s) = T(5)\,T(s),$$

if s is prime to 5; and for all values of s,

$$T(5^{\lambda+2}s) - T(5)\,T(5^{\lambda+1}s) + 5^{11}T(5^{\lambda}s) = 0.$$

Hence since $T(5) \equiv 0 \pmod 5$, it follows by induction that $T(5n) \equiv 0 \pmod 5$. An exactly similar proof shows that $T(7n) \equiv 0 \pmod 7$.

Noting next that

$$\theta_{11}' = 2r^{\frac{1}{8}} \prod_{1}^{\infty} (1-r^n)^3 = 2\sum_{0}^{\infty} (-1)^n (2n+1)\, r^{\frac{1}{8}(2n+1)^2},$$

and that $$r^{\frac{7}{8}} \prod_{1}^{\infty} (1-r^n)^{21} = \Sigma A_n\, r^{n+\frac{7}{8}},$$

where either $$A_n \equiv 0 \pmod 7 \quad \text{or} \quad n \equiv 0 \pmod 7,$$

it is clear by multiplication that

$$T(7n+a) \equiv 0 \pmod 7 \quad \text{if} \quad a = 2,\ 4,\ 6,$$

since no squares are congruent to 2, 4, 6 (mod 7).

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* This is a particular case of a general theorem given in my paper "On Mr. Ramanujan's Empirical Expansions of Modular Functions," *Proceedings of the Cambridge Philosophical Society*, Vol. 19, pp. 118, 119.

Noting again

$$r \prod_1^\infty (1-r^n) = \sum_0^\infty (-1)^n r^{\frac{1}{2}(3n^2 \pm n)+1} = \sum_0^\infty (-1)^n r^{[(6n \pm 1)^2+23]/24},$$

and that $$\prod^\infty (1-r^n)^{23} = \Sigma B^n r^n,$$

where either $$B_n \equiv 0 \pmod{23} \quad \text{or} \quad n \equiv 0 \pmod{23},$$

it follows by multiplication that $T(23n+b) \equiv 0 \pmod{23}$,

where $$b = 5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22$$

are the non-quadratic residues (mod 23). These congruences were given by Ramanujan in these *Proceedings*, Ser. 2, Vol. 17, pp. XIX, XX.

The result (6) is equivalent to

$$\sum_{\kappa=0}^4 \left[\frac{\Delta(5\omega_1, \omega_2)}{\Delta[(\omega_1+\kappa\omega_2)/5, \omega_2]}\right]^{\frac{1}{24}} = 5\left[\frac{\Delta(5\omega_1, \omega_2)}{\Delta(\omega_1, \omega_2)}\right]^{\frac{1}{4}} = \frac{\tau_5(\omega)}{25}.$$

The general term on the left-hand side can be written as the 24th root of

$$\frac{\Delta(5\omega_1, \omega_2)}{\Delta^5(\omega_1, \omega_2)} \bigg/ \frac{\Delta(-5\omega_2, \omega_1+\kappa\omega_2)}{\Delta^5(-\omega_2, \omega_1+\kappa\omega_2)},$$

since $$\Delta(\omega_1+\omega_2, \omega_2) = \Delta(\omega_1, \omega_2),$$

$$\Delta(-\omega_2, \omega_1) = \Delta(\omega_1, \omega_2),$$

and from K.F., Vol. 2, p. 67, equation (8), and p. 27, equation (7), it follows that the left-hand side* is an invariant of the group Γ_6, and hence can be rationally expressed in terms of $\tau_5(\omega)$. Its possible singularities in the polygon F_6 are at $\omega = i\infty$, 0, of which $\omega = i\infty$ is ruled out, as the expansion of the left-hand side involves no negative powers of r. At $\omega = 0$, writing $\omega_1 = -\Omega_2$, $\omega_2 = \Omega_1$, $\Omega = \Omega_1/\Omega_2$, so that $\omega = 0$ corresponds to $\Omega = i\infty$, we have

$$\sum_{\kappa=0}^4 \left[\frac{\Delta(-5\Omega_2, \Omega_1)}{\Delta(-\Omega_2+\kappa\Omega_1, 5\Omega_1)}\right]^{\frac{1}{24}} = \sum_{\kappa=0}^4 \left[\frac{\Delta(\Omega_1, 5\Omega_2)}{\Delta(5\Omega_1, \Omega_2-\kappa\Omega_1)}\right]^{\frac{1}{24}}.$$

Since $$\Delta(5\Omega_1, \Omega_2-\kappa\Omega_1) = \Delta(\Omega_1+\lambda\Omega_2, 5\Omega_2) \quad \text{or} \quad \Delta(5\Omega_1, \Omega_2),$$

for a suitable value of λ, it is clear that $\Omega = i\infty$ will be a singularity of the right-hand side whose expansion in powers of $R = e^{2\pi i\Omega}$ starts with a numerical multiple of

$$(R^{\frac{1}{5}-5})^{\frac{1}{24}} = R^{-\frac{1}{5}}.$$

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* It can be written as $\sum_{\kappa=1}^4 \frac{\sigma_{01}\sigma_{02}}{\sigma_{1\kappa}\sigma_{2\kappa}}$ in the rotation of K.F.

But since near $\omega = 0$, $\tau_5(\omega) = R^{-\frac{1}{5}}$, the left-hand side must be a numerical multiple of $\tau_5(\omega)$ which is easily found.

The proof of the result (7) is based on exactly the same ideas as that for (6). For it can be written as

$$\sum_{\kappa=0}^{6} \left[\frac{\Delta(7\omega_1,\, \omega_2)}{\Delta[(\omega_1+\kappa\omega_2)/7,\, \omega_2]}\right]^{\frac{1}{24}} = 7\left[\frac{\Delta(7\omega_1,\, \omega_2)}{\Delta(\omega_1,\, \omega_2)}\right]^{\frac{1}{6}} + 49\left[\frac{\Delta(7\omega_1,\, \omega_2)}{\Delta(\omega_1,\, \omega_2)}\right]^{\frac{1}{3}}.$$

Putting $\quad \tau_7(\omega) = [\Delta(\omega_1,\, \omega_2/7)/\Delta(\omega_1,\, \omega_2)]^{\frac{1}{6}} = 49r + \ldots,$

then* $\tau_7(\omega)$ is a "Haupt Modul" for the sub-group Γ_8 of the modular group defined by $c \equiv 0 \pmod 7$. The left-hand side, which is also an invariant of the sub-group Γ_8, can be expressed rationally in terms of $\tau_7(\omega)$, and has no singularities at $\omega = i\infty$. The expansion for the singularity at the origin is given by

$$(R^{\frac{1}{7}-7})^{\frac{1}{24}} = R^{-\frac{2}{7}},$$

that is, it is of order 2,† so that the left-hand side is a quadratic polynomial in τ_7 and is easily found to be

$$\frac{\tau_7}{7} + \frac{\tau_7^2}{49}.$$

A similar result cannot be expected when the 5 and 7 are replaced by 11 as the fundamental polygon F_{12}, associated with the corresponding Γ_{12}, is of genus one. A similar expansion, however, holds for 13 as F_{14} is of genus zero. (K.F., Vol. 2, p. 52.)

The result (1) and the similar formulæ by Prof. Rogers are the equations connecting $\zeta(\omega)$ and $\zeta(p\omega)$ for

$$p = 2,\ 3,\ 5,\ 7,\ \ldots.$$

The theory and the results in a slightly different form (*i.e.* using homogeneous variables $\zeta = \zeta_1/\zeta_2$) are given in K.F., Vol. 2, p. 137, and pp. 150, 151. The forms of the algebraic equations are known from *a priori* considerations for far more general functions than the modular functions of which $\zeta(\omega)$ is a very special case. For example, when $p = 2$, the equation between f and f_1 is of degree 3 in each of them, is irreducible, and remains the same when f, f_1 are replaced by $e^{2\pi i/5}f$, $e^{4\pi i/5}f_1$ correspond-

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* See K.F., Vol. 2, p. 52 and pp. 62–64.

† In the neighbourhood of the origin $\tau_7(\omega)$ starts with a multiple of $R^{-\frac{1}{7}}$ as is clear by putting $\omega_1 = -\Omega_2$, $\omega_2 = \Omega_1$ in the formula for $\tau_7(\omega)$.

ing to a change of ω into $\omega+1$. Since f and f_1 both vanish at $\omega=i\infty$, it easily follows that the required equation takes the form

$$f_1^3 f+af_1^2 f^3+bf_1+cf^2=0,$$

where a, b, c are numerical constants.

Since $f=r^{\frac{1}{3}}(1-r+r^2+r^3-r^4\ldots)$, $f_1=r^{\frac{2}{3}}(1-r^2+r^4\ldots)$,

$$r(1-3r^2\ldots)(1-r+r^2\ldots)+ar(1-3r+6r^2\ldots)+c(1-2r+3r^2\ldots)$$
$$+b(1-r^2\ldots)=0,$$

from which $b+c=0$, $1+a-2c=0$, $-1-3a-b+3c=0$,

so that $$f_1^3 f+f_1^2 f^3-f_1+f^2=0.$$

ON DOUBLE SURFACES

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SUMMARY OF RESULTS.

1. Under certain restrictions, all double surfaces fall into three classes.
2. Algebraic double surfaces have at least one double line.
3. Cubic and quartic double surfaces determined.
4. Condition for a double surface in tangential coordinates. It must be of odd class.
5. One of the centro-surfaces must also be a double surface.
6. Effect of a cross-cut on the connectivity of a double surface.
7. Bonnet's associates of double minimal surfaces are deforms, but are not one-sided.
8. Explanation of the anomaly. Remarks on deformation.

PREFACE.

One-sided or double surfaces are those on which it is possible to pass from one side to the other by a finite and continuous path. The simplest example, in a model form, occurs when a long rectangular strip of paper $ABCD$, of which AC and BD are the diagonals, is twisted once, or an odd number of times, and then joined into a twisted ring by making the edge AB coincide with the edge CD, so that A coincides with C and B with D.*

The only occasion on which these surfaces are mentioned in the standard works on Differential Geometry is in connection with Lie's minimal surfaces. It is the purpose of this paper to investigate some general types of such surfaces.

There is hardly any literature on the subject. The following are the only two references given in the Royal Society Catalogue of Scientific Papers:—

P. H. Schoute, *Proc. Roy. Soc. Edin.* (1892), p. 208.

M. Feldblum, *Wiad. Mat.* (1897), p. 101.

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* Forsyth, *Differential Geometry*, p. 295, footnote.

A summary of the latter paper is given in the *Jahrbuch über die Fortschritte der Mathematik* (1897), p. 579. Both of these deal with a particular surface which will be called the Möbius surface. There is a short note on the same surface in one of the *Bulletins of the American Mathematical Society.*

The notation used is generally that of Eisenhart's *Differential Geometry.*

1. *Mode of Formation.*—One-sided or double surfaces are those on which it is possible to pass from one side to the other by a finite and continuous path. The conditions of finiteness and continuity are of importance in this connection. Moreover, we shall assume that the path is not restricted to pass through any particular point on the surface, for such points are in general singularities of the surface, and the continuity of the path is destroyed. The analytical criterion of such surfaces is that, though after describing a finite and continuous path we come back to the same point on the surface, the direction of the normal is reversed.

Let u, v be the Gaussian parameters in terms of which the Cartesian coordinates (assumed rectangular) are expressed. Then, according to the usual convention, the positive directions of $v =$ const., $u =$ const., and the normal form a right-handed system of axes.

Now the Dupin indicatrix at any point is the same for both the sides of the surface; the lines of curvature, therefore, are the same for both the sides. Similarly, all other organic lines (the asymptotic lines for example) which are determined by the nature of the surface, are the same for both the sides. Let some such lines be taken for the parametric lines and the surface defined by the three equations

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v).$$

These functions will be assumed to be uniform, continuous and differentiable. All these limitations, except that of uniformity, are among the usual assumptions of Differential Geometry.

There exist, therefore, two functions ϕ and ψ of u, v such that

$$x = f_1(u, v) = f_1(\phi, \psi),$$

$$y = f_2(u, v) = f_2(\phi, \psi),$$

$$z = f_3(u, v) = f_3(\phi, \psi).$$

Since the parametric lines on the two sides are the same curves, ϕ and ψ must be functions of one of the variables only. We have, therefore, two cases to distinguish.

(A) If ϕ is a function of u only, say U, and ψ a function of v only, say V. Then

$$x = f_1(u, v) = f_1(U, V),$$
$$y = f_2(u, v) = f_2(U, V),$$
$$z = f_3(u, v) = f_3(U, V).$$

(B) But it may happen that the same curve may be designated differently on the two sides; a u-curve, for example, may be styled a v-curve on the other side. The second possibility, therefore, is that ϕ is a function of v only, say V, and ψ a function of u only, say U. Then

$$x = f_1(u, v) = f_1(V, U),$$
$$y = f_2(u, v) = f_2(V, U),$$
$$z = f_3(u, v) = f_3(V, U).$$

Now we shall assume that the parametric system is real, so that no imaginary value of the parameters will give a real point on the surface.

A. Allowing for a slight change of notation, we have

$$x = f(u, v) = f(U, V),$$
$$y = \phi(u, v) = \phi(U, V),$$
$$z = \psi(u, v) = \psi(U, V).$$

If X, Y, Z be the direction-cosines of the normal

$$X = \frac{1}{H}\frac{\partial(y, z)}{\partial(u, v)}, \quad Y = \frac{1}{H}\frac{\partial(z, x)}{\partial(u, v)}, \quad Z = \frac{1}{H}\frac{\partial(x, y)}{\partial(u, v)},$$

where $$H = +\left[\left\{\frac{\partial(y, z)}{\partial(u, v)}\right\}^2 + \left\{\frac{\partial(z, x)}{\partial(u, v)}\right\}^2 + \left\{\frac{\partial(x, y)}{\partial(u, v)}\right\}^2\right]^{\frac{1}{2}}$$

is positive for all possible values of u and v, the parametric system being real.

Now $$y_1 = \frac{\partial\phi}{\partial u} = U_1\frac{\partial\phi}{\partial U} = U_1\Phi_1,$$

where Φ_1 is the value of ϕ_1 when U, V are substituted for u, v. Similarly,

$$y_2 = V_2\Phi_2.$$

Therefore $$X = \frac{1}{H}\frac{\partial(y, z)}{\partial(u, v)} = \frac{U_1V_2}{H}(\Phi_1\Psi_2 - \Phi_2\Psi_1).$$

But $$X' = (\Phi_1\Psi_2 - \Phi_2\Psi_1)/H',$$

where H' is the value of H when U, V are substituted for u, v. H being always positive, the direction-cosines of the normal have opposite signs *if* $U_1 V_2$ *is negative.*

B. Here
$$x = f(u, v) = f(V, U),$$
$$y = \phi(u, v) = \phi(V, U),$$
$$z = \psi(u, v) = \psi(V, U).$$

Since $$y_1 = \frac{\partial\phi}{\partial u} = \Phi_2 U_1,$$

and $$y_2 = \frac{\partial\phi}{\partial v} = \Phi_1 V_2,$$

therefore $$X = U_1 V_2(\Phi_2\Psi_1 - \Phi_1\Psi_2)/H.$$

But $$X' = (\Phi_1\Psi_2 - \Phi_2\Psi_1)/H'.$$

The direction-cosines of the normal, therefore, have opposite signs *if* $U_1 V_2$ *be positive.*

2. *Types of Double Surfaces.*—Let P be a point on the surface with parameters (u, v), and P' the same point on the other side. Let χ denote the transformation (u, v) into (U, V) or (V, U). A repetition of the operation brings us back to the point P.

It is clear that a double surface will arise in the special case where the transformation is algebraic and there is a (1, 1)-correspondence between the parameters (U, V) and (u, v) or (v, u). Then U, V will be homographic functions of their respective parameters. Now consider the transformation x, $(ax+b)/(cx+d)$.

If $c = 0$, this may be reduced to the form x, $A(x+B)$.

If $c \neq 0$, it may be reduced to the form x, $A+B/(x+C)$; or, changing the variable, to the form x, $D/(x+C)$, or, after a further change, to x, $\pm 1/(x+C)$.

CLASS A.—Since $U_1 V_2$ is to be negative for the transformation (u, v) into (U, V), the following are the only possible combinations:

(1) $\begin{matrix} u, & 1/(u+A); \\ v, & -1/(v+B); \end{matrix}$ (2) $\left.\begin{matrix} u, & A(u+B) \\ v, & C(v+D) \end{matrix}\right\} (AC < 0);$ (3) $\begin{matrix} u, & A(u+B); \\ v, & \pm 1/(v+C); \end{matrix}$

the plus sign being taken when A is negative, and the minus sign when A is positive.

It follows, therefore, that under the above conditions double surfaces whose coordinates can be expressed as uniform, continuous and differentiable functions of two real variables, belong to one of these three classes; the passage through infinity has also to be avoided.

To see how double surfaces may be constructed we give particular values to the constants and obtain the types*

$$\begin{array}{llll} (1)\ u, \quad 1/u; & (2)\ u, \quad u+B; & (3)\ u, \ u+B; & (4)\ u, \ -(u+B). \\ \quad v, \ -1/v; & \quad v, \ -(v+D); & \quad v, \ 1/v, & \quad v, \ -1/v. \end{array}$$

With appropriate substitutions, as, for example, $v = \tan\theta$ in (1) and (4), $v = w - \frac{1}{2}D$ in (2), we reduce these to the two types

$$\begin{array}{ll} \text{(i)}\ u, \quad -u, \quad u \quad ; & \text{(ii)}\ u, \quad 1/u, \quad u, \\ \quad \theta, \ \theta+\pi, \ \theta+2\pi; & \quad \theta, \ \theta+\pi, \ \theta+2\pi, \end{array}$$

the third columns giving the effect of a second transformation. It will be found that all the well-known double surfaces come under these heads.

For the cylindroid given by the equations

$$x = u\cos v, \qquad y = u\sin v, \qquad z = a\sin 2v,$$

$$X = \frac{2a\sin v\cos 2v}{\sqrt{(u^2+4a^2\cos^2 2v)}}, \qquad Y = \frac{2a\cos v\cos 2v}{\sqrt{(u^2+4a^2\cos^2 2v)}},$$

$$Z = \frac{u}{\sqrt{(u^2+4a^2\cos^2 2v)}}.$$

The surface, therefore, comes under head (i).

The double minimal surfaces discussed later on fall under (ii).

It will not be possible to identify a given surface as unifacial by this method. We shall see later how this may be done.

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* It may be pointed out that these include all self-inverse linear transformations. For the condition that x, $(ax+b)/(cx+d)$ be self-inverse is that

$$x = \frac{a(ax+b)+b(cx+d)}{c(ax+b)+d(cx+d)}$$

identically, which gives $a+d=0$. Thus x, $(ax+b)/(cx-a)$ is the most general type. If $a \neq 0$, $c \neq 0$, changing the variable we have x, A/x as the most general type. After a further change, we get the most general types in the forms $(u, 1/u)$ and $(u, -1/u)$. If $c = 0$ (a, c cannot both be zero), $(u, -u+A)$ is the most general type; or, more simply, $(u, -u)$. If $a = 0$, the transformation is of the type $(u, B/u)$ which may be split up into $(u, 1/u)$ and $(u, -1/u)$.

All of these are included in the four special types.

To construct a function of u, v such that $f(u, v) = f[R(u), S(v)]$, where $R^2 = S^2 = E$, to use the notation of the Theory of Groups, we take any function of u, v, say $\phi(u, v)$. Form the function $\phi[R(u), S(v)]$. Adding these together or multiplying one by the other, we get a function having the required property. We can thus construct as many double surfaces as we like. It is necessary to see that the parametric system is real and that none of the coordinates pass through infinity during the transformation.

Class B.—These do not seem to exist; for if

$$\begin{array}{lll} u, & R(u), & RS(u), \\ v, & S(v), & SR(v) \end{array}$$

be the transformation, where $RS = SR = E$, it is impossible to make the parametric system exclusively real. If we take u, $R(v)$ conjugate imaginaries, then $S(u)$, $SR(v)$ are also conjugate imaginaries. If ϕ be a function of u, v [such that $\phi(u, v)$ is real when u, v are real] $\phi(u, v)$, $\phi[R(v), S(u)]$ are also conjugate imaginaries. Their sum or product is thus real.

3. *Singularities.*—It is natural to expect that these surfaces have some form of singularities. It will now be proved that algebraic double surfaces possess at least one double line.

Let $$F(x, y, z) = 0$$

be the Cartesian equation. Then the direction-cosines of the normal at any point are

$$\frac{\dfrac{\partial F}{\partial x}}{\left\{\left(\dfrac{\partial F}{\partial x}\right)^2 + \left(\dfrac{\partial F}{\partial y}\right)^2 + \left(\dfrac{\partial F}{\partial z}\right)^2\right\}^{\frac{1}{2}}}, \quad \frac{\dfrac{\partial F}{\partial y}}{\left\{\left(\dfrac{\partial F}{\partial x}\right)^2 + \left(\dfrac{\partial F}{\partial y}\right)^2 + \left(\dfrac{\partial F}{\partial z}\right)^2\right\}^{\frac{1}{2}}},$$

$$\frac{\dfrac{\partial F}{\partial z}}{\left\{\left(\dfrac{\partial F}{\partial x}\right)^2 + \left(\dfrac{\partial F}{\partial y}\right)^2 + \left(\dfrac{\partial F}{\partial z}\right)^2\right\}^{\frac{1}{2}}}.$$

After describing a finite path on the surface, we come back to the same point x, y, z, but the direction of the normal is changed. Therefore (on account of the restriction that the variables and functions are real and

finite), the square root must change its sign in the course of the path and

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

simultaneously somewhere on the surface. This generally cannot be a mere point, for we have not restricted our path to pass through any particular point. It is clear that the proposition has been understated.

As an example, consider the Möbius surface given by the equations

$$x = (a+\rho \sin \tfrac{1}{2}\theta) \cos \theta, \quad y = (a+\rho \sin \tfrac{1}{2}\theta) \sin \theta, \quad z = \rho \cos \tfrac{1}{2}\theta,$$

the transformation being

$$\begin{matrix} \rho, & -\rho, & \rho, \\ \theta, & \theta+2\pi, & \theta+4\pi. \end{matrix}$$

This is the surface generated by a straight line which moves in such a manner that a point on it describes a circle, the straight line always remaining perpendicular to the tangent to the circle, and the spin of the straight line about the tangent being half of the angular velocity of the point on the circle.

There is no general method of finding double lines on surfaces given by parametric equations. We have to find a function f such that

$$\rho = f(\theta)$$

gives the same point on the surface when $\theta = \theta_1$ or θ_2, $\theta_2-\theta_1$ not being a period of the transformation. On the surface we are considering, θ_2 must not be equal to $\theta_1+2\pi$.

Now $$\rho = 2a \sin \tfrac{1}{2}\theta \sec \theta$$

satisfies this condition. Substituting for ρ we obtain the following values for the coordinates

$$x = a, \quad y = a \tan \theta, \quad z = a \tan \theta.$$

If we change θ into $\theta+\pi$, we get the same point.

$$x = a, \quad y = z$$

are, therefore, the equations of a double line on the surface.

We shall now consider some general types of Lie's double minimal

surfaces. The Weierstrassian formulæ* are

$$x = \tfrac{1}{2}\int(1-u^2)\,F(u)\,du + \tfrac{1}{2}\int(1-v^2)\,G(v)\,dv,$$

$$y = \frac{i}{2}\int(1+u^2)F(u)\,du - \frac{i}{2}\int(1+v^2)G(v)\,dv,$$

$$z = \int u\,F(u)\,du + \int v\,G(v)\,dv.$$

For a real surface, u and v are conjugate imaginaries, and so are $F(u)$ and $G(v)$. Let

$$u = \rho \exp i\theta, \qquad v = \rho \exp(-i\theta).$$

The transformation† is

$$\begin{array}{ccc} \rho, & 1/\rho, & \rho, \\ \theta, & \theta+\pi, & \theta+2\pi. \end{array}$$

For a double minimal surface

$$F(u) = -\frac{1}{u^4}\,G\left(-\frac{1}{u}\right).$$

The general solution‡ of this functional equation is

$$F(u) = \frac{1}{u^2}\Big[ia + \sum_{m=0}^{\lambda} c_{2m+1}(u^{2m+1}e^{i\alpha_{2m+1}} + u^{-(2m+1)}e^{-i\alpha_{2m+1}})$$
$$+ \sum_{m=1}^{\mu} c_{2m}(u^{2m}e^{i\alpha_{2m}} - u^{-2m}e^{-i\alpha_{2m}})\Big]$$

and $$G(v) = \frac{1}{v^2}\Big[-ia + \sum_{m=0}^{\lambda} c_{2m+1}(v^{2m+1}e^{-i\alpha_{2m+1}} + v^{-(2m+1)}e^{i\alpha_{2m+1}})$$
$$+ \sum_{m=1}^{\mu} c_{2m}(v^{2m}e^{-i\alpha_{2m}} - v^{-2m}e^{i\alpha_{2m}})\Big],$$

where, a, c's and α's are all real.

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* Forsyth, *Differential Geometry*, p. 281.

† The transformation cannot be

$$\begin{array}{ccc} \rho, & -1/\rho, & \rho, \\ \theta, & \theta, & \theta, \end{array}$$

for then the coordinates would become infinite during the transformation, which moreover does not change the direction of the normal, U_1V_2 being positive.

‡ Forsyth, *Differential Geometry*, p. 296.

We shall assume that λ and μ are finite.

We can now show that the double minimal surfaces for which $c_{2m+1}=0$ (so that odd powers of u, v do not occur in F and G) have the z-axis for a double line.

$$2x = \int \frac{1-u^2}{u^2}\left[ia+\Sigma c_{2m}(u^{2m}e^{ia_{2m}}-u^{-2m}e^{-ia_{2m}})\right]du$$

$$+\int \frac{1-v^2}{v^2}\left[-ia+\Sigma c_{2m}(v^{2m}e^{-ia_{2m}}-v^{-2m}e^{ia_{2m}})\right]dv$$

$$= -2a\sin\theta\left(\rho-\frac{1}{\rho}\right)+2\Sigma\frac{c_{2m}}{2m-1}\cos\{(2m-1)\theta+a_{2m}\}\left(\rho^{2m-1}-\frac{1}{\rho^{2m-1}}\right)$$

$$+2\Sigma\frac{c_{2m}}{2m+1}\cos\{(2m+1)\theta+a_{2m}\}\left(\rho^{2m+1}-\frac{1}{\rho^{2m+1}}\right),$$

putting $\qquad u=\rho\exp i\theta, \qquad v=\rho\exp -i\theta.$

Similarly,

$$2y = -2a\cos\theta\left(\rho-\frac{1}{\rho}\right)-2\Sigma\frac{c_{2m}}{2m+1}\sin\{(2m+1)\theta+a_{2m}\}\left(\rho^{2m+1}-\frac{1}{\rho^{2m+1}}\right)$$

$$-2\Sigma\frac{c_{2m}}{2m-1}\sin\{(2m-1)\theta+a_{2m}\}\left(\rho^{2m-1}-\frac{1}{\rho^{2m-1}}\right),$$

$$z = -2a\theta+2\Sigma\frac{c_{2m}}{2m}\cos(2m\theta+a_{2m})\left(\rho^{2m}+\frac{1}{\rho^{2m}}\right).$$

Unless $a=0$, the surface is periodic, and therefore transcendental. We take

$$a=0.$$

If $\rho=\pm 1$, then $\qquad x=0, \quad y=0,$

and $\qquad z=4\sum_{m=1}^{\mu}\frac{c_{2m}}{2m}\cos(2m\theta+a_{2m}).$

The right-hand side is a continuous function of θ and has the period π. Therefore, to every value of θ (say θ_1) there is another value θ_2, $(0<\theta_1<\pi,\ \theta_2<\pi)$ such that z is the same for both.

$$\rho=\pm 1$$

is, therefore, a double line, or, in other words, the z-axis is a double line on the surface.

As a particular case, the Henneberg surface* for which

$$F(u) = 1-u^{-4} = (u^2-u^{-2})/u^2,$$

and $$a = 0,\ c_2 = 1,\ c_4,\ c_6 \dots = 0,\ a_2 = a_4 \dots = 0,$$

has the z-axis for a double line.

4. *Cubic and quartic double surfaces.*—The fact that double surfaces have at least one double line may be used to find the cubic and quartic double surfaces. A cubic having a double line must have it straight and must be a ruled surface. Consider a ruled surface having the line of striction for the guiding curve for convenience. Take the generator through a point on the line of striction, a line perpendicular to it in the tangent plane, and the normal to the surface for a moving set of right-handed system of axes. As the origin moves on the guiding curve, the first axis generates the surface. It is a double surface, if, when the moving origin comes back to its original position, the positive direction of the generator coincides with the initial negative direction. Or, in other words, if the total spin about the second axis is an odd multiple of π. It is also necessary to see that the moving origin does not go off to infinity. The line of striction, therefore, must be a closed curve.

If we take the double straight line of the cubic for the z-axis, the equation of the surface can be written in the form

$$(ax^3+3bx^2y+3cxy^2+dy^3)+z\,(a'x^2+2b'xy+c'y^2) \\ +(a''x^2+2b''xy+c''y^2) = 0.$$

Put $$x = \rho\cos\theta, \quad y = \rho\sin\theta.$$

Then

$$z = -\frac{\rho(a\cos^3\theta+3b\cos^2\theta\sin\theta+3c\cos\theta\sin^2\theta+d\sin^3\theta) \\ +(a''\cos^2\theta+2b''\cos\theta\sin\theta+c''\sin^2\theta)}{a'\cos^2\theta+2b'\cos\theta\sin\theta+c'\sin^2\theta}.$$

It is clear that all the conditions are satisfied if the equation

$$a'\cos^2\theta+2b'\cos\theta\sin\theta+c'\sin^2\theta = 0$$

has no real roots; that is, if $b'^2-a'c'$ is negative, the transformation being

$$\begin{matrix} \rho, & -\rho, & \rho, \\ \theta, & \theta+\pi, & \theta+2\pi. \end{matrix}$$

* Forsyth, *loc. cit.*, p. 289.

The cylindroid and the Henry Smith surface* are particular cases. It may be noted, by the way, that the latter is a mere algebraic transform of the former.

A quartic surface with a non-planar double line is ruled. We can, therefore, find the double quartics with non-planar double lines. But the algebra is formidable.

5. *Tangential equation.*—Let the irreducible homogeneous equation of degree n,

$$F(X, Y, Z, T) = 0,$$

be the tangential equation of an algebraic surface. Then the point of contact of any tangent plane is given by the equations

$$x : y : z : 1 = \partial F/\partial X : \partial F/\partial Y : \partial F/\partial Z : \partial F/\partial T.$$

Now for a double surface it will be possible, starting from a set of initial values, say X_0, Y_0, Z_0, T_0, to change X, Y, Z, T continuously, subject to the condition $F = 0$, into $-X_0$, $-Y_0$, $-Z_0$, $-T_0$ without making any of the coordinates infinite. Therefore, $\partial F/\partial T$ must not vanish during the transformation. Now

$$\partial F/\partial T = 0$$

will represent a *cone* in the four-dimensional space in which X, Y, Z, T are the Cartesian coordinates. The necessary and sufficient condition for a double surface is that the *cone* be imaginary. It follows, therefore, that $\partial F/\partial T$ must be of even degree. F must, therefore, be of odd degree. A double surface will be of odd class. For example, the tangential equation of the cylindroid is

$$T(X^2+Y^2) = 2aXYZ,$$

which satisfies the required condition. Henneberg's surface† has the equation

$$(T-4Z)(X^2+Y^2)^2 = 4Z(3X^2+3Y^2+2Z^2)(X^2-Y^2).$$

It will be seen that as the point of contact describes a continuous path returning to its initial position on the other side, the tangent plane sweeps out the whole of space. Through every point in space there is a real tangent cone.

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* Forsyth, *loc. cit.*, p. 308.

† Forsyth, *loc. cit.*, p. 287.

6. *Centro-surfaces.*—The centro-surfaces of double surfaces are interesting. It is necessary to recall some well-known properties of centro-surfaces. Let M be a point on a surface, and MC_1 and MC_2 the lines of curvature through it. Then the normals to the surface along the curve C_1 form a developable surface D_1 and the normals along C_2 form another developable D_2. Let S_1 and S_2 be the centro-surfaces traced by the centres of curvature corresponding to the systems C_1 and C_2 respectively. Then the developable D_1 has its edge of regression on the surface S_1 and envelops the surface S_2. The tangent to the line of curvature C_1 is therefore parallel to the normal to S_1 at the corresponding point, and the tangent to the line C_2 is parallel to the normal to S_2 at the corresponding point.

Now consider a moving trihedral formed by the tangent to the lines C_1, C_2 in the positive sense and the normal, forming a right-handed system of axes. If the surface be one-sided, we can make the origin describe a continuous path on the surface, returning to the same point on the opposite side. The direction of the normal having been changed, the direction of one of the other axes (the tangents to the lines of curvature) is also reversed. Supposing that the direction of the tangent to the line C_1 is reversed, the direction of the normal to S_1 has also been reversed. The centro-surface S_1 is, therefore, one-sided.

It may happen, however, that some surfaces have a line of parabolic points which has to be crossed. One of the radii of curvature becomes infinite, and the corresponding point on the centro-surface goes off to infinity.

For Henneberg's surface*

$$E = G = 0, \qquad F = 18(1-u^4)(1-v^4)(1+uv)^2\, u^{-4}v^{-4},$$

$$D = 6(u^{-4}-1), \qquad D' = 0, \qquad D'' = 6(v^{-4}-1).$$

The Gaussian measure of curvature K, is

$$-\tfrac{1}{9}\,\frac{u^4v^4}{\{1-(u^4+v^4)+u^4v^4\}(1+uv)^4}$$

$$= -\tfrac{1}{9}\,\frac{\rho^8}{\{1-2\rho^4\cos 4\theta+\rho^8\}(1+\rho^2)^4}.$$

* Forsyth, *loc. cit.*, p. 289.

It is clear that K does not vanish during the transformation

$$\begin{array}{ccc} \rho, & 1/\rho, & \rho, \\ \theta, & \theta+\pi, & \theta+2\pi. \end{array}$$

One of the centro-surfaces is, therefore, unifacial.

It is easily seen that the above proposition holds for double minimal surfaces in general. For, if the coordinates be expressed in the customary Weierstrassian form

$$ds^2 = \tfrac{1}{2}F(u)\,G(v)(1+uv)^2,$$

and
$$K = -\frac{1}{F(u)\,G(v)(1+uv)^4},$$

which clearly does not vanish anywhere on the path, u, v being conjugate imaginaries, as are also $F(u)$, $G(v)$.

Again, for a ruled surface,*

$$K = -\frac{\beta^2}{[(u-a)^2+\beta^2]^2},$$

where β is the parameter of distribution, and $u-a$ the distance along a generator measured from the line of striction. Therefore

$$K = 0 \quad \text{when} \quad \beta = 0.$$

On the Möbius surface β is a constant. It has, therefore, the property mentioned above. But for the cylindroid

$$x = u\cos\theta, \qquad y = u\sin\theta, \qquad z = a\sin 2\theta,$$

$$\beta = 0 \quad \text{when} \quad \theta = \pi/4 \text{ or } 3\pi/4.$$

The cylindroid has, therefore, two parabolic lines.

7. *Inversion.*—If we invert a double surface with respect to any point, the new surface is also unifacial. We shall verify the statement by inverting the cylindroid

$$z(x^2+y^2) = 2axy,$$

with respect to the origin, taking the radius of inversion to be a for simplicity. Then the new surface is given by the equations

$$x = \tfrac{1}{4}a\sin\phi\,\mathrm{cosec}\,\theta, \qquad y = \tfrac{1}{4}a\sin\phi\sec\theta, \qquad z = \tfrac{1}{4}a(1+\cos\phi)\sec\theta\,\mathrm{cosec}\,\theta,$$

* Eisenhart, *Differential Geometry*, p. 247.

the transformation being

$$\begin{array}{ccc} \phi, & -\phi, & \phi, \\ \theta, & \theta+\pi, & \theta+2\pi. \end{array}$$

We may describe the surface in the following way. Consider a plane rotating about the axis of z, which lies in it. Let θ be the angle which the plane makes with its initial position, which is taken as the y-plane. The surface is generated by a circle of radius $\frac{1}{2}a \operatorname{cosec} 2\theta$ lying in the plane, and passing through the origin, the centre always lying on the z-axis. The z-axis is a double line except the portion between $z = \pm a$.

8. *Connectivity.*—A double surface is, by definition, multiply-connected. There is a remarkable analogy between a double surface and a two-sheeted Riemann surface, the double line of the former corresponding to the branch line of the latter. We consider now the effect of a cross-cut on a double surface.

On any bounded surface, a cross-cut may join two points on the same boundary line, or two points on two distinct boundary lines, or it may proceed from a point on a boundary line and come back to itself. In the second case the number of distinct boundary lines is diminished by unity. For the first case, let us start from any point on the boundary line in question and mark with arrow-heads the sense of the boundary line as we come back to the starting point. Now it may happen that the arrow-heads at the two extremities of the cross-cut may point in opposite directions (Fig. 1), or in the same (Fig. 2).

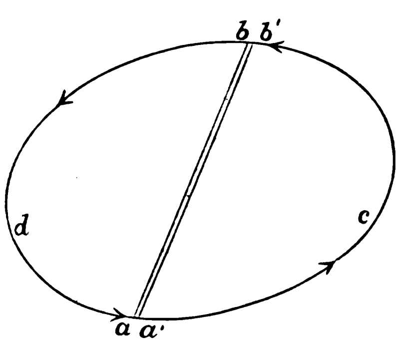

Fig. 1.

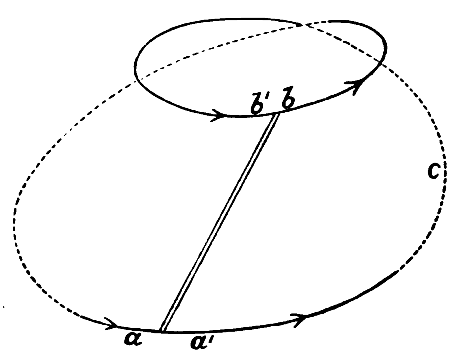

Fig. 2.

$aa'cb'bd$ is the boundary line. In Fig. 1, ab and $a'b'$ are the two sides of the cut, and in Fig. 2, ab' and $a'b$. In the former case it is obvious that

the surface is divided into two distinct parts $a'cb'a'$ and $abda$. We show that in the second alternative, Fig. 2, the surface must have been unifacial and that the effect of the cross-cut is to convert it into a bifacial one.

Suppose that a man starts from a' keeping the surface to his left, follows the boundary line till he comes to b', and then, still keeping the surface to his left, follows the edge $b'a$ of the cut. If he now proceeds along the boundary line against the direction of the arrow-heads, he will come to the point b and following the edge ba' of the cut, back finally to a'. The number of boundary lines is not affected by the cut. It is obvious that if there had not been a cut, and the man had followed his path $a'cba'$ as before, he would have been on the opposite side of the surface from which he started. The surface must, therefore, have been a unifacial one, and it has been converted into a bifacial one by the cross-cut.

In the third case, when the cross-cut comes back to itself, the number of boundary lines is increased by unity, and the surface is divided into two distinct parts.

A cross-cut, therefore, increases by unity, or diminishes by unity, or keeps intact, the number of distinct boundary lines of a surface. In the latter case, however, the surface is converted from a unifacial into a bifacial one.

9. *Deformation.*—The word deformation is used in several distinct senses in Mathematics. In Riemann's development of the Theory of Functions, it is used to denote the modification of a flexible and extensible surface provided there is no tearing or joining. This we may call conformal deformation. In Differential Geometry, on the other hand, it is used to denote the modification of a flexible, but inextensible surface provided there is no tearing or joining. This we may call continuous deformation. It is, therefore, a special form of conformal deformation. Let us examine Bonnet's associates of double minimal surfaces. If a surface be given by the Weierstrassian formulæ,* Bonnet's associated surfaces are obtained by substituting $\phi(u)$, $\psi(v)$ for $F(u)$, $G(v)$ respectively, where

$$\phi(u) = e^{i\alpha} F(u), \qquad \psi(v) = e^{-i\alpha} G(v).$$

Then the arc-element of the original as well as the associated surface is

* *Vide* § 3.

given by the equation

$$ds^2 = (1+uv)^2 F(u)\, G(v)\, du\, dv.$$

It is, therefore, concluded that the associated surfaces are continuous deforms of the original surface.

It is easily seen that ϕ and ψ do not satisfy the equation

$$F(u) = -\frac{1}{u^4}\, G\left(-\frac{1}{u}\right),$$

satisfied by F and G. This, of course, proves that either the surfaces are not one-sided, or the equation is not a necessary condition.

Among these associated surfaces there is one particularly important, the adjoint surface obtained by putting

$$a = \pi/2.$$

It is easily shown that for the original surface the asymptotic lines are given by the equation

$$F(u)\, du^2 + G(v)\, dv^2 = 0,$$

and the lines of curvature by

$$F(u)\, du^2 - G(v)\, dv^2 = 0.$$

For the adjoint surface the asymptotic lines are given by the latter equation and the lines of curvature by the former. It follows, therefore, that the parametric lines on the adjoint surface are related to the lines of curvature in the same way as the parametric lines on the original surface to the asymptotic lines. The direction-cosines of the normal are

$$X = \frac{2\cos\theta}{\rho+\dfrac{1}{\rho}}, \qquad Y = \frac{2\sin\theta}{\rho+\dfrac{1}{\rho}}, \qquad Z = \frac{\rho-\dfrac{1}{\rho}}{\rho+\dfrac{1}{\rho}},$$

for both the surfaces. It will be seen that no linear transformation of ρ and θ is possible which will alter the signs of X, Y, Z but leave unaltered the values of the coordinates for the adjoint surface. For, since both X, Y change their signs, $\tan\theta$ will be unaltered; therefore (θ, θ), $(\theta, \pi+\theta)$ are the only two alternatives. In either case it will be found that no appropriate transformation for ρ can be determined. The adjoint surface of a double minimal surface, therefore, is not a double surface with linear transformation, to which type the latter surface belongs.

We shall now apply the tangential equation. It will be convenient to

take the simplest of the double minimal surfaces, the Henneberg surface. If we put

$$F(u) = f'''(u), \qquad G(v) = g'''(v),$$

the tangential equation of a double minimal surface can be written in the form*

$$T = f'\left(\frac{X+iY}{1-Z}\right) + g'\left(\frac{X-iY}{1-Z}\right) - (X-iY)f\left(\frac{X+iY}{1-Z}\right) - (X+iY)g\left(\frac{X-iY}{1-Z}\right).$$

The equation of the associates of the Henneberg surface is

$$T = 2 \text{ R.P. } e^{i\alpha}\left[\left\{3\left(\frac{X+iY}{1-Z}\right)^2 - 2 - \left(\frac{1-Z}{X+iY}\right)^2\right\} - (X-iY)\left\{\left(\frac{X+iY}{1-Z}\right)^3 - 2\frac{X+iY}{1-Z} + \frac{1-Z}{X+iY}\right\}\right],$$

where R.P. denotes the real part of the following expression. Simplifying we obtain

$$\{(T-4Z\cos\alpha)(X^2+Y^2)^2 - 4Z\cos\alpha\,(X^2-Y^2)(3X^2+3Y^2+2Z^2)\}^2 = 256\sin^2\alpha\, X^2Y^2(X^2+Y^2+Z^2)^3,$$

which agrees with Forsyth's result when $\alpha = 0$. These being of an even class cannot be one-sided.

10. *Topology.*—It is a well-known proposition in topology that unifacial surfaces can be deformed into unifacial surfaces only, the deformation being conformal.† It is difficult to see why continuous deformation which does not allow even stretching should destroy the unifaciality of the surface.

The explanation seems to be the fundamental difference between the Cartesian and the Gaussian method of representing a surface. In the Gaussian method we admit the possibility of an infinite number of sheets superposed on one another (as, for example, when one of the parameters enters as periodic functions). But in the Cartesian method there is no such possibility. This difference may give rise to wide diversity when the surface is deformed. We can illustrate the point with a paper model of two superposed sheets of the surface mentioned in the preface. So long as the two sheets are kept together, any deformation will preserve

* Forsyth, *loc. cit.*, p. 287.

† Forsyth, *Theory of Functions*, p. 362.

the unifaciality of the surface. But if the two sheets are separated, the surface becomes a bifacial one. This explanation is also borne out by the tangential equation of the associates of Henneberg's surface, which becomes a perfect square when a is zero.

If this explanation is accepted, it follows that the solutions of the partial differential equation of the second order which governs the deformation of a surface will depend on the independent variables employed. If a surface be given by the equation

$$z = f(x, y),$$

and also by three equations of the Gaussian type, in which one of the variables enters as periodic functions, it stands to reason that the deforms of the surface given by the two distinct equations will not be identical.

I hope to follow this up in another paper.

THE ALGEBRAIC THEORY OF ALGEBRAIC FUNCTIONS OF ONE VARIABLE

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Introduction.

In 1906 Dr. J. C. Fields published a book* containing a purely algebraic theory of the algebraic functions of one variable. During the succeeding five years various papers by the same author appeared, some in the *American Journal of Mathematics* and others in the *Transactions of the American Mathematical Society.*

In 1912 Dr. Fields published in the *Transactions of the Royal Society of London* a new treatment† of the subject and followed this up with three other papers.‡

The present paper is a development of a shorter one§ by the writer. The majority of the proofs given depend upon properties of rational functions of (z, u) built upon special bases subject to choice. The first seven sections consist for the most part of definitions of terms and statements of fact. A sketch of Dr. Fields' proof of the first existence theorem of (5) is given in a footnote. The second existence theorem of (5) is less comprehensive and differs in one other particular from a similar theorem in Dr. Fields' papers, but the proof along similar lines is so immediate that

* "Theory of the Algebraic Functions of a Complex Variable," Mayer and Müller, Berlin.

† "On the Foundations of the Theory of Algebraic Functions of One Variable," Ser. A, Vol. 212.

‡ "Direct Derivation of the Complementary Theorem from Elementary Properties of the Rational Functions," *Proceedings of the International Congress of Mathematicians*, Cambridge, 1912; "Proofs of certain general Theorems relating to Orders of Coincidence," *Proc. London Math. Soc.*, Ser. 2, Vol. 12; "Proof of the Complementary Theorem," *Proc. London Math. Soc.*, Ser. 2, Vol. 15.

§ "Derivation of the Complementary Theorem from the Riemann-Roch Theorem," *American Journal of Math.*, Vol. 39, No. 3.

it too is given in a footnote. The paper is characterized by the use made of non-positive bases and the role of u^{n-1} and by the fact that the complementary theorem becomes in the final sections an instrument of proof.

I. *Rational Functions of* (z, u) *built on an Order-Number for a Cycle relative to a Given Value of* z.

(1) $f(z, u)$ will denote $u^n f_0 + u^{n-1} f_1 + \ldots + f_n$, in which f_0 is unity and the remaining coefficients are rational functions of z. If $f(z, u)$ is reducible in the domain of rational functions of z, it is supposed that all its irreducible factors in that domain are different.

(2) The fundamental equation $f(z, u) = 0$ defines u as an algebraic function of z, for which there are n expansions $u_1, u_2, \ldots, u_n$ in the vicinity of a given value of z. The expansions are series in powers of the element $z-a$ or $1/z$, according as the given value of z is a or ∞. The series may contain fractional and negative powers of the element. The fundamental equation is said to be of type m relative to a given value of z, if $-m$ is the least power of the element appearing when $f_0, f_1, \ldots, f_n$ are expanded in the vicinity of the given value of z.

(3) An integral rational function of (z, u) is formed by applying the operations of addition and multiplication to u and rational functions of z. A rational function of (z, u) is the quotient of two integral rational functions of (z, u), in which the denominator has no factor in common with $f(z, u)$. A given rational function of (z, u) is equal for all values of (z, u) for which $f(z, u) = 0$ to one and only one rational function of (z, u) in reduced form

$$u^{n-1} g_1 + u^{n-2} g_2 + \ldots + g_n.$$

A representation* of $R(z, u)$, a rational function of (z, u), for values of z in the vicinity of a given value of z, is afforded by the expression

$$\sum_{t=1}^{n} \frac{R(z, u_t)}{Q_t(z, u_t)} Q_t(z, u),$$

in which for the vicinity of the given value of z, the function $Q_t(z, u)$, not in general a rational function of (z, u), is the product of all the linear factors of $f(z, u)$ except $u - u_t$. This type of representation for $f_u(z, u)$,

* "On the Foundations, etc.," formula (8).

the partial derivative of $f(z, u)$ with respect to u, is

$$\sum_{t=1}^{n} Q_t(z, u).$$

In a polynomial in u with coefficients, functions of z, which are expansible in the vicinity of a given value of z involving only integral (with at most only a finite number of negative) powers of the element, each product of u to a power, and the element to a power is called a term relative to the given value of z.

(4) The expansions $u_1, u_2, \ldots, u_n$ in the vicinity of a given value of z fall into groups or cycles. The number of expansions in the various cycles may be denoted by $\nu_1, \nu_2, \ldots, \nu_r$. An expansion in a cycle of ν expansions proceeds according to ascending integral powers of some ν-th root of the element, and on replacing in it this particular ν-th root by another, the result is another expansion in the same cycle. An integral multiple of $1/\nu$ is called an *order-number* for the cycle. An expansion of u in the vicinity of a given value of z, with respect to which the fundamental equation is of type m, does not involve the element to a power less than $-m$, and if l denotes the least power of the element in any of the terms of the reduced form of a rational function of (z, u), an expansion of such function in the vicinity of the given value of z does not involve the element to a power less than $l-m(n-1)$. Usually not all the coefficients in the expansion of the function are zero, and the least power of the element present is then called the *order of the function* for the expansion of u employed. If, however, all the coefficients in the expansion are zero, infinity is said to be the order of the function for the expansion of u employed. The order of a rational function of (z, u) is the same for expansions of u from the same cycle and is called *the order of the function for the cycle*; moreover, this order if finite is an order-number for the cycle. If u_t belongs to a cycle of ν_s expansions in the vicinity of a given value of z, *the order of* $f'_u(z, u)$ *for the cycle* is the least power of the element in $Q_t(z, u_t)$ which is finite, and *is denoted by* μ_s. An order-number for the cycle which is not less than $m(n-1)+\mu_s-1+1/\nu_s$, where the equation is of type m relative to the given value of z, will be said to be *adjoint of type* m. A rational function of (z, u) is *built on* a given order-number for a cycle, if the order of the rational function for cycle is not less than the order-number.

(5) It has been observed that the order supposed finite of a rational

function of (z, u) for a cycle relative to a given value of z is an order-number for the cycle. An important converse in the form of an existence theorem* has been noted by Dr. Fields. It may be stated as follows:

On assuming an order-number for a cycle relative to a given value of z, there exists a rational function of (z, u) possessing as order for the cycle the assumed order-number and possessing as great orders as desired for the remaining cycles relative to the given value of z.

A second existence theorem† of somewhat similar nature admits of the statement: If the fundamental equation is of type m relative to a given value of z, then on assuming an order-number not adjoint of type m for a cycle relative to the given value of z, there exists a rational function of (z, u) built on the assumed order-number, possessing as great orders as desired for the remaining cycles relative to the given value of z, and containing in its reduced form among other terms u to the power $n-1$ and the element to the power $m(n-1)-1$.

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* If the cycle is made up of ν_s expansions of u in the vicinity of the given value of z, and if τ_s is the assumed order-number for the cycle, then in the function

$$\Sigma R_t^{\tau_s \nu_s - \mu_s \nu_s} Q_t(z, u),$$

u_t is one of the ν_s expansions of u belonging to the cycle, R_t is the particular ν_s-th root of the element appearing in u_t, and the summation extends to the ν_s expansions belonging to the cycle. The least power of the element in the expansion of this function for each of the ν_s expansions belonging to the cycle is τ_s, while the expansions of this function are all identically zero for the remaining expansions of u in the vicinity of the given value of z. If only this function were a rational function of (z, u) it would possess all the properties required. On writing it in the form

$$u^{n-1} g_1 + u^{n-2} g_2 + \ldots + g_n,$$

it appears that the coefficients $g_1, g_2, \ldots, g_n$, when expanded in the vicinity of the given value of z, involve only integral (with at most only a finite number of negative) powers of the element. It is possible to split for the various values of t the expanded form of g_t into g_t', g_t'', the powers of the element being all greater in the latter part than in the former, so that the term in $u^{n-t} g_t''$ of least degree in the element has for any of the ν_s expansions belonging to the cycle an order greater than τ_s and for the remaining expansions of u in the vicinity of the given value of z orders as great as previously agreed upon. The function

$$u^{n-1} g_1' + u^{n-2} g_2' + \ldots + g_n'$$

is then such as required in the statement of the theorem.

† If τ_s is the assumed order number, it suffices to proceed as in the former case except by employing $m(n-1) + \mu_s - 1$ instead of τ_s and by choosing g_1'' to be identically zero, thereby leaving g_1' to be ν_s times the element to the power $m(n-1)-1$.

II. *Rational Functions of (z, u) built on a Basis relative to a Given Value of z.*

(6) A basis $\tau_1, \tau_2, \ldots, \tau_r$ relative to a given value of z is an aggregate of order-numbers, one for each cycle, relative to the given value of z. The zero basis relative to a given value of z is one in which all the order-numbers are zero. A non-positive basis relative to a given value of z is one not containing a positive order-number. A basis adjoint of type m relative to a given value of z, with respect to which the fundamental equation is of type m, contains only order-numbers adjoint of type m. A rational function of (z, u) whose orders relative to a given value of z are all finite furnishes therewith a basis relative to the given value of z. A rational function of (z, u) is built on a basis relative to a given value of z, if it is built on each of the order-numbers comprising the basis relative to the given value of z.

(7) If $R(z, u)$ is a rational function of (z, u) built on a basis $\tau_1, \tau_2, \ldots, \tau_r$ relative to a given value of z, with respect to which the fundamental equation is of type m, and if λ denotes a number not greater than any of the numbers $\tau_1-\mu_1, \tau_2-\mu_2, \ldots, \tau_r-\mu_r$, it appears from the second expression in (3) that the reduced form of $R(z, u)$ contains no term with a degree in the element less than $\lambda-m(n-1)$. On the other hand, a term involving u to a power p and the element to a power q is itself a rational function of (z, u) built on the basis relative to the given value of z, provided $q-mp$ is not less than any of the order-numbers, comprising the basis; and if any finite number of any such terms are multiplied by arbitrary constants the sum of such products is included under the reduced form of the general rational function of (z, u) built on the basis relative to the given value of z. Also there are no terms in the reduced form of such general function possessing a degree in the element less than $\lambda-m(n-1)$. There is at most only a finite number of terms for which q is not less than $\lambda-m(n-1)$ and $q-mp$ is less than some order-number in the basis. On multiplying these terms by arbitrary constants and subjecting the sum of such products to conditions sufficient to build it on the basis relative to the given value of z, there results a function likewise included under the reduced form of the general rational function of (z, u) built on the basis relative to the given value of z.

(8) It is now proposed to prove the following lemma :—

In the reduced form of the general rational function of (z, u) built on a non-positive basis $\tau_1, \tau_2, \ldots, \tau_r$ relative to a given value of z the coeffi-

cients of terms involving the element to negative powers are expressible linearly in terms of arbitrary constants, in number not less than

$$-\sum_{s=1}^{r} \tau_s \nu_s + \tfrac{1}{2} \sum_{s=1}^{r} \left(\mu_s - 1 + \frac{1}{\nu_s}\right) \nu_s.$$

It will be supposed that the fundamental equation is of type m relative to the given value of z. The reduced form of the general rational function of (z, u) built on the zero basis relative to the given value of z may be written as the sum of reduced forms $F'(z, u)$ and $S'(z, u)$, the former of these being made up of all those terms in the reduced form of the general function in which the power of the element is less than m times the power of u. The aggregate of arbitrary constants in terms of which the coefficient of terms in $F'(z, u)$ are linearly expressible will be denoted by (F') and the number of such constants by B. The coefficients of terms in $S'(z, u)$ are arbitrary, and (S') will be employed to denote their aggregate. It is to be supposed that $F''(z, u)$ and $S''(z, u)$ are other such reduced forms, and (F'') and (S'') the corresponding aggregates, and that in the reduced form of the product of $F'(z, u)+S'(z, u)$ and $F''(z, u)+S''(z, u)$ the coefficient of the term of degree $n-1$ in u and $m(n-1)-1$ in the element is equated to zero for arbitrary values of the constants in the aggregates (F') and (S'). This product is the sum of three products $S'(z, u)$, $S''(z, u)$, $F'(z, u)\{F''(z, u)+S''(z, u)\}$ and $S'(z, u)F''(z, u)$. The first of the three products, like the forms $S'(z, u)$, $S''(z, u)$ themselves, contains no term in which the power of the element is less than m times the power of u, and in it the reduction is effected by successively replacing as often as necessary $-u^n$ by $u^{n-1}f_1+\ldots+f_n$, which operation when applied to a single term not in reduced form gives terms in which the power of the element has been decreased by m at most and the power of u by 1 at least. Consequently, in the reduced form of the first of the three products there is no term in which the power of the element is less than m times the power of u, and hence the coefficient of the term of degree $n-1$ in u and $m(n-1)-1$ in the element is zero for arbitrary values of the constants in all four aggregates. In the reduced form of the second of the three products the coefficient of the term of degree $n-1$ in u and $m(n-1)-1$ in the element, is the sum of at most B expressions, each of which is obtained by multiplying a number of (F') by a linear form of numbers from (F''), (S''). In the reduced form of the last of the three products the coefficient of the term of degree $n-1$ in u and $m(n-1)-1$ in the element is the sum of expressions, each of which is obtained by multiplying a number of (S') by a linear form of numbers from (F''). Putting, as above directed, the coefficient equal to zero means nothing more nor less than

equating to zero each of these linear forms. The number of equations resulting from equating to zero the linear forms of the first type is at most equal to B, and the number of linearly independent equations resulting from equating to zero the linear forms of the second type, which involve only numbers from (F''), an aggregate of B constants, is at most equal to B. Hence the total number of linearly independent conditions applied to (F''), (S'') is at most equal to $2B$.

As a result of applying the above conditions to (F''), (S'') the orders of the function $F''(z, u)+S''(z, u)$ are all adjoint of type m relative to the given value of z, for if not it is possible to give values to the constants in (F''), (S'') remaining arbitrary, so that the resulting specific function $F''(z, u)+S''(z, u)$ possesses the orders of a basis which is not adjoint of type m relative to the given value of z. The reduced form of the general rational function of (z, u) on this basis is the reduced form of the product of $F'(z, u)+S'(z, u)$ and the resulting specific function $F''(z, u)+S''(z, u)$, and in it the coefficient of the term of degree $n-1$ in u and $m(n-1)-1$ in the element is zero, which conflicts with the second existence theorem of (5). Hence the independent conditions, $2B$ in number at most, have produced adjointness of type m relative to the given value of z, having been applied to the coefficients in the reduced form of the general rational function of (z, u) built on the zero basis relative to the given value of z. Therefore, on employing the first existence theorem of (5), it appears that these independent conditions are in number not less than

$$\sum_{s=1}^{r} \left\{ m(n-1)+\mu_s-1+\frac{1}{\nu_s} \right\} \nu_s,$$

from which it follows that

$$B \geqslant \tfrac{1}{2}mn(n-1)+\tfrac{1}{2}\sum_{s=1}^{r}\left(\mu_s-1+\frac{1}{\nu_s}\right)\nu_s.$$

The reduced form of the general rational function of (z, u) built on a non-positive basis $\tau_1, \tau_2, \ldots, \tau_r$ relative to the given value of z may be written as the sum of reduced forms $F(z, u)$ and $S(z, u)$, the former of these being made up of all those terms in the reduced form of the general function in which the power of the element is less than m times the power of u. The coefficients of terms in $F(z, u)$ are expressible linearly in terms of an aggregate of arbitrary constants to be denoted by (F), while coefficients of terms in $S(z, u)$ are arbitrary. As a consequence of the first existence theorem of (5) it appears that the aggregate (F) is made up of

$$B-\sum_{s=1}^{r}\tau_s\nu_s$$

arbitrary constants. Hence the aggregate (F) is made up of arbitrary constants, in number not less than

$$\tfrac{1}{2}mn(n-1)-\sum_{s=1}^{r}\tau_s\nu_s+\tfrac{1}{2}\sum_{s=1}^{r}\left(\mu_s-1+\frac{1}{\nu_s}\right)\nu_s.$$

The function $s(z, u)+S(z, u)$ is now to be arranged in the form $P(z, u)+Q(z, u)$ in which $P(z, u)$ is made up of all those terms in the function in which the power of the element is negative. The terms of $S(z, u)$ are all included under $Q(z, u)$, and a term of $F(z, u)$ of degree p in u and q in the element is also included under $Q(z, u)$ provided $0 \leqslant q < mp$, $0 < p < n$. Since there are at most $\tfrac{1}{2}mn(n-1)$ such terms, the coefficients in the function $P(z, u)$ are expressible linearly in terms of arbitrary constants in number not less than

$$-\sum_{s=1}^{r}\tau_s\nu_s+\tfrac{1}{2}\sum_{s=1}^{r}\left(\mu_s-1+\frac{1}{\nu_s}\right)\nu_s,$$

which proves the lemma.

It is, of course, evident that the number here written down might not even be positive. However, it will be shown in (15) that if the non-positive basis does not possess an order-number greater than the corresponding order of u^{n-1} relative to the given value of z, the number of arbitrary constants in question is precisely the number appearing in the statement of the lemma.

III. *Rational Functions of* (z, u) *built on a Basis.*

(9) A basis τ is made up of bases, hereafter known as constituent bases, one relative to each value of z, and all, unless perhaps a finite number, being zero bases. A rational function of (z, u) possessing none but finite orders relative to one value of z possesses none but finite orders relative to all values of z and furnishes therewith a basis. In a zero basis τ the constituent bases are all zero bases. In a non-positive basis τ, if there are constituent bases not zero bases they are non-positive bases. The basis μ furnished by $f'_u(z, u)$ will always be included among bases being discussed, and those values of z relative to which not all the constituent bases are zero bases, and those values of z for which there are less than n cycles, will be paired off with the elements κ of a finite aggregate (κ). The basis τ contains as constituent basis relative to the value of z paired off with a given element κ,

$$\tau_1^{(\kappa)},\ \tau_2^{(\kappa)},\ \ldots,\ \tau_{r_\kappa}^{(\kappa)},$$

and the number of expansions of u in the various cycles corresponding are

$$\nu_1^{(\kappa)},\ \nu_2^{(\kappa)},\ \ldots,\ \nu_{r_\kappa}^{(\kappa)}.$$

Bases $\tau+1/\nu$ and τ differ only through the order-number of the former for one cycle relative to a given value of z exceeding by the least possible the corresponding order-number of the latter. The same distinction applies in the case of bases τ and $\tau-1/\nu$.

(10) A rational function of (z, u) is built on a basis τ if it is built on each constituent basis. Of course, the reduced form in (3) in which $g_1, g_2, \ldots, g_n$ are all identically zero is built on any basis τ, and it will be called the zero form. Reduced forms $F_1(z, u)$, $F_2(z, u)$, ..., $F_l(z, u)$ of rational functions of (z, u) are said to be linearly dependent if there exist constants $c_1, c_2, \ldots, c_l$, not all of which are zero, so that

$$c_1 F_1(z, u)+c_2 F_2(z, u)+\ldots+c_l F_l(z, u)$$

is the zero form. If no such constants exist, the reduced forms are said to be linearly independent.

If there are less than l linearly independent reduced forms of rational functions of (z, u) built on a basis τ, there are less than $l+1$ linearly independent reduced forms of rational functions of (z, u) built on a basis $\tau-1/\nu$. For if it is supposed that $F_1(z, u)$, $F_2(z, u)$, ..., $F_{l+1}(z, u)$ are $l+1$ linearly independent reduced forms of rational functions of (z, u) built on the basis $\tau-1/\nu$ one of them, say $F_{l+1}(z, u)$, not built on the basis τ may be selected and constants $c_1, c_2, \ldots, c_l$ chosen, so that

$$F_1(z, u)+c_1 F_{l+1}(z, u),\ F_2(z, u)+c_2 F_{l+2}(z, u),\ \ldots,\ F_l(z, u)+c_l F_{l+1}(z, u)$$

are all built on the basis τ. From the linear dependence of these l forms follows the linear dependence of the original $l+1$ forms, which contradicts the supposition already made with regard to them.

It will be supposed that $f(z, u)$ breaks up into ρ irreducible factors. On denoting one of such factors by $f_\sigma(z, u)$, the reciprocal of the product of the $\rho-1$ remaining factors is a rational function of (z, u) with respect to the equation $f_\sigma(z, u) = 0$, and will have in connection with that equation a reduced form. The product of such reduced form and the $\rho-1$ remaining factors is a rational function of (z, u) in its reduced form with respect to the fundamental equation, and is built on the zero basis. In fact its orders for expansions of u satisfying $f_\sigma(z, u) = 0$ are all zero, while its orders for remaining expansions of u are all infinity. The ρ such reduced forms are consequently linearly independent, and, moreover,

the reduced form of any rational function of (z, u) built on the zero basis is the sum of constant multiples of these ρ forms.

On combining the results of the two previous paragraphs, it appears that there are not more than

$$\rho-\Sigma_\kappa \sum_{s=1}^{r_\kappa} \tau_s^{(\kappa)} \nu_s^{(\kappa)}$$

linearly independent reduced forms of rational functions of (z, u) built on a non-positive basis τ. If each positive order-number in a basis is replaced by zero the result is a non-positive basis. Since a rational function of (z, u) built on the basis τ is also built on the non-positive basis furnished above, and since of the rational functions of (z, u) built on the latter basis not more than a finite number have linearly independent reduced forms, it appears that of the rational functions of (z, u) built on the basis τ not more than this same finite number have linearly independent reduced forms. The actual number of linearly independent reduced forms of rational functions of (z, u) built on a basis τ is denoted by N_τ. On multiplying each of these N_τ reduced forms by an arbitrary constant, the sum of such products is the reduced form of the general rational function of (z, u) built on the basis τ and contains N_τ arbitrary constants. A conclusion from a result already established in the present section is that $N_{\tau-1/\nu}$ is either the same as N_τ or exceeds it by unity.

(11) It will be supposed that t is a non-positive basis not possessing an order-number greater than the corresponding order furnished by u^{n-1}. In the reduced form of the general rational function of (z, u) built on the constituent basis relative to the value of z paired off with κ, the coefficients of terms of negative degree in the element are expressible linearly in terms of arbitrary constants, in number at least

$$-\sum_{s=1}^{r_\kappa} t_s^{(\kappa)} \nu_s^{(\kappa)} + \tfrac{1}{2} \sum_{s=1}^{r_\kappa} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}\right) \nu_s^{(\kappa)},$$

which latter may be denoted by $\lambda^{(\kappa)}$. If the coefficients are so expressed, then in such reduced form the part composed of terms involving the element to none but negative powers appears as the sum of arbitrary constant multiples of at least $\lambda^{(\kappa)}$ rational functions of (z, u), each in reduced form and built on the basis t, and none expressible linearly with constant coefficients in terms of the rest. On applying this argument to all values of z paired off with the elements of (κ) and taking account of the n linearly independent functions $1, u, u^2, \ldots, u^{n-1}$ built on the basis t, it appears

that there is a total of at least

$$n-\Sigma_\kappa \sum_{s=1}^{r_\kappa} t_s^{(\kappa)}\nu_s^{(\kappa)}+\tfrac{1}{2}\Sigma_\kappa \sum_{s=1}^{r_\kappa}\left(\mu_s^{(\kappa)}-1+\frac{1}{\nu_s^{(\kappa)}}\right)\nu_s^{(\kappa)}$$

reduced forms of rational functions of (z, u) built on the basis t. It is clear from the above and from the way in which each reduced form involves the elements for all values of z, that the forms are linearly independent, and hence N_t is not less than the number immediately preceding. Since the sum of the orders of $f'_u(z, u)$ for all expansions of u is zero, what has been proved is that

$$N_t \geqq n-\Sigma_\kappa \sum_{s=1}^{r_\kappa} t_s^{(\kappa)}\nu_s^{(\kappa)}-\tfrac{1}{2}\Sigma_\kappa \sum_{s=1}^{r_\kappa}(\nu_s^{(\kappa)}-1).$$

(12) The residual order-number relative to a given value of z is the order possessed by the element raised to such power as to have a residue for that value of z. The residual order-number is then -1 or $+1$ according as the given value of z is a or ∞. It is to be supposed that $W(z, u)$ is a rational function of (z, u) possessing none but finite orders; the basis which it furnishes will be denoted by ω. Bases τ and $\bar{\tau}$ are said to be complementary to the level of ω if for each and every cycle the sum of the order-numbers of τ and $\bar{\tau}$ exceeds by the least possible the sum of the residual order-number and the order-number of ω.

Bases $\tau-1/\nu$ and $\bar{\tau}$ satisfy the requirements of being complementary to the level of ω except for one cycle, known as the excepted cycle. There is not a rational function of (z, u) built on the former basis and another built on the latter basis each possessing relative to the excepted cycle the precise order of the basis on which it is built, for if so the function obtained on dividing their product by $W(z, u)$ would have as order for the excepted cycle and for none else the residual order-number, which conflicts with the fact that the sum of the residues of a rational function of (z, u) is zero. This result stated in numerical form is

$$(N_{\tau-1/\nu}-N_\tau)+(N_{\bar{\tau}}-N_{\bar{\tau}+1/\nu})-1 \leqq 0.$$

If t is a basis not possessing an order-number greater than the corresponding order-number of τ, then by repeated application of this type of formula and by addition of the results, it follows that

$$(N_t-N_\tau)+(N_{\bar{\tau}}-N_{\bar{t}})-\Sigma_\kappa \sum_{s=1}^{r_\kappa}(\tau_s^{(\kappa)}-t_s^{(\kappa)})\nu_s^{(\kappa)} \leqq 0.$$

(13) It is now proposed to establish the complementary theorem,* which is contained in the complementary formula

$$N_\tau + \tfrac{1}{2}\Sigma_\kappa \sum_{s=1}^{r_\kappa} \tau_s^{(\kappa)} \nu_s^{(\kappa)} = N_{\bar{\tau}} + \tfrac{1}{2}\Sigma_\kappa \sum_{s=1}^{r_\kappa} \bar{\tau}_s^{(\kappa)} \nu_s^{(\kappa)}.$$

If it is supposed that the expression on the left is less than the one on the right, then on selecting a non-positive basis t not possessing an order-number greater than either the corresponding order-number of τ or the corresponding order furnished by u^{n-1}, and such that that part of the sum

$$\Sigma_\kappa \sum_{s=1}^{r_\kappa} \bar{t}_s^{(\kappa)} \nu_s^{(\kappa)}$$

relating to each irreducible equation is positive, it follows from employing the final formula in (12) and from the fact that $N_{\bar{t}} = 0$, that

$$N_t - \tfrac{1}{2}\Sigma_\kappa \sum_{s=1}^{r_\kappa} (\tau_s^{(\kappa)} + \bar{\tau}_s^{(\kappa)})\, \nu_s^{(\kappa)} + \Sigma_\kappa \sum_{s=1}^{r_\kappa} t_s^{(\kappa)} \nu_s^{(\kappa)} < 0.$$

But since the sum of the orders of $W(z, u)$ for all expansions of u is zero, and since τ and $\bar{\tau}$ are complementary to the level of ω, it follows that

$$N_t < n - \Sigma_\kappa \sum_{s=1}^{r_\kappa} t_s^{(\kappa)} \nu_s^{(\kappa)} - \tfrac{1}{2}\Sigma_\kappa \sum_{s=1}^{r_\kappa} (\nu_s^{(\kappa)} - 1),$$

which conflicts with the final formula of (11), thereby completing the proof of the complementary theorem.

IV. *Applications of the Complementary Theorem.*

(14) From the complementary formulæ stated for bases τ, $\bar{\tau}$ and $\tau - 1/\nu$, $\bar{\tau} + 1/\nu$, it follows that

$$(N_{\tau-1/\nu} - N_\tau) + (N_{\bar{\tau}} - N_{\bar{\tau}+1/\nu}) = 1.$$

This may be called the unit theorem and affirms that of the general rational functions of (z, u) built on bases $\tau - 1/\nu$, $\bar{\tau}$, one and only one possesses for the excepted cycle the precise order of the basis on which it is built. From the unit theorem the complementary theorem follows. For, on stating the unit theorem in the form

$$(N_{\tau-1/\nu} - N_{\bar{\tau}+1/\nu}) = (N_\tau - N_{\bar{\tau}}) + 1,$$

* "On the Foundations, etc.," formula (83).

an immediate corollary of it is that

$$(N_t - N_{\bar{t}}) = (N_\tau - N_{\bar{\tau}}) + \Sigma_\kappa \sum_{s=1}^{r_\kappa} (\tau_s^{(\kappa)} - t_s^{(\kappa)})\, \nu_s^{(\kappa)},$$

in which t, $\bar{t}$ are any complementary bases. This is equivalent to the complementary formula, if t, $\bar{t}$ are chosen $\bar{\tau}$, τ respectively.

The Riemann-Roch theorem is a particular case of the complementary theorem obtained by taking one of the bases non-positive. It, too, is equivalent to the complementary theorem. For on supposing that

$$N_\tau + \tfrac{1}{2}\Sigma_\kappa \sum_{s=1}^{r_\kappa} \tau_s^{(\kappa)} \nu_s^{(\kappa)}$$

is less than the corresponding expression for $\bar{\tau}$, then as a result of applying successively formulæ of the type of the first formula in (12) it appears that

$$N_t + \tfrac{1}{2}\Sigma_\kappa \sum_{s=1}^{r_\kappa} t_s^{(\kappa)} \nu_s^{(\kappa)}$$

is less than the corresponding expression for $\bar{t}$, in which no order-number of t exceeds the corresponding order-number of τ, which conflicts with the Riemann-Roch theorem on t being chosen non-positive.

(15) If $\tau_1, \tau_2, \ldots, \tau_r$ is a non-positive basis relative to a given value of z not possessing an order-number greater than the corresponding order furnished by u^{n-1}, then in the reduced form of the general rational function of (z, u) built on the basis relative to the given value of z, the coefficients of terms of negative degree in the element are expressible linearly in terms of arbitrary constants, in number not less than

$$-\sum_{s=1}^{r} \tau_s \nu_s + \tfrac{1}{2} \sum_{s=1}^{r} \left(\mu_s - 1 + \frac{1}{\nu_s}\right) \nu_s.$$

But the number of such arbitrary constants is also not greater than this, for if so then on associating with the basis relative to the given value of z bases relative to remaining values of z, the aggregate constituting a non-positive basis τ not possessing an order-number greater than the corresponding order furnished by u^{n-1}, and such that that part of the sum

$$\Sigma_\kappa \sum_{s=1}^{r_\kappa} \bar{\tau}_s^{(\kappa)} \nu_s^{(\kappa)},$$

relating to each irreducible equation is positive, it follows by the argu-

ment in (11) that

$$N_\tau > n - \Sigma_\kappa \sum_{s=1}^{r_\kappa} \tau_s^{(\kappa)}\nu_s^{(\kappa)} - \tfrac{1}{2}\Sigma_\kappa \sum_{s=1}^{r_\kappa} (\nu_s^{(\kappa)} - 1),$$

which conflicts with the complementary formula, as the latter involves the sign of equality not the sign greater than.

It is to be supposed that $\tau_1, \tau_2, \ldots, \tau_r$ is a basis relative to a given value of z. As indicated in (7), integers i, j can be determined so that in the reduced form of the general rational function of (z, u) built on the basis relative to the given value of z, coefficients of terms of degree less than i in the element are all zero and coefficients of terms of degree j in the element are all arbitrary. The integer j will also be required to be zero or positive, while i is necessarily equal to or less than j. A non-positive basis $t_1, t_2, \ldots, t_r$ relative to the given value of z can now be selected, not possessing an order-number greater than either the corresponding order-number in the basis $\tau_1, \tau_2, \ldots, \tau_r$ or the corresponding order furnished by u^{n-1} relative to the given value of z. An integer h can be determined so that in the reduced form of the general rational function of (z, u) built on the basis $t_1, t_2, \ldots, t_r$ relative to the given value of z, coefficients of terms of degree less than h in the element are all zero. The integer h is necessarily zero or negative and will be chosen not to exceed the integer i. A general rational function of (z, u) in reduced form in which no power of the element is less than h nor as great as j and in which coefficients of terms are all arbitrary, is to be considered. As a result of applying conditions to this function to build it on the basis $t_1, t_2, \ldots, t_r$ relative to the given value of z, coefficients of terms of zero or positive degree in the element remain arbitrary, while coefficients of terms of negative degree in the element are expressible linearly in terms of

$$-\sum_{s=1}^{r} t_s\nu_s + \tfrac{1}{2}\sum_{s=1}^{r}\left(\mu_s - 1 + \frac{1}{\nu_s}\right)\nu_s$$

arbitrary constants. The number of such conditions which are linearly independent is, therefore,

$$-nh + \sum_{s=1}^{r} t_s\nu_s - \tfrac{1}{2}\sum_{s=1}^{r}\left(\mu_s - 1 + \frac{1}{\nu_s}\right)\nu_s.$$

On employing this result and the first existence theorem of (5), it appears that the number of linearly independent conditions applicable to the general function above to build it on the basis $\tau_1, \tau_2, \ldots, \tau_r$ relative to the given value of z, is

$$-nh + \sum_{s=1}^{r} \tau_s\nu_s - \tfrac{1}{2}\sum_{s=1}^{r}\left(\mu_s - 1 + \frac{1}{\nu_s}\right)\nu_s.$$

Of these linearly independent conditions, $n(i-h)$ are accounted for by equating to zero coefficients of terms of degree less than i in the element. Therefore, the number* of linearly independent conditions applicable to a general rational function of (z, u) in reduced form in which no power of the element is less than i nor as great as j and in which coefficients of terms are all arbitrary, in order to build it on the basis $\tau_1, \tau_2, \ldots, \tau_r$ relative to the given value of z, is

$$-ni+\sum_{s=1}^{r}\tau_s\nu_s-\tfrac{1}{2}\sum_{s=1}^{r}\left(\mu_s-1+\frac{1}{\nu_s}\right)\nu_s.$$

†Corresponding to only a finite number of constituent bases of a basis τ is i necessarily negative. In that case, the general function made up of terms of degree in the element negative but not less than the corresponding value of i may be called the preparatory function relative to such constituent basis. The sum of all such preparatory functions and the general function made up of terms of zero degree in every element will be called the preparatory function relative to the basis τ. The sum of numbers of the type of the preceding, in which i is zero if not negative, exceeds by N_τ the number of linearly independent conditions applicable to the preparatory function relative to the basis τ to convert it into the reduced form of the general rational function of (z, u) built on the basis τ.

* "Proofs of certain, etc.," formulæ (21) and (24).

† This paragraph differs merely in statement from the corresponding discussion on pp. 228-230, "Proofs of certain, etc."

ON THE INTEGRALS OF THE DIFFERENTIAL EQUATIONS OF THE FIRST ORDER DERIVABLE FROM AN IRREDUCIBLE ALGEBRAIC PRIMITIVE

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1. Let $\phi(x, y, c)$ be any polynomial in x, y and c, which cannot be broken up into two or more polynomial factors in x, y and c, then the equation

$$\phi(x, y, c) = 0 \tag{I}$$

is said to be irreducible.

2. It may however happen that $\phi(x, y, c)$ can be broken up into factors which are polynomials in x and y, but are not rational in c:—*e.g.* the equation

$$c^2(x^2-1)-2cxy+y^2-1 = 0 \tag{II}$$

does not represent a proper curve of the second degree.

The left-hand side breaks up into the factors

$$y-cx-(1+c^2)^{\frac{1}{2}}, \quad y-cx+(1+c^2)^{\frac{1}{2}},$$

each of which equated to zero represents a straight line. This kind of reducibility is not important in what follows, and will not be referred to again.

3. On the other hand it may be possible by substituting for c some function of c, which may be called C, to replace the equation (I) by another of the form

$$\psi(x, y, C) = 0, \tag{III}$$

where $\psi(x, y, C)$ is a polynomial in x, y and C, which is of lower degree in C than $\phi(x, y, c)$ is in c.

In this case equation (I) will be regarded as *reducible in the degree of the arbitrary constant necessarily involved.* So far as the relation between x and y is concerned the two equations (I) and (III) are equivalent,

but the differential equations, to which they give rise, do not appear in exactly the same form, *if a strict adherence to the rules of elimination is maintained.*

Consider, for example, the equation

$$c^2y-(c^3+c)x-(c^4+2c^2+1) = 0. \tag{IV}$$

This gives $$y-\left(c+\frac{1}{c}\right)x-\left(c+\frac{1}{c}\right)^2 = 0.$$

Replacing $c+\dfrac{1}{c}$ by C it becomes

$$y-Cx-C^2 = 0. \tag{V}$$

The differential equation corresponding to (V) is

$$y-px-p^2 = 0, \tag{VI}$$

but that corresponding to (IV), if the rules of elimination are strictly adhered to, is

$$(y-px-p^2)^2 = 0,$$

which is of course equivalent to (VI), but appears in a slightly different form. And in the general case, if there are m values of c corresponding to each value of C, and if the differential equation corresponding to (III) be

$$f(x, y, p) = 0, \tag{VII}$$

then the differential corresponding to (I) is

$$[f(x, y, p)]^m = 0. \tag{VIII}$$

It will be seen in what follows that the kind of reducibility described in this section is important.

4. Suppose that the degree of $\phi(x, y, c)$ in c is n, and suppose that the equation (I) may or may not be reducible in the degree of the arbitrary constant necessarily involved in the manner described in the preceding section. The differential equation is found by eliminating c between (I) and

$$\frac{\partial\phi(x, y, c)}{\partial x}+p\frac{\partial\phi(x, y, c)}{\partial y} = 0. \tag{IX}$$

Treating (I) as an equation for c, let its roots be $c_1, c_2, \ldots, c_n$. Then the eliminant is

$$\prod_{r=1}^{n}\left(\frac{\partial\phi(x, y, c)}{\partial x}+p\frac{\partial\phi(x, y, c)}{\partial y}\right)_{c=c_r} = 0. \tag{X}$$

When the factors on the left-hand side of (X) have been multiplied

out, the values of $c_1, c_2, \ldots, c_n$ inserted and a factor, which is a function of x and y only, introduced if necessary to avoid fractional expressions, the left-hand side of (X) becomes a polynomial in x, y and p of degree n in p.

If $c = c_r$ be any root of (I) it is always possible to express

$$\frac{\partial\phi(x, y, c_r)}{\partial x}\Big/\frac{\partial\phi(x, y, c_r)}{\partial y}$$

as a rational function of x, y and c_r in a form which is integral in c_r and of degree $(n-1)$ at most in c_r.

When this has been done let its value be denoted by $-\Theta(c_r)$. Also let

$$\Phi(p) \equiv [p-\Theta(c_1)][p-\Theta(c_2)]\ldots[p-\Theta(c_n)], \tag{XI}$$

so that

$$\Phi(p) = 0 \tag{XII}$$

gives the same values of p as (X).

I proceed to investigate the reducibility of $\Phi(p)$, in the manner explained in Weber's *Algebra*, Vol. 1, pp. 461–2. If $\Phi(p)$ is reducible in p it must have an irreducible factor. Call this factor, if it exist, $\chi(p)$; and suppose the coefficient of the highest power of p which it contains is taken to be unity. Then $\chi(p)$ must vanish when p has one (at least) of the values $\Theta(c_1), \Theta(c_2), \ldots, \Theta(c_n)$.

The equations

$$\chi[\Theta(c)] = 0, \tag{XIII}$$

and

$$\phi(x, y, c) = 0, \tag{I}$$

are simultaneously satisfied by one or more values of c. But $\phi(x, y, c)$ is irreducible. Therefore equation (XIII) is satisfied by all the n values of c which satisfy (I). Now, if $\chi(p)$ be of degree $s\,(<n)$ in p, let

$$\chi(p) = [p-\Theta(c_1)][p-\Theta(c_2)]\ldots[p-\Theta(c_s)], \tag{XIV}$$

so that

$$\chi[\Theta(c)] = [\Theta(c)-\Theta(c_1)][\Theta(c)-\Theta(c_2)]\ldots[\Theta(c)-\Theta(c_s)]. \tag{XV}$$

Now (XIII) is satisfied by all the values of c which satisfy (I). Hence if s be less than n, we must have $\Theta(c_{s+1})$ equal to one of the values $\Theta(c_1), \Theta(c_2), \ldots, \Theta(c_s)$. If therefore all the values $\Theta(c_1), \Theta(c_2), \ldots, \Theta(c_n)$ are different from one another, then we must have $s = n$, and then $\chi(p)$ is identical with $\Phi(p)$: and, as $\chi(p)$ is irreducible, therefore in this case $\Phi(p)$ is irreducible. If however only $s\,(<n)$ of the values

$$\Theta(c_1),\ \Theta(c_2),\ \ldots,\ \Theta(c_n),$$

are different from one another, then

$$\chi(p) = [p-\Theta(c_1)][p-\Theta(c_2)]\ldots[p-\Theta(c_s)]. \tag{XIV}$$

Also every factor of $\Phi(p)$ other than $\chi(p)$ must be identical with $\chi(p)$ because it vanishes for one at least of the values $\Theta(c_1), \Theta(c_2), \ldots, \Theta(c_s)$ of p, and therefore, since $\chi(p)$ is irreducible, for all of them. Therefore $\Phi(p)$ is a power of $\chi(p)$.

This involves the fact that n is divisible by s. Let $n = sm$, then

$$\Phi(p) = [\chi(p)]^m.$$

It follows that $c_1, c_2, \ldots, c_n$ fall into m groups of s each, such that the values of $\Theta(c)$ for each group are the same as $\Theta(c_1), \Theta(c_2), \ldots, \Theta(c_s)$.

5. Up to the end of the preceding article the line of Weber's argument has been followed. The significance of p has not yet come into play. By considering what it is, further information can be obtained.

It will now be proved that when the integer m of the preceding article is greater than unity, it will be possible to replace the primitive

$$\phi(x, y, c) = 0, \tag{I}$$

by another of the form

$$\psi(x, y, C) = 0, \tag{III}$$

where m values of c correspond to each value of C, whilst the equations (I) and (III) express the same relation between x and y. It has been shown that when m is greater than unity,

$$\Theta(c_{s+1}) = \Theta(c_r),$$

where r is some one of the values $1, 2, \ldots, s$. Consequently, using the value of $\Theta(c_r)$ given in § 4,

$$\left[\frac{\partial\phi(x, y, c)}{\partial x}\Big/\frac{\partial\phi(x, y, c)}{\partial y}\right]_{c=c_r} = \left[\frac{\partial\phi(x, y, c)}{\partial x}\Big/\frac{\partial\phi(x, y, c)}{\partial y}\right]_{c=c_{s+1}}. \tag{XVI}$$

Now c_r and c_{s+1} both satisfy (I), from which, if δ denote partial differentiation with regard to x and y, it follows that

$$\left.\begin{aligned}
\frac{\partial\phi(x, y, c_r)}{\partial x} + \frac{\partial\phi(x, y, c_r)}{\partial c_r}\,\frac{\delta c_r}{\delta x} &= 0\\
\frac{\partial\phi(x, y, c_r)}{\partial y} + \frac{\partial\phi(x, y, c_r)}{\partial c_r}\,\frac{\delta c_r}{\delta y} &= 0\\
\frac{\partial\phi(x, y, c_{s+1})}{\partial x} + \frac{\partial\phi(x, y, c_{s+1})}{\partial c_{s+1}}\,\frac{\delta c_{s+1}}{\delta x} &= 0\\
\frac{\partial\phi(x, y, c_{s+1})}{\partial y} + \frac{\partial\phi(x, y, c_{s+1})}{\partial c_{s+1}}\,\frac{\delta c_{s+1}}{\delta y} &= 0
\end{aligned}\right\}. \tag{XVII}$$

Since (I) considered as an equation for c has no repeated roots it follows that

$$\frac{\partial\phi(x, y, c_r)}{\partial c_r} \quad \text{and} \quad \frac{\partial\phi(x, y, c_{s+1})}{\partial c_{s+1}}$$

do not vanish. Hence from (XVI) and (XVII) it follows that

$$\frac{\delta(c_r, c_{s+1})}{\delta(x, y)} = 0, \tag{XVIII}$$

so that c_{s+1} is a function of c_r, the functional form not involving x and y.

Let
$$c_{s+1} = \lambda(c_r). \tag{XIX}$$

It will now be proved that the curve

$$\phi(x, y, c_r) = 0 \tag{XX}$$

is the same as the curve
$$\phi(x, y, c_{s+1}) = 0. \tag{XXI}$$

Take a point ξ, η in the plane of the variables x, y. Consider the values $c = c_r$ and $c = c_{s+1}$ at $x = \xi$, $y = \eta$, so that ξ, η is a point on both curves (XX) and (XXI). Also $c_{s+1} = \lambda(c_r)$ by (XIX), the form of λ not depending on ξ, η.

Take another point ξ', η' on the curve (XX), so that $\phi(\xi', \eta', c_r) = 0$. Then the value of c_{s+1} at ξ', η' is still equal to $\lambda(c_r)$, and therefore to the value of c_{s+1} at ξ, η; so that

$$\phi(\xi', \eta', c_{s+1}) = 0.$$

Hence ξ', η' lies on both curves (XX) and (XXI). But ξ', η' is any point whatever on the curve (XX). Hence every point on (XX) lies on (XXI). But these two equations are of the same degree. Therefore the curves (XX) and (XXI) are identical.

Let us suppose that the values of c corresponding to the second of the groups of s each into which the n values of c are divided, are c_{s+1}, c_{s+2}, ..., c_{2s}. We may suppose that

$$\Theta(c_{s+1}) = \Theta(c_1),\ \Theta(c_{s+2}) = \Theta(c_2),\ \ldots,\ \Theta(c_{2s}) = \Theta(c_s).$$

Then the curves corresponding to

$$c = c_{s+1},\ c = c_{s+2},\ \ldots,\ c = c_{2s},$$

are identical with the curves corresponding to

$$c = c_1,\ c = c_2,\ \ldots,\ c = c_s,$$

respectively; and so on. The curves corresponding to the values of c in

any group are identical with the curves corresponding to

$$c = c_1,\ c = c_2,\ \ldots,\ c = c_s.$$

Hence through each point of the plane there pass only s *distinct* curves. Hence the equation (I) represents a family of curves such that through every point in their plane there pass only s distinct curves.

It will next be proved that the parameters of these s distinct curves sasisfy an equation of degree s in the parameter, the coefficients being polynomials in x and y.

It is convenient to make a slight change in the notation. Instead of $c_1, c_2, \ldots, c_s$, write $c_{1,1}, c_{1,2}, \ldots, c_{1,s}$; instead of $c_{s+1}, c_{s+2}, \ldots, c_{2s}$, write $c_{2,1}, c_{2,2}, \ldots, c_{2,s}$; and so on up to $c_{m,1}, c_{m,2}, \ldots, c_{m,s}$. Then the parameters

$$c_{1,r},\ c_{2,r},\ \ldots,\ c_{m,r} \quad (r = 1, 2, \ldots, s)$$

correspond to the same curve, *i.e.* the polynomials

$$\phi(x, y, c_{1,r}),\ \ \phi(x, y, c_{2,r}),\ \ \ldots,\ \ \phi(x, y, c_{m,r}),$$

can at most differ by a constant factor only.

Suppose that after dividing $\phi(x, y, c)$ by the coefficient of some specified term, the coefficient of any* other specified term which happens to contain c is selected. Call it $F(c)$.

Then since the curves

$$\phi(x, y, c_{1,r}) = 0,\ \phi(x, y, c_{2,r}) = 0,\ \ldots,\ \phi(x, y, c_{m,r}) = 0,$$

are the same, it follows that

$$F(c_{1,r}) = F(c_{2,r}) = \ldots = F(c_{m,r}) \quad (r = 1, 2, \ldots, s).$$

Call each of these values $\qquad C_r \quad (r = 1, 2, \ldots, s)$.

Now form the equation

$$\Pi\,[C - F(c_{q,r})] = 0 \quad (q = 1, 2, \ldots, m \text{ and } r = 1, 2, \ldots, s). \qquad \text{(XXII)}$$

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* If some other term than the one first selected be chosen, it may affect the form of equation (III), viz.:—this equation may be reducible in the degree of the coordinates, but not the parameter, *e.g.* if in equation (IV) we take

$$C = \left(c + \frac{1}{c}\right)^2,$$

the primitive appears in the form $\quad C^2 - C\,(2y + x^2) + y^2 = 0,$

which reduces to $\qquad y - C = \pm x\sqrt{C}.$

The curves represented are necessarily the same because the parameter in the one case is a function of that in the other case, each being a function of c.

The left-hand side is a symmetrical function of the values of c which satisfy (I). Hence it is a polynomial of degree n in C with coefficients which are rational in x and y. But each value of C which satisfies it is repeated m times. Hence the left-hand side is the m-th power of

$$[C-F(c_{1,1})][C-F(c_{1,2})]\dots[C-F(c_{1,s})],$$

and this product can be found by rational operations only. It is a polynomial in C with coefficients which are rational in x and y.

Equating it to zero we obtain an equation of degree s in C with coefficients rational in x and y. The values of C which satisfy it are the values of a rational function of c. Hence there is a function of c, which is rational but not necessarily integral, which satisfies an equation of degree s, with coefficients rational in x and y.

The values of C are the parameters of the s distinct curves represented by

$$\phi(x, y, c) = 0. \qquad \text{(I)}$$

Suppose the equation satisfied by C to be written

$$\psi(x, y, C) = 0. \qquad \text{(III)}$$

It is of degree s in C, and to each value of C there correspond m values of c, each of which gives the same curve as is given by the value of C.

6. It will now be shown that if a differential equation is derivable from a primitive such as (I) involving an arbitrary constant, it cannot possess another primitive, also a polynomial in x, y and c, which is independent of the first primitive.

The two primitives if they exist can always, in virtue of the preceding sections, be reduced so that the integer denoted by m may be regarded as having the value unity. They must then be of the same degree in the arbitrary constant. If this were not so they would give rise to differential equations which were of different degrees in p.

Suppose that the two primitives are

$$\phi(x, y, c) = 0 \qquad \text{(I)}$$

and

$$\chi(x, y, k) = 0. \qquad \text{(XXIII)}$$

Suppose that they are of the same degree s in c and k respectively. Then, since they give the same value of p at any point x, y, it must be possible to find a value of c satisfying (I) and a value of k satisfying

(XXIII), such that

$$\frac{\partial\phi(x, y, c)}{\partial x}\bigg/\frac{\partial\phi(x, y, c)}{\partial y} = \frac{\partial\chi(x, y, k)}{\partial x}\bigg/\frac{\partial\chi(x, y, k)}{\partial y}. \tag{XXIV}$$

Since c and k satisfy (I) and (XXIII) it follows that

$$\left.\begin{aligned}
\frac{\partial\phi(x, y, c)}{\partial x}+\frac{\partial\phi(x, y, c)}{\partial c}\,\frac{\delta c}{\delta x} &= 0\\
\frac{\partial\phi(x, y, c)}{\partial y}+\frac{\partial\phi(x, y, c)}{\partial c}\,\frac{\delta c}{\delta y} &= 0\\
\frac{\partial\chi(x, y, k)}{\partial x}+\frac{\partial\chi(x, y, k)}{\partial k}\,\frac{\delta k}{\delta x} &= 0\\
\frac{\partial\chi(x, y, k)}{\partial y}+\frac{\partial\chi(x, y, k)}{\partial k}\,\frac{\delta k}{\delta y} &= 0
\end{aligned}\right\}. \tag{XXV}$$

Since (I) and (XXIII) have no repeated roots in c and k respectively, it follows that

$$\frac{\partial\phi(x, y, c)}{\partial c} \quad\text{and}\quad \frac{\partial\chi(x, y, k)}{\partial k}$$

do not vanish.

Hence, from (XXIV) and (XXV) it follows that

$$\frac{\delta(c, k)}{\delta(x, y)} = 0. \tag{XXVI}$$

Therefore k is a function of c.

If we call the values of c at x, y, $c_1, c_2, \ldots, c_s$, and those of k, $k_1, k_2, \ldots, k_s$, then it is proved that k_1 is a function of one of the c's, say c_1; and in like manner that k_2 is a function of c_2, k_3 of c_3, and so on.

Suppose that the relation between c_1 and k_1 is

$$\lambda(c_1, k_1) = 0. \tag{XXVII}$$

Now eliminate c between
$$\phi(x, y, c) = 0 \tag{I}$$
and
$$\lambda(c, k) = 0. \tag{XXVIII}$$
Let the result be
$$\omega(x, y, k) = 0. \tag{XXIX}$$

Then it follows from (I), (XXVII) and (XXVIII) that $k = k_1$ satisfies (XXIX). But $k = k_1$ satisfies (XXIII), which is irreducible in k. Hence all the values of k, which satisfy (XXIII), also satisfy (XXIX).

If therefore k_2 is any root of (XXIII) it is also a root of (XXIX). Therefore there is a value of c, say c_2, which satisfies (I) and (XXVIII) when

$k = k_2$; or

$$\lambda(c_2, k_2) = 0.$$

In like manner each value of k satisfying (XXIII) is connected with a value of c satisfying (I) in such a manner that corresponding values of c and k satisfy (XXVIII).

It will now be proved that the curve

$$\chi(x, y, k_1) = 0 \qquad \text{(XXX)}$$

is the same as the curve

$$\phi(x, y, c_1) = 0. \qquad \text{(XXXI)}$$

Consider a point ξ, η on both curves and the values of c and k corresponding to this point, viz. c_1 and k_1; and take any other point ξ', η' on the curve (XXXI), so that

$$\phi(\xi', \eta', c_1) = 0. \qquad \text{(XXXII)}$$

Now the relation between c_1 and k_1 is independent of the values of x and y. Consequently k_1 is one of the values of k which satisfy

$$\chi(\xi', \eta', k) = 0, \qquad \text{(XXXIII)}$$

and the curve

$$\chi(x, y, k_1) = 0, \qquad \text{(XXX)}$$

passes through ξ', η'. Now ξ', η' is any point whatever on the curve (XXXI). Hence the curve

$$\chi(x, y, k_1) = 0, \qquad \text{(XXX)}$$

passes through all the points on

$$\phi(x, y, c_1) = 0. \qquad \text{(XXXI)}$$

That is to say, $\phi(x, y, c_1)$ is a factor of $\chi(x, y, k_1)$.

But if $\chi(x, y, k_1)$ differed from $\phi(x, y, c_1)$ by any polynomial factor containing x, y, c_1, it would be reducible, and would not be, as is supposed, an integral which leads solely to the differential equation

$$f(x, y, p) = 0. \qquad \text{(VII)}$$

Hence $\chi(x, y, k_1)$ can differ from $\phi(x, y, c_1)$ by a constant factor only.

Consequently the curves

$$\chi(x, y, k_1) = 0 \quad \text{and} \quad \phi(x, y, c_1) = 0,$$

are identical. Similarly the curves

$$\chi(x, y, k_2) = 0 \quad \text{and} \quad \phi(x, y, c_2) = 0,$$

are identical, and so on.

It remains to prove that the equation

$$\lambda(c, k) = 0 \qquad \text{(XXVIII)}$$

is a lineo-linear relation between c and k.

Let us now divide the equation

$$\phi(x, y, c) = 0$$

by the coefficient of some specified term.

Then there must be at least two terms in which the coefficients are distinct functions of c, differing from each other by something more than a factor independent of c. For if that were not the case it would be possible, by replacing the coefficient containing c by a single arbitrary constant, to make the equation one of the first degree in the arbitrary constant, and this is supposed not to be possible as the differential equation is supposed to be of a degree higher than the first. Call these two distinct coefficients, each of which is rational, but not necessarily integral, $f(c)/g(c)$ and $h(c)/l(c)$, where $f(c)$, $g(c)$, $h(c)$ and $l(c)$ are polynomials in c.

Let the coefficients of the corresponding terms of the equation $\chi(x, y, k) = 0$, when it has been treated in the same way as $\phi(x, y, c) = 0$, be $a(k)/b(k)$ and $j(k)/t(k)$, where $a(k)$, $b(k)$, $j(k)$ and $t(k)$ are polynomials in k. Then the relation

$$\lambda(c, k) = 0, \qquad \text{(XXVIII)}$$

transforms $f(c)/g(c)$ into $a(k)/b(k)$ and $h(c)/l(c)$ into $j(k)/t(k)$. Hence the equations

$$\left.\begin{aligned} f(c)\, b(k) - g(c)\, a(k) &= 0 \\ \text{and} \qquad h(c)\, t(k) - l(c)\, j(k) &= 0 \end{aligned}\right\}, \qquad \text{(XXXIV)}$$

are true in virtue of (XXVIII).

If we treat the equations (XXXIV) as polynomials in c we shall *in general* obtain at length by elimination two equations of the first degree in c, the coefficients being functions of k. These two equations must be the same. If they were not then eliminating c we should obtain an equation in k, which would give a finite number of values for k, to each of which would correspond one value of c. So that there would be only a finite number of values of c and k satisfying (XXXIV), whereas we know that for every value of c there is at least one corresponding value of k and conversely. Taking therefore one of the equations of the first degree in c, we know that to every value of k corresponds only one value of c.

In like manner if, instead of solving the equations (XXXIV) for c, we had solved them for k, we could show that to every value of c corresponds

only one value of k. Hence to every value of c corresponds one value of k, and to every value of k corresponds one value of c. Hence since the relation between c and k is rational, it must be a lineo-linear relation.

It still remains to consider what would happen if it were possible to satisfy the equations (XXXIV) by a relation which would give two (or more) values of k corresponding to each value of c. It is sufficient to take the case where two values of k correspond to each value of c. In this case the equation

$$\lambda(c, k) = 0$$

would be such that when $c = c_1$, then $k = k_1$ or k_2. And then by the preceding argument the curves $\chi(x, y, k_1) = 0$ and $\chi(x, y, k_2) = 0$ would each be the same as $\phi(x, y, c_1) = 0$. Hence the curves $\chi(x, y, k_1) = 0$ and $\chi(x, y, k_2) = 0$ would be identical.

In this case the equation $f(x, y, p) = 0$ would have equal values for p, and would therefore be reducible, contrary to what has already been proved. Thus the two equations (I) and (XXIII) are not independent, there being a lineo-linear relation between their respective parameters.

7. It remains to be seen whether any other solution of the differential equation derived from the primitive (I), but not involving an arbitrary constant, can exist. If so let it be

$$\mu(x, y) = 0. \tag{XXXV}$$

Since it satisfies the same differential equation, it must give the same value of p at any point as that given by one of the curves $\phi(x, y, c) = 0$ passing through that point. Suppose the curve which gives the same value of p at x, y is

$$\phi(x, y, c_1) = 0. \tag{XXXI}$$

It is assumed that x, y is not a double point, so that $\phi(x, y, c_1)$, $\frac{\partial\phi(x, y, c_1)}{\partial x}$ and $\frac{\partial\phi(x, y, c_1)}{\partial y}$ do not simultaneously vanish. Equating the values of p, it follows that

$$\frac{\delta\mu}{\delta x}\Big/\frac{\delta\mu}{\delta y} = \frac{\partial\phi(x, y, c_1)}{\partial x}\Big/\frac{\partial\phi(x, y, c_1)}{\partial y}. \tag{XXXVI}$$

Now take a point $x+\Delta x$, $y+\Delta y$ on $\mu(x, y) = 0$ near to x, y. Then

$$\frac{\delta\mu}{\delta x}\Delta x + \frac{\delta\mu}{\delta y}\Delta y = 0, \tag{XXXVII}$$

and therefore, since $\frac{\partial\phi(x, y, c_1)}{\partial x}$, $\frac{\partial\phi(x, y, c_1)}{\partial y}$ do not simultaneously vanish,

it follows by (XXXVI) that

$$\frac{\partial\phi(x, y, c_1)}{\partial x}\Delta x+\frac{\partial\phi(x, y, c_1)}{\partial y}\Delta y=0. \qquad \text{(XXXVIII)}$$

Now, since c is defined as a continuous function of x, y by (I), there will be at $x+\Delta x$, $y+\Delta y$ a value of c differing infinitesimally from c_1, which may be called $c_1+\Delta c_1$. Hence

$$\phi(x+\Delta x, y+\Delta y, c_1+\Delta c_1)=0. \qquad \text{(XXXIX)}$$

But $$\phi(x, y, c_1)=0.$$

Hence $$\frac{\partial\phi(x, y, c_1)}{\partial x}\Delta x+\frac{\partial\phi(x, y, c_1)}{\partial y}\Delta y+\frac{\partial\phi(x, y, c_1)}{\partial c_1}\Delta c_1=0. \qquad \text{(XL)}$$

From (XXXVIII) and (XL) it follows that

$$\frac{\partial\phi(x, y, c_1)}{\partial c_1}\Delta c_1=0. \qquad \text{(XLI)}$$

Hence either $$\frac{\partial\phi(x, y, c_1)}{\partial c_1}=0, \qquad \text{(XLII)}$$

or $$\Delta c_1=0. \qquad \text{(XLIII)}$$

Now, if $\frac{\partial\phi(x, y, c_1)}{\partial c_1}=0$, then, since $\phi(x, y, c_1)=0$, it follows that x, y satisfy the result of eliminating c_1 from these equations. The eliminant consists of the envelope-locus, the node-locus (twice) and the cusp-locus (thrice). As we have supposed that $\frac{\partial\phi(x, y, c_1)}{\partial x}$ and $\frac{\partial\phi(x, y, c_1)}{\partial y}$ do not vanish simultaneously with $\phi(x, y, c_1)$ we may put aside the node- and cusp-loci. Hence x, y, which is any point on $\mu(x, y)=0$, is a point on the envelope-locus. Hence in this case $\mu(x, y)=0$ represents the envelope-locus of the system of curves (I).

Before proceeding to consider the second alternative $\Delta c_1=0$, I will examine the value of Δc_1 at a point on the envelope-locus.

If therefore $x+\Delta x$, $y+\Delta y$ be a point near to x, y on the envelope-locus, and if $c_1+\Delta c_1$ be the value at $x+\Delta x$, $y+\Delta y$ of c, which differs infinitesimally from c_1, then

$$\phi(x, y, c_1)=0 \qquad \text{(XXXI)}$$

and $$\frac{\partial\phi(x, y, c_1)}{\partial c_1}=0, \qquad \text{(XLII)}$$

are both satisfied when x, y, c_1 are replaced by $x+\Delta x$, $y+\Delta y$, $c_1+\Delta c_1$ respectively.

From (XXXI) we get (XL), which by using (XLII) reduces to

$$\frac{\partial\phi(x, y, c_1)}{\partial x}\Delta x+\frac{\partial\phi(x, y, c_1)}{\partial y}\Delta y=0. \qquad \text{(XXXVIII)}$$

From (XLII) we get

$$\frac{\partial^2\phi(x, y, c_1)}{\partial x\partial c_1}\Delta x+\frac{\partial^2\phi(x, y, c_1)}{\partial y\partial c_1}\Delta y+\frac{\partial^2\phi(x, y, c_1)}{\partial c_1^2}\Delta c_1=0. \qquad \text{(XLIV)}$$

From (XXXVIII) and (XLIV) it follows that

$$\frac{\partial^2\phi(x, y, c_1)}{\partial c_1^2}\Delta c_1$$
$$=(\Delta x)\left(\frac{\partial\phi(x, y, c_1)}{\partial x}\frac{\partial^2\phi(x, y, c_1)}{\partial y\partial c_1}-\frac{\partial\phi(x, y, c_1)}{\partial y}\frac{\partial^2\phi(x, y, c_1)}{\partial x\partial c_1}\right)\Big/\frac{\partial\phi(x, y, c_1)}{\partial y}. \qquad \text{(XLV)}$$

Now $\frac{\partial^2\phi(x, y, c_1)}{\partial c_1^2}$ and $\frac{\partial\phi(x, y, c_1)}{\partial y}$ are finite or at special points zero. Hence, if Δc_1 vanish, and Δx is not zero, which can only happen at special points, we must have

$$\frac{\partial\phi(x, y, c_1)}{\partial x}\frac{\partial^2\phi(x, y, c_1)}{\partial y\partial c_1}-\frac{\partial\phi(x, y, c_1)}{\partial y}\frac{\partial^2\phi(x, y, c_1)}{\partial x\partial c_1}=0, \qquad \text{(XLVI)}$$

i.e.

$$\frac{\partial}{\partial c_1}\left[\frac{\partial\phi(x, y, c_1)}{\partial x}\Big/\frac{\partial\phi(x, y, c_1)}{\partial y}\right]=0, \qquad \text{(XLVII)}$$

or

$$\frac{\partial\phi(x, y, c_1)}{\partial x}\Big/\frac{\partial\phi(x, y, c_1)}{\partial y},$$

must be independent of c_1. Suppose

$$\phi(x, y, c)=u_0c^n+u_1c^{n-1}+\ldots+u_rc^{n-r}+\ldots+u_n, \qquad \text{(XLVIII)}$$

where $u_0, u_1, \ldots, u_r, \ldots, u_n$ are polynomials in x, y but do not contain c. Then we must have

$$\left(\frac{\partial u_0}{\partial x}c_1^n+\ldots+\frac{\partial u_r}{\partial x}c_1^{n-r}+\ldots+\frac{\partial u_n}{\partial x}\right)\Big/\left(\frac{\partial u_0}{\partial y}c_1^n+\ldots+\frac{\partial u_r}{\partial y}c_1^{n-r}+\ldots+\frac{\partial u_n}{\partial y}\right)$$

independent of c_1. Therefore

$$\left(\frac{\partial u_0}{\partial x}\Big/\frac{\partial u_0}{\partial y}\right)=\ldots=\left(\frac{\partial u_r}{\partial x}\Big/\frac{\partial u_r}{\partial y}\right)=\ldots=\left(\frac{\partial u_n}{\partial x}\Big/\frac{\partial u_n}{\partial y}\right).$$

Consider the equation $\qquad \frac{\partial u_r}{\partial x}\Big/\frac{\partial u_r}{\partial y}=\frac{\partial u_s}{\partial x}\Big/\frac{\partial u_s}{\partial y}.$

From this it follows that u_r, u_s are connected by a functional relation. Hence all the polynomials $u_0, u_1, \ldots, u_r, \ldots, u_n$ which are not constants, may be regarded as functions of one of their number, say u_r. This being so, the equation (I), which is

$$u_0 c^n + \ldots + u_r c^{n-r} + \ldots + u_n = 0,$$

is equivalent to $\qquad u_r =$ arbitrary constant.

This is of the first degree in the arbitrary constant, and then there can be no envelope-locus.

The conclusion is that the equation

$$\frac{\partial^2 \phi(x, y, c_1)}{\partial x \partial c_1} \frac{\partial \phi(x, y, c_1)}{\partial y} - \frac{\partial^2 \phi(x, y, c_1)}{\partial y \partial c_1} \frac{\partial \phi(x, y, c_1)}{\partial x} = 0 \qquad \text{(XLVI)}$$

can only be satisfied when there is no envelope-locus. Hence Δc_1 cannot vanish at a point on the envelope.

We can now consider the alternative

$$\Delta c_1 = 0.$$

In this case the curve $\qquad \mu(x, y) = 0$

is a particular case of the complete primitive. Hence all the solutions of the differential equation satisfied by $\phi(x, y, c) = 0$ are obtained (1) by giving to c any arbitrary constant value; (2) by eliminating c between

$$\phi(x, y, c) = 0 \qquad \text{(I)}$$

and

$$\frac{\partial \phi(x, y, c)}{\partial c} = 0, \qquad \text{(XLIX)}$$

but excluding from the eliminant any factor which represents a node-locus or cusp-locus of the curves (I). *There can be no solution not included amongst these two sets of solutions.*

8. The usual method of obtaining the Singular Solution is as follows.

If $\phi(x, y, c) = 0$ be the primitive, then the differential equation is obtained by eliminating c between

$$\phi(x, y, c) = 0 \qquad \text{(I)}$$

and

$$\frac{\partial \phi(x, y, c)}{\partial x} + p \frac{\partial \phi(x, y, c)}{\partial y} = 0. \qquad \text{(IX)}$$

Now let C be any function of x and y which satisfies

$$\frac{\partial \phi(x, y, C)}{\partial C} = 0. \tag{L}$$

Then consider the primitive

$$\phi(x, y, C) = 0. \tag{LI}$$

It gives on differentiation, and using (L),

$$\frac{\partial \phi(x, y, C)}{\partial x} + \frac{\partial \phi(x, y, C)}{\partial y} p = 0, \tag{LII}$$

and if we eliminate C between (LI) and (LII) we get the same differential equation as when we eliminate c between (I) and (IX). This only shows that we *may* get a solution of the differential equation in this way. It does not show, what is proved in the preceding section, that no other kind of solution can exist. The assumption that no other kind of solution can exist was made by Lagrange in his memoir "Sur les intégrales particulières des équations différentielles" (*Nouveaux Mémoires de l'Académie royale des Sciences et Belles Lettres de Berlin*, année 1774), printed in the fourth volume of his collected works, see p. 12, § 5, of this memoir, where he says:—"Il est facile de démontrer qu'il n'y a pas d'autres combinaisons possibles qui puissent fournir des intégrales de cette espèce non comprises dans l'intégrale complète."

THE INVARIANT THEORY OF THREE QUADRICS

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Introduction.

The following pages give in outline a complete system of concomitants of three quadrics. In §§ 20–22 the invariants are dealt with, and a complete list of these is given in § 23. In § 5, the *prepared system* of bracket types is explained, and in § 14 tabulated.

A geometrical discussion of these results is deferred.

I. *Notation.*

1. In symbolic form let the point, plane, and line equations of the quadrics be

$$\left.\begin{aligned} f = a_x^2 = a_x'^2 = \ldots, &\qquad \phi = u_\alpha^2 = u_{\alpha'}^2 = \ldots, \\ f_1 = b_x^2 = b_x'^2 = \ldots, &\qquad \phi_1 = u_\beta^2 = \ldots, \\ f_2 = c_x^2 = c_x'^2 = \ldots, &\qquad \phi_2 = u_\gamma^2 = \ldots, \\ \text{and} \qquad \pi &= (Ap)^2 = \ldots, \\ \pi_1 &= (Bp)^2 = \ldots, \\ \pi_2 &= (Cp)^2 = \ldots. \end{aligned}\right\} \tag{1}$$

These symbols refer to quaternary forms wherein

$$a_x = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4,$$

$$A = (aa') \text{ a second degree element,}$$

$$\alpha = (aa'a'') \text{ a third degree element,}$$

$$u_x = 0,$$

$$p = (uv),$$

$$x = (uvw).$$

Any single term concomitant of f, f_1, f_2 is denoted by P. The word *member* will be used to signify a concomitant.

The symbols a, A, α, u, p, x are called *elements* of various *degrees* one, two, or three; and these three degrees are distinguished respectively by (1) small italic letters, (2) capital italic letters, and (3) Greek letters, together with x.

Reducibility.

2. Following Gordan* in his theory of two quadrics we introduce the symbols c_i, $c_{\nu\mu}$ to denote the character of a form P. Let c_1, c_2, c_3 denote the degree of P in the coefficients of f, f_1, f_2 respectively. Let $c_{1\mu}, c_{2\mu}, c_{3\mu}$ refer to f, f_1, f_2 respectively: and in $c_{1\mu}$ let μ denote the number of brackets in P, each of which contains μ symbols $a, a', \ldots$ or the equivalent of μ symbols in the higher currencies A, α. Then μ may not exceed 4.

Then a form P_1 is held to be simpler than P_2 if one of c_1, c_2, c_3 in P_1 is less than the corresponding degree in P_2, while the other two are not greater. In this sense, forms are considered in ascending degree.

To distinguish forms of the same degree, P_1 is simpler than P_2 if in P_1 one of c_{14}, c_{24}, c_{34} is greater than in P_2, the other two being not less. If this test fails, then c_{i3}, c_{i2} are examined in succession.†

If $c_{\nu 4} > 0$, P_1 is reducible.

3. As before, the symbol a_α implies the factor a_α^2.

Equivalent Forms.

4. P_1, P_2 are equivalent if $P_1 - P_2$ is reducible. This is symbolised by

$$P_1 - P_2 \equiv 0 \bmod R,$$

or

$$P_1 - P_2 \equiv 0,$$

or

$$P_1 \equiv P_2.$$

Prepared Forms.

5. To begin with, P consists of four types of bracket factor: $(dd_1 d_2 d_3)$,

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* *Math. Ann.*, Bd. 56.

† Cf. Turnbull, "System of Two Quadratics," *Proc. London Math. Soc.*, Ser. 2, Vol. 18, p. 74.

(dd_1d_2u), (dd_1p), d_x, where d denotes a, b, or c. Wherever in a factor two or three d's refer to one quadric they are replaced by D or δ respectively. Now every symbol d must occur twice in P. But if, say, dd_1 stand for aa_1 in one bracket, it does not follow that the complementary a, a_1 will be found to be also convolved in another bracket. Yet, by a proper introduction of new bracket types, we arrive at an alternative form of P in which *every* symbol d, D, or δ is explicitly paired. This is called the prepared form of P, and must now be investigated.

II. *The Prepared System.*

6. A bracket of P may have four or less a's: *i.e.* it may contain a_α, a, A, or α, or no reference to the quadric f. The first of these implies the invariant a_α^2, so we pass on to the second case, where $\alpha = (aa'a'')$ occurs in a bracket. By the use of new brackets

$$(\alpha\beta p), \quad (\alpha\gamma p), \quad (\alpha\beta\gamma x),$$

we may collect the complements of $aa'a''$ which occur bracketed once.

The proof is the same as for two quadrics* with the additional case of

$$(aa'a''i)\, a_\delta a'_{\delta'} a''_x.$$

This is seen, by interchanging the a's in every way, to be

$$= \tfrac{1}{6}(aa'a''i)(aa'a''.\delta\delta'x)$$

$$= \tfrac{1}{6} i_\alpha (\alpha\delta\delta' x).$$

The bracket $(\alpha\delta\delta'x)$ is $(\alpha\beta\gamma x)$ or else is zero.

The bracket $(\alpha\beta\gamma x)$.

7. This bracket is the reciprocal or dual of $(abcu)$ and does not appear for less than three quadrics. It obeys the same rule of interchange as its dual, and, expressed in the original form, is a six-term series

$$\dot{a}_\beta \dot{a}'_\gamma \dot{a}''_x \quad (aa'a'' = \alpha),$$

where the dots indicate a determinantal permutation.

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* Cf. Turnbull, *ibid.*, p. 75, § 10.

Interchangeability of a.

8. Since $e_a e'_{a'} - e'_a e_{a'} \equiv 0 \bmod c_{14}$, any two single a's in P may be interchanged. Nor would this reduction break down for an a contained in the new brackets $(a\delta p)$, $(a\beta\gamma x)$. We may then suppress any distinguishing marks between the a's; so also for β, γ. A form P will now contain an even number, or none, of each of a, β, γ. Moreover this is true for n quadrics if we add the new bracket $(a\beta\gamma\delta)$.

The Element A.

9. The next step is more complicated: we must consider the pairing of A. Let a^i, a^j denote any two of a, a', a'', a'''. As in the case of two quadrics, if P contain brackets $(aa'kl)(a^i a^j mn)$, then we may express this in terms of $(aa'kl)(aa'mn)$, and terms with more than two symbols a in the second bracket. It is important to notice that the other symbols kl, mn of the original brackets are undisturbed in the equivalent brackets.

As P will originally contain either an even or an odd number of brackets (c_{12}), each with two symbols like a, a', we may thus pair off all such to become pairs of A's except possibly one odd pair. This gives two cases:—

$$\text{(i)}\quad P = \{\Pi(Aij)(Akl)\}\, M,$$

$$\text{(ii)}\quad P = \{\Pi(Aij)(Akl)\}\,(aa'mn)\, a_\rho a'_\sigma M,$$

where both ρ, σ involve b, c, u, p, x, but no reference to the quadric f.

The same applies to B and C. Hence P has at most one of each sort (aa'), (bb'), (cc') unpaired, which leads to three cases:—

Case I.—One, (aa') say, occurs, but all symbols b, b' are in separate factors: as also c, c'.

Case II.—Two are unpaired, (aa'), (bb') say.

Case III.—Three are unpaired, (aa'), (bb'), (cc').

Case I.—P contains $(aa'mn)\, a_\rho a'_\sigma$. Here we may write

$$2(aa'mn)\, a_\rho a'_\sigma = (aa'mn)(a_\rho a'_\sigma - a'_\rho a_\sigma) = (aa'mn)(aa'\rho\sigma),$$

introducing the new bracket $(aa'\rho\sigma)$, which is unnecessary if ρ or σ may be broken up, *i.e.* if ρ or $\sigma = (bcu)$. Besides this, the bracket $(aa'\rho\sigma)$ resolves itself into two simpler ones, or to zero, if $\rho = \sigma$, or if ρ, σ both

contain B, C, or p. The cases wherein there is no reduction are given in the following table :—

	(1)		(2)	(4)′			(3)	(4)	(5)	(6)	(7)					(8)	(9)	
$\rho =$	x	x	x	x	x	x	β	β	β	β	β	γ	γ	γ	γ	Bu	Bc	Cb
$\sigma =$	β	γ	Bu	Bc	Cu	Cb	γ	Cu	cp	Cb	bp	Bu	bp	cp	Bc	Cu	cp	bp

In these tabulated cases, any attempt to bracket aa' in one or other factor a_ρ or a'_σ fails to simplify.

10. *Case* II.—This may be dealt with as Case I, unless the odd symbols a, a', b, b' are convolved at least once. P therefore may contain (aa'), (bb'), together with

either $$(abQ)(a'b'R), \tag{1}$$

or $$(abQ)\,a'_\rho b'_\sigma; \tag{2}$$

where Q, R, containing neither a nor b, can only be C, p, or cu: the last of which at once reduces. Since $Q \neq R$ only one possibility is left, $Q = C$, $R = p$. Hence the bracket pair (1) is $(abC)(a'b'p)$, which is conveniently written as
$$(ABCp). \tag{3}$$

Again, in form (2), if $Q = C$, the form may be written
$$(\dot{a}\dot{b}C)\,\dot{a}'_\rho \dot{b}'_\sigma,$$

since the complementary elements aa', and bb', are convolved. This form is symmetrical in A, B, C as regards its first bracket. For either A or B may be explicitly bracketed by breaking C up. This shows that ρ, σ must be independent of a, b and c. So they are both equal to x. This gives one new bracket type $(\dot{a}\dot{b}C)\,\dot{a}'_x \dot{b}_x$ which may be written
$$(ABCxx). \tag{4}$$

Exactly the same argument shows that if $Q = p$, then ρ, σ can only be γ, γ: leading to $(AB\gamma\gamma p)$. Similarly for
$$(BCaap), \quad (CA\beta\beta p). \tag{5}$$

11. *Case* III.—Here the symbols a, a', b, b', c, c' are left over after pairing existing sets A, B, C: and unless they are all convolved they may

be treated as in Cases I and II. This leaves only the following to be considered:—

(i) $(abcu)\, a'b'c'$, which reduces by bracketing aa' in $(abcu)$;

(ii) $(\dot{a}\dot{b}Q)(\dot{a}'\dot{c}R)\,\dot{b}'_\rho \dot{c}'_\sigma$;

(iii) $(\dot{a}\dot{b}Q)(\dot{a}'\dot{c}R)(\dot{b}'\dot{c}'S)$.

Here the symbols Q, R, S can only be A, B, C, or p [else the bracket at once reduces as in (i)]; and no two of Q, R, S are equal; so that at least one of them is A, B, or C. By convolving one or other of aa', bb', cc' into the bracket not containing p, we effect a reduction. So no new form of bracket is needed.

12. New types of bracket are indicated by the table of § 9, and by (3), (4), (5) of § 10. By means of these new types we have now explicitly paired off all the A, B, C symbols of P, and further have proved that among the symbols A, any two may be interchanged indifferently. Such a member P is now *prepared.*

In the prepared form P, the first degree symbols belonging to one form f, say, may be interchanged. For let $I(a, a')P$ denote the effect on P of interchanging *one* a with *one* a'. Then

$$P - I(a, a')P \equiv 0 \bmod c_{12},$$

for $\overline{aa'}$ will be bracketed and give rise to an increase in c_{12}, provided that neither the a nor the a' occur in the new types of bracket given in Case I of § 9. Yet even in this case, the pair aa' may be bracketed for the same reasons as those considered in Cases II and III.

It follows that for the last two values of ρ, σ in the table of § 9 there is no need to consider the case where two different first degree elements c, c' occur. By using $I(c, c')P$ the difference is eliminated from this bracket.

The Bracket $(ABCp) \equiv 0$.

13. For let (AB, Cp) denote $(A\dot{c}u)(B\dot{c}'v)$ and for brevity let

$$g = (BC, Ap), \quad h = (CA, Bp), \quad k = (AB, Cp).$$

Then clearly k is unaltered if B, A are interchanged. Now if we bracket

C in the first bracket of k, we obtain by the fundamental identity,

$$k = (Acc')(Buv) + (\dot{a}c'c\dot{u})(B\dot{a}'v);$$

thus $$k = 2(AC)(Bp) - (C\dot{a}\dot{u})(B\dot{a}'v);$$

transposing, this is $$k+g = 2(AC)(Bp).$$

Similarly for $g+h$, $h+k$: hence

$$k = (AC)(Bp) + (BC)(Ap) - (AB)(Cp),$$

which reduces k at once.

Statement of the Prepared System.

14. We may now sum up the preceding results and give special notations for the various groups of new brackets introduced. The table of § 9 gives these types:—

(1) $(A\beta x) = a_\beta a'_x - a'_\beta a_x = \dot{a}_\beta \dot{a}'_x$,

(2) $(ABux) = (\dot{a}Bu)\dot{a}'_x = (BAux) = (\dot{b}Au)\dot{b}'_x$,

(3) $(A\beta\gamma) = \dot{a}_\beta \dot{a}'_\gamma$,

(4) $(ACu\beta) = (\dot{a}Cu\dot{a}'_\beta = (CAu\beta) = H_2$, and (4)$'$ $(ABxc) = \dot{a}_x(\dot{a}'Bc) = h_3$,

(5) $(Apc\beta) = (\dot{a}cp)\dot{a}'_\beta = G_{13}$,

(6) $(ACb\beta) = (\dot{a}Cb)\dot{a}'_\beta = (CAb\beta) = F'_4$,

(7) $(Apb\beta) = (\dot{a}bp)\dot{a}'_\beta = F_{12}$,

(8) $(ABCuu) = (\dot{a}Bu)(\dot{a}'Cu) = k$,

(9) $(ABccp) = (\dot{a}Bc)(\dot{a}'cp)$.

To these must be added the results of § 10,

$$(ABCxx) = (A\dot{b}\dot{c})\dot{b}'_x\dot{c}'_x = (B\dot{a}\dot{c})\dot{a}'_x\dot{c}'_x = k,$$

$$(AB\gamma\gamma p) = (\dot{a}\dot{b}p)\dot{a}'_\gamma\dot{b}'_\gamma.$$

The symbols H, h, G, F, k, etc. are found useful for reference, and in the above list several alternative ways of writing each type of bracket are given. These and all the original brackets may now be classified in four

groups F_1, F_2, F_3, F_4; the suffix denoting the number of unpaired symbols explicitly found in the prepared bracket. Thus under F_2 would fall $(ABccp)$ which requires the two symbols A, B to be paired elsewhere in the member. All the brackets, old and new, of the prepared system are given in the following table.

The Prepared System.

12	F_1	$a_x \quad b_x \quad c_x \quad u_\alpha \quad u_\beta \quad u_\gamma \quad (Ap) \quad (Bp) \quad (Cp) \quad a_\alpha \quad b_\beta \quad c_\gamma$
36	F_2	$a_\beta \quad a_\gamma \quad (bcp) \quad (\beta\gamma p) \quad (Abu) \quad (Acu) \quad (A\beta x) \quad (A\gamma x) \quad (BC)$ $b_\gamma \quad b_\alpha \quad (cap) \quad (\gamma\alpha p) \quad (Bcu) \quad (Bau) \quad (B\gamma x) \quad (B\alpha x) \quad (CA)$ $c_\alpha \quad c_\beta \quad (abp) \quad (\alpha\beta p) \quad (Cau) \quad (Cbu) \quad (C\alpha x) \quad (C\beta x) \quad (AB)$ $(BC)' = (BCux) \quad (BC)'' = (BCaap) \quad (BC)''' = (BC\alpha\alpha p)$ $(CAux) \quad (CAbbp) \quad (CA\beta\beta p)$ $(ABux) \quad (ABccp) \quad (AB\gamma\gamma p)$
28	F_3	$(abcu) \quad (\alpha\beta\gamma x)$ $(Abc) \quad (A\beta\gamma) \quad (Apb\beta) = F_{12} \quad (Apc\gamma) = F_{13} \quad (Apc\beta) = G_{13} \quad (Apb\gamma) = G_{12}$ $(Bca) \quad (B\gamma\alpha) \quad (Bpc\gamma) = F_{23} \quad (Bpa\alpha) = F_{21} \quad (Bpc\alpha) = G_{23} \quad (Bpa\gamma) = G_{21}$ $(Cab) \quad (C\alpha\beta) \quad (Cpa\alpha) = F_{31} \quad (Cpb\beta) = F_{32} \quad (Cpa\beta) = G_{31} \quad (Cpb\alpha) = G_{32}$ $(BCu\alpha) = H_1 \quad (BCxa) = h_1 \quad (ABCuu) = K$ $(CAu\beta) = H_2 \quad (CAxb) = h_2 \quad (ABCxx) = k$ $(ABu\gamma) = H_3 \quad (ABxc) = h_3$
3	F_4	$(BCa\alpha) = F_4 \quad (CAb\beta) = F_4' \quad (ABc\gamma) = F_4''$

III. *Generalised Identities.*

15. The prepared system, now tabulated, shows clearly a principle of duality, the algebraic equivalent of reciprocation. For this system is symmetrical in regard to the line coordinate elements p, A, B, C: and to every group involving any of a, b, c, u corresponds a group of α, β, γ, x. Some of the factors, *e.g.* (Ap), $(BC)'$, F_{12}, F_4, are self-reciprocal: others form pairs of reciprocals H_1 with h_1, K with k, and so on.

This duality goes further: it may be affirmed that whatever identity or syzygy exists between symbolic forms, has consequently a dual identity or syzygy. For example, the fundamental identity

$$(abcd)\, e_x - (abce)\, d_x + \ldots = 0 \tag{1}$$

implies the existence of

$$(\alpha\beta\gamma\delta)\,u_\epsilon - (\alpha\beta\gamma\epsilon)\,u_\delta + \ldots = 0, \tag{2}$$

where each of α, β, γ, δ, ϵ are third degree elements in the coefficients of the quadrics. The second identity is readily established by resolving each 12 degree bracket $(\alpha\beta\gamma\delta)$ into factors $a_\beta a'_\gamma a''_\delta$. The same holds true of

$$(abp)\,c_x + (bcp)\,a_x + (cap)\,b_x = 0 \tag{3}$$

and

$$(\alpha\beta p)\,u_\gamma + (\beta\gamma p)\,u_\alpha + (\gamma\alpha p)\,u_\beta = 0. \tag{4}$$

Again, the identity $a_\pi b_\rho - a_\rho b_\pi = (ab\pi\rho)$ is self-reciprocal; whereas

$$(\dot{a}\dot{b}K)(\dot{c}\dot{d}L) = (abcd)(KL) \tag{5}$$

leads to the dual form

$$(\dot{\alpha}\dot{\beta}K)(\dot{\gamma}\dot{\delta}L) = (\alpha\beta\gamma\delta)(KL). \tag{6}$$

It is a straightforward matter to write down all the linear types of quaternary identities, and then to copy the dual forms such as (2), (4), (6) above. By resolving the component parts of these latter into their elementary brackets, they can all be proved true. Now whatever process of reduction is used to test the reducibility of a member of a quaternary system, this process must ultimately depend upon two—and only two—things, (1) the fundamental linear identities, and (2) the interchange of equivalent symbols. Since both these principles apply to either type of symbol a or α, it follows that any identity or syzygy whatever may be reciprocated.

Reducibility.

16. The criteria $c_1 \ldots c_{33}$ of § 2 must now be supplemented. When two members P_1 and P_2 have the same characters $c_1 \ldots c_{33}$, let the number of F_i brackets ($i = 1, 2, 3, 4$) be counted, i being the greatest suffix in P_1 or P_2. Then P_1 is simpler than P_2 if its number of F_i brackets is less than that of P_2.

Failing this, let W_3, W_2, W_1 denote the number of brackets in P containing, respectively, three, two, one of the symbols A, B, C. Then P_1 is simpler than P_2, if for P_1, W_3 is less than it is for P_2. Failing this, W_2 is similarly examined.

This gives an order of precedence among the F_3 brackets which require

one further discrimination, viz. that the six brackets (Abc), $(A\beta\gamma)$, etc. are the simplest F_3 brackets with *one* symbol A, B or C; next come the six F_{ij}; and next G_{ij}. Other F_3 brackets precede or follow this group because less or more symbols A, B, C occur.

The Reduction System.

17. The prepared system contains 79 elements, but a product of two of these elements is often reducible. Thus the product of two F_3 brackets $(abcu)(a\beta\gamma x)$ is identically equal to $\Sigma \dot{a}_x \dot{b}_\beta \dot{c}_\gamma \dot{u}_x$, which eliminates the F_3 brackets and therefore reduces the product. It is possible to carry out a systematic examination of every such product, and to construct a table in which any such product of two of these 79 factors is shown to be either (i) reducible, or (ii) irreducible, or (iii) equivalent to another product. This table consists of 79 rows and columns—one row and one column for each different factor, from a_x to F_4''. The following fragment of the complete table should make clear the method of classification :—

	H_1	H_2	H_3	h_1	h_2	h_3
H_1	0					
H_2	.	0				
H_3	.	.	0			
h_1	.	x	x	0		
h_2	x	.	x	.	0	
h_3	x	x	.	.	.	0

x = reducible, 0 = irreducible, · = equivalent to another product.

Here, for example, it is shown that the product $h_1 H_2$ is reducible, that $H_3 H_3$ is irreducible, and $H_1 H_2$ is equivalent to another product. The whole table is a large triangle with an hypotenuse of 79 marks of irreducibility which indicate the squares of 79 factors $a_x \ldots F_4''$. This table is called the Reduction System.

Construction of the Reduction System.

18. This table is constructed by examining a product of factors, for example $(Abu)(p\beta\gamma)$. Here, by permuting bu, p we arrive at the identity

$$(Abu)(p\beta\gamma) \equiv G_{12} u_\beta - F_{12} u_\gamma - (pA)\, b_\gamma u_\beta,$$

suppressing reducible terms involving b_β. In accordance with the conditions of § 16, the reducible mark x is placed opposite G_{12} and u_β in the table, and the mark · is placed twice, to correspond with $(Abu)(p\beta\gamma)$ and with $F_{12}u_\gamma$. The third term $(pA)\,b_\gamma u_\beta$ has three factors and is analysed independently.

By interchanging symbols a, A, a with b, B, β or c, C, γ this one identity implies five others. By reciprocating these we get six others, as, for example,

$$(A\beta x)(pbc) \equiv G_{13}\,b_x - \ldots .$$

And further, by interchanging in a *linear* identity the symbols a, A, a with u, p, x we obtain a new identity, equally valid, since the convolution of two of u, p, x is reducible, and also since the symbols u, p, x behave analytically in the same way as a, A, a. For example, by interchanging b, B, β with u, p, x in the above identity we may forecast the new relation

$$(Abu)(B\gamma x) \equiv H_3\,b_x - (AB)'\,b_\gamma - (AB)\,u_\gamma\,b_x.$$

Thus from one product $(Abu)(p\beta\gamma)$ a large number of other products may be dealt with at considerable economy of labour.

Below is subjoined the table of the reduction system, broken up for convenience into three parts: these deal respectively with (i) F_1F_2 brackets, (ii) one F_1 or F_2 with one F_3 or F_4, and (iii) F_3F_4 brackets. The detailed proofs are not given, for they are tedious but all of the same kind: and it is easy to test any assertion made in the table by applying one or other linear identity.

I.

	a_x	b_x	c_x	u_α	u_β	u_γ	a_β	a_γ	b_γ	b_α	c_α	c_β	bcp	cap	abp	$\beta\gamma p$	$\gamma\alpha p$	$\alpha\beta p$	Ab	Ac	Bc	Ba	Ca	Cb	$A\beta$	$A\gamma$	$B\gamma$	$B\alpha$	$C\alpha$	$C\beta$	BC'	CA'	AB'	BC''	CA''	AB''			
g*		0			0			0			0																												
bcp	x	0	0																																				
cap	0	.	0		0				0					0																									
abp	0	0	.																																				
$\beta\gamma p$				x	0	0																																	
$\gamma\alpha p$		0		0	.	0			0					0			0																						
$\alpha\beta p$				0	0	.																																	
Abu							0	0	0	0	.	0	0	.	0	.	.	.																					
Acu							0	0	0	.	0	0	0	0	.	.	.	.				0																	
Bcu		0			0		.	0	0	0	0	0	0	0	.	.	.	.																					
Bau							0	0	0	0	0	.	.	0	0	.	.	.																					
Cau							0	0	.	0	0	0	.	0	0	.	.	.				0																	
Cbu							0	.	0	0	0	0	0	.	0	.	.	.																					
$A\beta x$							0	.	0	0	0	0	.	.	.	0	.	0	0	0	.	.	.	.															
$A\gamma x$							.	0	0	0	0	0	.	.	.	0	0	.	0	0	.	.	.	.															
$B\gamma x$		0			0		0	0	0	.	0	0	.	.	.	0	0	.	.	.	0	0	.	.				0											
$B\alpha x$							0	0	.	0	0	0	.	.	.	.	0	0	.	.	0	0	.	.															
$C\alpha x$							0	0	0	0	0	.	.	.	.	.	0	0	.	.	.	.	0	0															
$C\beta x$							0	0	0	0	.	0	.	.	.	0	.	0	.	.	.	.	0	0													Ap	Bp	Cp
$BCux$							.	.	x	.	.	x							.	.	0	0	0	0	.	.	0	0	0	0							x	0	0
$CAux$		0			0		.	x	.	.	x	.		x			x		0	0	.	.	0	0	0	0	.	.	0	0		0					0	.	0
$ABux$							x	.	.	x	.	.							0	0	0	0	.	.	0	0	0	0	.	.							0	0	.
$BCaap$	0	.	.	.	.	.	0	0	.	.	.	.	x	0	0	.	.	.	.	.	x	0	0	x	x	x	.	0	0	.	x	.	.	0					
$CAbbp$	.	0	.	.	.	.	.	.	0	0	.	.	0	x	0	.	.	.	0	x	.	.	x	0	0	.	x	x	.	0	.	x	.	.	0				
$ABccp$	.	.	0	.	.	.	.	.	.	.	0	0	0	0	x	.	.	.	x	0	0	x	.	.	.	0	0	.	x	x	.	.	x	.	.	0	BC'''	CA'''	AB'''
$BC\alpha\alpha p$	.	.	.	0	.	.	.	.	.	0	0	.	.	.	.	x	0	0	x	x	.	0	0	.	.	.	x	0	0	x	x	.	.	.	x	x	0		
$CA\beta\beta p$	.	.	.	.	0	.	0	.	.	.	.	0	.	.	.	0	x	0	0	.	x	x	.	0	0	x	.	.	x	0	.	x	.	x	.	x	.	0	
$AB\gamma\gamma p$	.	.	.	.	.	0	.	0	0	.	.	.	.	.	.	0	0	x	.	0	0	.	x	x	x	0	0	x	.	.	.	.	x	x	x	.	.	.	0

* The factors (BC), (CA), (AB) do not reduce with any of the above F_1 and F_2 factors. Also g in the above denotes any of the twelve factors $a_x, b_x, \ldots, c_\alpha, c_\beta$.

II.

	$a_x\ b_x\ c_x$	$u_\alpha\ u_\beta\ u_\gamma$	$Ap\ Bp\ Cp$	$a_\beta\, a_\gamma\, b_\gamma\, b_\alpha\, c_\alpha\, c_\beta$	$bcp\ cap\ abp$	$\beta\gamma p\ \gamma\alpha p\ \alpha\beta p$	$Ab\ Ac\ Bc\ Ba\ Ca\ Cb$	$A\beta\ A\gamma\ B\gamma\ B\alpha\ C\alpha\ C\beta$	$BC\ CA\ AB$	$BC'\ CA'\ AB'$	$BC''\ CA''\ AB''$	$BC'''\ CA'''\ AB'''$
$abcu$	0	0	x	0	0	x	0	x	x	x	x	x
$\alpha\beta\gamma x$					x	0	x	0				
Abc	0	x 0 0	0	0	0	. . .	0 0 0 . . 0	0 0	0	. . .	. 0 0	x . .
Bca		0 x 0				. . .	. 0 0 0 0 .	. . 0 0 . .		. . .	0 . 0	. x .
Cab		0 0 x				. . .	0 . . 0 0 0	. . . . 0 0		. . .	0 0 .	. . x
$A\beta\gamma$	x 0 0	0	0	0	. . .	0	0 0	0 0 0 . . 0	0	. . .	x . .	. 0 0
$B\gamma\alpha$	0 x 0				. . .		. . 0 0 . .	. 0 0 0 0 .		. . .	. x .	0 . 0
$C\alpha\beta$	0 0 x				. . .		. . . . 0 0	0 . . 0 0 0		. . .	. . x	0 0 .
F_{12}	x 0 .	x 0 .	0	0 . 0 0 . 0	0 . 0	0 . 0	0 . x x . .	0 . x x . .	. 0 0	0	. 0 .	. 0 .
F_{13}	x . 0	x . 0		. 0 0 . 0 0	0 0 .	0 0 .	. 0 . . x x	. 0 . . x x	. 0 0		. . 0	. . 0
F_{23}	. x 0	. x 0		. 0 0 . 0 0	0 0 .	0 0 .	. . 0 . x x	. . 0 . x x	0 . 0		. . 0	. . 0
F_{21}	0 x .	0 x .		0 0 . 0 0 .	. 0 0	. 0 0	x x . 0 . .	x x . 0 . .	0 . 0		0 . .	0 . .
F_{31}	0 . x	0 . x		0 0 . 0 0 .	. 0 0	. 0 0	x x . . 0 .	x x . . 0 .	0 0 .		0 . .	0 . .
F_{32}	. 0 x	. 0 x		0 . 0 0 . 0	0 . 0	0 . 0	. . x x . 0	. . x x . 0	0 0 .		. 0 .	. 0 .
G_{12}	x 0 x	x x 0	0	x 0 0 0 x x	0 x 0	0 0 x	0 x x x x x	x 0 x x . x	0	. x x	x 0 .	x . 0
G_{13}	x x 0	x 0 x		0 x . x 0 0	0 0 x	0 x 0	x 0 x x . x	0 x x . x x		. x x	x . 0	x 0 .
G_{23}	x x 0	0 x x		x x x 0 0 0	0 0 x	x 0 0	x x 0 x x x	. x x 0 x x		x . x	. x 0	0 x .
G_{21}	0 x x	x x 0		0 0 0 x . x	x 0 0	0 0 x	. x x 0 x x	x x 0 x x .		x . x	0 x .	. x 0
G_{31}	0 x x	x 0 x		0 0 x x x 0	x 0 0	0 x 0	x x x x 0 x	x x . x x 0		x x .	0 . x	. 0 x
G_{32}	x 0 x	0 x x		. x 0 0 0 x	0 x 0	x 0 0	x x . x x 0	x . x x 0 x		x x .	. 0 x	0 . x
H_1	. x x	0	0	. . x 0 0 x	x	x	. . 0 0 0 0	x x x 0 0 x	0 . .	0 . .	0 x x	0 . .
H_2	x . x			0 x . . x 0			0 0 . . 0 0	0 x x x x 0	. 0 .	. 0 .	x 0 x	. 0 .
H_3	x x .			x 0 0 x . .			0 0 0 0 . .	x 0 0 x x x	. . 0	. . 0	x x 0	. . 0
h_1	0	x x x	0	0 0 x . . x	x	x	x x x 0 0 x	. . 0 0 0 0	0 . .	0 . .	0 . .	0 x x
h_2		x x x		. x 0 0 x .			0 x x x x 0	0 0 . . 0 0	. 0 .	. 0 .	. 0 .	x 0 x
h_3		x x x		x . . x 0 0			x 0 0 x x x	0 0 0 0 . .	. . 0	. . 0	. . 0	x x 0
$ABCuu$	x x x	0 0 0	0	x	x	x	0 0 0 0 0 0	x x x x x x	0	0	x	x
$ABCxx$	0 0 0	x x x					x x x x x x	0 0 0 0 0 0				
F_4	0 x x	0 x x	x 0 0	0 0 x 0 0 x	x	x	x x x 0 0 x	x x x 0 0 x	0	x x x	0 x x	0 x x
F_4	x 0 x	x 0 x	0 x 0	0 x 0 0 x 0			0 x x x x 0	0 x x x x 0		x x x	x 0 x	x 0 x
F_4''	x x 0	x x 0	0 0 x	x 0 0 x 0 0			x 0 0 x x x	x 0 0 x x x		x x x	x x 0	x x 0

III.

	abc	$\alpha\beta\gamma$	Abc	Bca	Cab	$A\beta\gamma$	$B\gamma\alpha$	$C\alpha\beta$	F_{12}	F_{13}	F_{23}	F_{21}	F_{31}	F_{32}	G_{12}	G_{13}	G_{23}	G_{21}	G_{31}	G_{32}	H_1	H_2	H_3	h_1	h_2	h_3	K	k	F_4	F_4'	F_4''
$abcu$	0																														
$\alpha\beta\gamma x$	x	0																													
Abc																															
Bca	0	x		0																											
Cab																															
$A\beta\gamma$			0	.	.																										
$B\gamma\alpha$	x	0	.	0	.		0																								
$C\alpha\beta$			.	.	0																										
F_{12}			0	x	0	0	x	0	0																						
F_{13}			0	0	x	0	0	x	.	0																					
F_{23}	x		0	0	x	0	0	x	x	0	0																				
F_{21}			x	0	0	x	0	0	x	x	.	0																			
F_{31}			x	0	0	x	0	0	x	x	x	0	0																		
F_{32}			0	x	0	0	x	0	0	x	x	x	.	0																	
G_{12}			0	.	x	0	x	.	0	0	x	x	x	x	0																
G_{13}			0	x	.	0	.	x	0	0	x	x	x	x	x	0															
G_{23}	x		x	0	.	.	0	x	x	x	0	0	x	x	x	x	0														
G_{21}			.	0	x	x	0	.	x	x	0	0	x	x	x	x	x	0													
G_{31}			.	x	0	x	.	0	x	x	x	x	0	0	x	x	x	x	0												
G_{32}			x	.	0	.	x	0	x	x	x	x	0	0	x	x	x	x	x	0											
H_1			x	.	.	.	0	0	x	x	x	.	.	x							0										
H_2	x		.	x	.	0	x	0	.	x	x	x	x	.	x						.	0									
H_3			.	.	x	0	0	x	x	.	.	x	x	x							.	.	0								
h_1			.	0	0	x	.	.	x	x	x	.	.	x							.	x	x	0							
h_2	x		0	x	0	.	x	.	.	x	x	x	x	.	x						x	.	x	.	0						
h_3			0	0	x	.	.	x	x	.	.	x	x	x							x	x	.	.	.	0					
K	x		x			x			x						x						0	0	0	x	x	x	0				
k																					x	x	x	0	0	0	x	0			
F_4			x	0	0	x	0	0	x	x	x	0	0	x							0	x	x	0	x	x	x	x	0		
F_4'	x		0	x	0	0	x	0	0	x	x	x	x	0	x						x	0	x	x	0	x	x	x	x	0	
F_4''			0	0	x	0	0	x	x	0	0	x	x	x							x	x	0	x	x	0	x	x	x	x	0

IV. *The Complete System.*

19. From the prepared system of § 14 we may in theory proceed to the complete system for three quadrics. This may be sub-divided into four groups K_1, K_2, K_3, K_4 say, corresponding to the four kinds of factors F_1, F_2, F_3, F_4 of the Prepared System. Each K group is defined as a group containing no factor F_i if i is greater than the suffix of K, while at least one factor with the suffix of K is present in the form.

It appears that the groups K_1, K_4 are small, whereas K_2 and K_3 are unwieldy. No effort will be made to count the members of K_2 and K_3, but it will be shown that they are strictly finite.

As for special types of members, all the *invariants* will be found.

The K_1 Group.

This consists of 12 forms made by squaring the 12 factors of the prepared system F_1 (§ 14).

The K_2 Group.

This consists of the forms made by squaring the 36 F_2 brackets (§ 14), together with all possible chains (i, i) where $i = a, b, c, \alpha, \beta, \gamma, A, B, C$; and also chains whose end elements are either x, p or u. A chain* has much the same significance as in the case of ternary forms, being a convenient abbreviation of a lengthy product. An example should make this clear :—

$\begin{pmatrix} & a & & b & & c & & \alpha & & \gamma & \\ x & & C & & A & & \beta & & B & & u \end{pmatrix}$ is a chain of grade 9, representing

$$a_x(aCu)(Cbu)(bAu)(Acu)(c_\beta)(\alpha\beta p)(\alpha Bx)(B\gamma x)u_\gamma.$$

The grade is the number of different elements not reckoning x, p, u. Each element a, etc. may stand in the upper or lower line. Manifestly all the elements of a chain must differ except possibly the end elements. The grade of a chain may be anything between two and nine inclusive. Theoretically then the K_2 system can be written out: it is finite but

---

* Cf. Turnbull, "Ternary Quadratic Types," *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 83, and Vol. 18, p. 79.

numerous. It is indeed limited further, since no pair of the three elements a, A, α may be adjacent, the same applying to b, B, β and c, C, γ. On the other hand the juxtaposition of BC would indicate four possible factors (BC), $(BCux)$, $(BCaap)$, $(BCaap)$.

This procedure does not guarantee that all the remaining members of K_2 are irreducible. A detailed application of the fundamental identities would eliminate a considerable number more. One useful step further may be taken by seeking the invariants of the group.

Invariants of the K_2 Group.

The six factors a_β, a_γ, ... together with (BC), (CA), (AB), alone lead to invariants. There are only two invariants properly belonging to *three* quadrics:

$$(BC)(CA)(AB) \text{ denoted by } \Phi_{123},$$

and

$$\begin{pmatrix} a & c & b & a \\ & \beta & \alpha & \gamma \end{pmatrix} \quad \text{,,} \quad \Omega.$$

The latter may be written as $\frac{1}{6}\,(\overline{abc}\,.\,\alpha\beta\gamma)^2$.

Before proceeding with the remaining K_3 and K_4 groups, the invariants of the whole system will be calculated.

The Invariants.

20. These forms are composed of the six factors a_β, a_γ, ..., three factors (BC), (CA), (AB), the six F_3 factors (Abc), $(A\beta\gamma)$, ..., and the three F_4 factors $(BCa\alpha)$, etc.

In the reduction system the following relations are relevant:—

$$\left.\begin{aligned} F_4(Abc) &\equiv (Bac)(AC)\,b_\alpha + (Cab)(AB)\,c_\alpha \\ F_4'(Bca) &\equiv (Cab)(AB)\,c_\beta + (Abc)(BC)\,a_\beta \\ F_4''(Cab) &\equiv (Abc)(BC)a_\gamma + (Bac)(CA)\,b_\gamma \end{aligned}\right\}. \tag{I}$$

Reciprocally

$$F_4(A\beta\gamma) \equiv (Ba\gamma)(AC)\,a_\beta + (Ca\beta)(AB)\,a_\gamma \text{ and two others.} \tag{II}$$

Again $\quad F_4 c_\beta \equiv (Bac)(Ca\beta) - (BC)\,a_\beta c_\alpha$ and five others, (III)

including $\quad F_4 b_\gamma \equiv (Ba\gamma)(Cab) - (BC)\,a_\gamma b_\alpha$. (IV)

Multiplying (I) by (Abc) and dropping reducible terms,

$$(Bac)(Abc)(AC)b_a + (Cab)(Abc)(AB)\,c_a \equiv 0 \text{ and reciprocally.} \tag{V}$$

Likewise from (III) there follows

$$(Bac)(Ca\beta)\,c_\beta \equiv 0\,; \tag{VI}$$

and from (IV) there follows, since $F_4(Abc)$ is reducible in (I),

$$(Ba\gamma)(Cab)(Abc) \equiv 0. \tag{VII}$$

Finally the product F_4F_4' is reducible thus:—

$F_4F_4' = (BCa\alpha)(CAb\beta) = (B\dot{c}a)\,\dot{c}'_\alpha(A\dot{c}b)\,\dot{c}'_\beta$: and now by bracketing A in the first bracket this simplifies.* (VIII)

The invariants are found in the K_2, K_3, K_4 groups. Those in the K_2 group have already been discussed.

As for the other invariants, they may be written as a product MN, where M consists of F_3 and F_4 factors, while N has only F_2 factors. A reference to the possible F_2 factors shows that N may consist of chains of the following types—A, B of course standing for any two of the three quadrics—

$$(A,\,B),\quad (a,\,\beta),\quad (a,\,b),\quad (\alpha,\,\beta),\quad (a,\,\alpha).$$

Moreover these chains can only be each of two sorts,

$$\begin{cases}(AB),\\ (AC)(CB),\end{cases}\quad \begin{cases}a_\beta,\\ \begin{pmatrix}a & & b & & c & \\ & & \gamma & & a & & \beta\end{pmatrix},\end{cases}\quad \begin{cases}\begin{pmatrix}a & & b\\ & \gamma & \end{pmatrix},\\ \begin{pmatrix}a & & c & & b\\ & \beta & & a & \end{pmatrix},\end{cases}\quad \begin{cases}\begin{pmatrix}a & & \beta\\ & c & \end{pmatrix},\\ \begin{pmatrix}a & & \gamma & & \beta\\ & b & & a & \end{pmatrix},\end{cases}\quad \begin{cases}\begin{pmatrix}a & & c & \\ & & & \beta & a\end{pmatrix},\\ \begin{pmatrix}a & & b & \\ & & & \gamma & a\end{pmatrix};\end{cases}$$

any others being immediately reducible.

21. Again, since N consists of chains, there are in N an even number of unpaired symbols standing as end links of these chains. Hence M also must have an even number of unpaired symbols; whence it follows that M has an even number of F_3 brackets. Also M may have F_4 brackets or not: suppose in the first case that M consists entirely of F_3 brackets.

* Analytically this is analogous to the formula (J) in reducing two quadrics. Cf. Turnbull, *ibid.*, p. 81.

Excluding the cases reducible by (VII), M may have two or four F_3 brackets, but cannot have six brackets: when the complementary factors of N are inserted this gives the following forms:—

(i) $(Abc)^2$ and its dual $(A\beta\gamma)^2$.

(ii) $(Abc)(Bca)(A, B)(a, b)$ and its dual $(A\beta\gamma)(B\gamma a)(A, B)(a, \beta)$.

(iii) $(Abc)(A\beta\gamma)(b, \gamma)(c, \beta)$, $(Abc)(A\beta\gamma)(b, \beta)(c, \gamma)$ and $(Abc)(A\beta\gamma)(b, c)(\beta, \gamma)$.

(iv) $(Abc)(B\gamma a)(A, B)(b, \gamma)(c, a)$

,, ,, ,, $(b, a)(c, \gamma)$

,, ,, ,, $(b, c)(\gamma, a)$.

(v) $(Abc)(A\beta\gamma)(Bac)(Ba\gamma)\, N$.

Of these, (i) is irreducible; as also is (ii) for the case when (A, B) is (AB). But the other type

$$(Abc)(Bca)(AC)(CB)(a, b)$$

reduces when the final chain is either $\begin{pmatrix} a & & b \\ & \gamma & \end{pmatrix}$ or $\begin{pmatrix} a & & c & & b \\ & \beta & & a & \end{pmatrix}$ by squaring the third of identities (I) or by using (V), respectively.

The next type (iii) gives $(Abc)(A\beta\gamma)\, b_\gamma c_\beta$ and $(Abc)(A\beta\gamma)\begin{pmatrix} b & & c \\ & a & \end{pmatrix}\begin{pmatrix} & a & \\ \beta & & \gamma \end{pmatrix}$ only: any other possible form of chain at once duplicates a link.

The next type (iv) must not contain the link b_a, owing to identity (VI). This leaves only two forms for the chains

$$(AB)\, b_\gamma c_a \quad \text{and} \quad (AC)(CB)\, b_\gamma c_a,$$

of which the former reduces by squaring an identity of type (IV).

Similarly by forming the product of identities (III) and (IV), type (v) reduces.

22. In the second case, suppose M to contain F_4 brackets. By (VIII) it is seen that only one such bracket, say F_4'' may occur. Excluding pro-

ducts reducible by identities (I)–(IV), the invariant is composed of

$$F_4'', \text{ i.e. } (ABc\gamma) \text{ with } (Abc), (A\beta\gamma), (Bac), (Ba\gamma), a_\gamma, b_\gamma, c_\alpha, c_\beta,$$

$$(BC), (CA), (AB).$$

In no case can an invariant be built of F_4'' followed by a product of these other factors, as is seen by trial. So no more invariants exist, except the squares of F_4'', F_4', and F_4.

23. *List of Invariants of Three Quadrics.*

	No. of forms.		Degree.
1	12	Forms Δ, Θ, etc. involving one, or two of the quadrics.	
2	1	$(BC)(CA)(AB) = \Phi_{123}$	(2, 2, 2)
3	1	$\begin{pmatrix} a & c & b & a \\ \beta & \alpha & \gamma & \end{pmatrix} = \Omega = a_\beta c_\beta c_\alpha b_\alpha b_\gamma a_\gamma$	(4, 4, 4)
4	6	$(Abc)^2$ and its dual $(A\beta\gamma)^2$	(2, 1, 1) (2, 3, 3)
5	3	$(BCa\alpha)^2 = F_4^2, F_4'^2, F_4''^2$	(4, 2, 2)
6	6	$(Abc)(Bca)(AB)\, a_\gamma b_\gamma$ and its dual $(A\beta\gamma)(B\gamma\alpha)(AB)\, c_\alpha c_\beta$	(3, 3, 4) (5, 5, 4)
7	6	$(Abc)(Bca)(AB)\begin{pmatrix} a & c & b \\ \beta & \alpha & \end{pmatrix}$ and its dual $(A\beta\gamma)(B\gamma\alpha)(AB)\begin{pmatrix} \alpha & \gamma & \beta \\ b & a & \end{pmatrix}$	(6, 6, 2) (6, 6, 6)
8	3	$(Abc)(A\beta\gamma)\, b_\gamma c_\beta$	(2, 4, 4)
9	6	$(Abc)(B\gamma\alpha)(AC)(CB)\, b_\gamma c_\alpha$	(5, 3, 6)
10	3	$(Abc)(A\beta\gamma)\begin{pmatrix} b & c \\ & \alpha \end{pmatrix}\begin{pmatrix} & a \\ \beta & \gamma \end{pmatrix}$	(6, 4. 4)

This gives 47 invariants in all.

The K_3 Group.

24. F_3 brackets are of these types

(i) $(abcu)$, $(\alpha\beta\gamma x)$.

(ii) (Abc), $(A\beta\gamma)$.

(iii) F_{ij} $(ij = 1, 2, 3$ and differ).

(iv) G_{ij}.

(v) H_i, h_i.

(vi) K, k.

In accordance with § 16, these six sets may be considered to be of increasing complexity; and to express members of one set in terms of earlier sets is to reduce them. We shall quote results without detailing every proof, as the work is tedious. Investigation shows that no irreducible member can have more than four F_3 brackets, and the cases when 3 or 4 occur are comparatively rare.

(i) $(abcu)$, $(\alpha\beta\gamma x)$.

25. The (r.s.)* shows that only the F_3 factors (Abc), (Bca), (Cab) can exist along with $(abcu)$. The complete system is

$$(abcu)(Abc)(Bca)(A_\epsilon B)c_x, \quad (abcu)(Abc)(A, a),$$

$(abcu)N$, where N consists of F_2 or F_1 brackets and (A, a) likewise; and where a, b, c may be rearranged. That all three F_3 factors cannot appear simultaneously is proved in § 26.

There is a dual set for $(\alpha\beta\gamma x)$.

(ii) (Abc), $(A\beta\gamma)$.

26. The (r.s.) rules out type $(abcu)$, so this group consists of members involving the six brackets $(Abc)\ldots(A\beta\gamma)$ with F_2 or F_1 brackets. Leaving

* A convenient abbreviation for *reduction system*.

out cases reduced in (VII), § 20, and (v), § 21, there may be the following general types :—

$$
\text{(B)} \quad \begin{cases} (Bac)N, \qquad (Ba\gamma)N, \\ (Bac)(Ca\beta)N, \\ (Bac)(Bu\gamma)N, \\ (Bca)^2 \text{ and } (B\gamma a)^2, \\ (Bac)(Cab)N \qquad \text{and its dual,} \\ (Bac)(Cab)(Abc)N \qquad \text{,,} \\ (Bca)(Abc)(A\beta\gamma)N \qquad \text{,,} \end{cases}
$$

where N consists of F_2 or F_1 brackets. The two latter forms reduce, leaving in this group the forms containing at most two F_3 brackets. Further reduction is not obvious. Herewith is a proof that $(Bca)(Cab)(Abc)$ reduces, which is typical of subsequent reductions, and shorter than that for $(Bca)(Abc)(A\beta\gamma)$.

$$(Bca)(Cab)(Abc)N \equiv 0.$$

From the (r.s.) we select these identities

(1) $(Abc)(Bau) \equiv -(Bca)(Abu)+(abcu)(BA)$,

(2) $(Abc)(Bax) \equiv h_3 b_a$ and also $h_3(Cab) \equiv 0 \bmod (Bac)$,

(3) $(Bca)(Cab)Ap \equiv (Abc)(BC)''+\text{etc.} \equiv 0$, § 16.

If N contains (Bau), the form reduces by multiplying (1) by (Bac). Hence by symmetry N cannot have any of the six (Bau), (Bcu), ...

If N contains (Bax), identity (2) applies. This rules out six more factors.

Since (3) rules out Ap, Bp, Cp it follows that the only factors in N involving A, B, C are $(BC)^i$, $(CA)^i$, $(AB)^i$, which cannot possibly be paired with the odd A, B, C of the first three brackets. Hence $N \equiv 0$.

(iii) F_{ij}, where $F_{12} = (Apb\beta)$.

27. The six F_{ij} factors reduce in product except for types F_{12}^2, $F_{12}F_{13}$, $F_{12}F_{32}$, which lead to four cases,

$$(\alpha)\ F_{ij}^2, \quad (\beta)\ F_{12}F_{13}M_1, \quad (\gamma)\ F_{12}F_{32}M_2, \quad (\delta)\ F_{12}M_3.$$

Now, by (r.s.),

M_1 may contain (Abc), $(A\beta\gamma)$, and F_2, F_1 factors,

M_2 " (Abc), (Cab), $(A\beta\gamma)$, $(C\alpha\beta)$ "

M_3 " " " "

Further investigation admits only the following to be retained :—

$$(C)\ \begin{cases} F_{ij}^2 \text{ and } F_{12}F_{13}(Abc)(A\beta\gamma) \text{ and the like, all quadratic complexes,} \\ {}^*F_{12}F_{13}(Abc)(A\gamma x)[\beta] \text{ where } [\beta] = u_\beta \text{ or } \begin{pmatrix}\beta & x \\ & a\end{pmatrix}, \\ F_{12}F_{13}(bcp)(\beta\gamma p) \text{ and } F_{12}F_{13}\, b_\gamma c_\beta, \\ F_{12}F_{32}(A, C) \text{ where } (A, C) = \begin{pmatrix}A & C \\ & p\end{pmatrix} \text{ or } (AC)^i, \\ {}^*F_{12}(Cab)N, \\ F_{12}(Abc)(A\beta\gamma)N, \\ {}^*F_{12}(Abc)N, \\ F_{12}N, \quad \text{where } N \text{ has } F_2 \text{ or } F_1 \text{ factors.} \end{cases}$$

(iv) G_{ij}, where $G_{12} = (Apb\gamma)$.

28. A form containing G_{ij} is reduced when expressed in terms of preceding factors. Taking G_{12} as typical, the (r.s.) admits of

.... F_{12}, F_{13}, (Abc), (Bca), $(A\beta\gamma)$, $(C\alpha\beta)$, N.

But $G_{12}G_{13} \equiv F_{12}F_{13}$; so that $G_{12}F_{12}F_{13} \equiv 0$. Accordingly we need

---

* These have dual forms.

only consider the types

(1) $G_{12}F_{12}M$,

(2) $G_{12}F_{13}M$,

(3) $G_{12}M$, where M contains neither G_{ij} nor F_{ij}.

Since G_{13} is dual of G_{12} and F_{13} is its own dual, then types (1) and (2) are dual. So (1) and (3) need only be considered. Ultimately we are left with

(D)
$$\left\{\begin{array}{l} {}^*G_{12}F_{12}(A\beta\gamma)(A) \text{ where } (A) = (Ap), \begin{pmatrix} A & x \\ & b \end{pmatrix} \text{ or } \begin{pmatrix} A & p \\ & B \end{pmatrix}, \\ G_{12}(A\beta\gamma)(Abc)\,c_\beta\,(A), \\ G_{12}(Abc)\begin{pmatrix} c & \gamma \\ & \beta \end{pmatrix}, \text{ where no independent dual exists, and there are only three of this type for all } G_{ij}. \\ G_{12}\begin{pmatrix} b & \beta & \\ & c & \gamma \end{pmatrix}(A), \\ G_{12}N, \text{ where } N \text{ consists of } F_3, F_1 \text{ factors but contains neither } c \text{ nor } \beta. \\ G_{12}^2. \end{array}\right.$$

The brevity of the above list is largely due to identities of the type

$$G_{12}\,i_\beta j_c \equiv G_{13}\,i_\gamma j_b,$$

where i, j are any two different symbols u, a, A, a.

(v) H_i, h_i: where $H_1 = (BCua)$.

29. The factor H_1 reduces with any F_3 bracket except

$$F_{21},\ F_{31},\ H_2,\ H_3,\ h_1,\ (A\beta\gamma),\ (B\gamma a),\ (Ca\beta),\ (Bca),\ (Cab).$$

But owing to relations such as

$$H_1F_{21} \equiv F_4\,(Bp)\,u_a, \qquad H_1(Bca) \equiv F_4(Bcu),$$

$$h_1F_{21} \equiv F_4\,(Bp)\,a_x, \qquad H_1h_1 \equiv F_4(BC)',$$

$$H_1H_2 \equiv K(Ca\beta),$$

$$H_1(A\beta\gamma) \equiv H_2(B\gamma a) \equiv H_3(Ca\beta),$$

one further discrimination, viz. that the six brackets (Abc), $(A\beta\gamma)$, etc. are the simplest F_3 brackets with *one* symbol A, B or C; next come the six F_{ij}; and next G_{ij}. Other F_3 brackets precede or follow this group because less or more symbols A, B, C occur.

The Reduction System.

17. The prepared system contains 79 elements, but a product of two of these elements is often reducible. Thus the product of two F_3 brackets $(abcu)(\alpha\beta\gamma x)$ is identically equal to $\Sigma \dot{a}_\alpha b_\beta \dot{c}_\gamma \dot{u}_x$, which eliminates the F_3 brackets and therefore reduces the product. It is possible to carry out a systematic examination of every such product, and to construct a table in which any such product of two of these 79 factors is shown to be either (i) reducible, or (ii) irreducible, or (iii) equivalent to another product. This table consists of 79 rows and columns—one row and one column for each different factor, from a_x to F_4''. The following fragment of the complete table should make clear the method of classification:—

	H_1	H_2	H_3	h_1	h_2	h_3
H_1	0					
H_2	.	0				
H_3	.	.	0			
h_1	.	x	x	0		
h_2	x	.	x	.	0	
h_3	x	x	.	.	.	0

x = reducible, 0 = irreducible, · = equivalent to another product.

Here, for example, it is shown that the product $h_1 H_2$ is reducible, that $H_3 H_3$ is irreducible, and $H_1 H_2$ is equivalent to another product. The whole table is a large triangle with an hypotenuse of 79 marks of irreducibility which indicate the squares of 79 factors $a_x \ldots F_4''$. This table is called the Reduction System.

Construction of the Reduction System.

18. This table is constructed by examining a product of factors, for example $(Abu)(p\beta\gamma)$. Here, by permuting bu, p we arrive at the identity

$$(Abu)(p\beta\gamma) \equiv G_{12} u_\beta - F_{12} u_\gamma - (pA)\, b_\gamma u_\beta,$$

suppressing reducible terms involving b_β. In accordance with the conditions of § 16, the reducible mark x is placed opposite G_{12} and u_β in the table, and the mark · is placed twice, to correspond with $(Abu)(p\beta\gamma)$ and with $F_{12}u_\gamma$. The third term $(pA)\,b_\gamma u_\beta$ has three factors and is analysed independently.

By interchanging symbols a, A, α with b, B, β or c, C, γ this one identity implies five others. By reciprocating these we get six others, as, for example,

$$(A\beta x)(pbc) \equiv G_{13}\,b_x - \dots .$$

And further, by interchanging in a *linear* identity the symbols a, A, α with u, p, x we obtain a new identity, equally valid, since the convolution of two of u, p, x is reducible, and also since the symbols u, p, x behave analytically in the same way as a, A, α. For example, by interchanging b, B, β with u, p, x in the above identity we may forecast the new relation

$$(Abu)(B\gamma x) \equiv H_3\,b_x - (AB)'\,b_\gamma - (AB)\,u_\gamma\,b_x.$$

Thus from one product $(Abu)(p\beta\gamma)$ a large number of other products may be dealt with at considerable economy of labour.

Below is subjoined the table of the reduction system, broken up for convenience into three parts: these deal respectively with (i) F_1F_2 brackets, (ii) one F_1 or F_2 with one F_3 or F_4, and (iii) F_3F_4 brackets. The detailed proofs are not given, for they are tedious but all of the same kind: and it is easy to test any assertion made in the table by applying one or other linear identity.

I.

	a_x	b_x	c_x	u_α	u_β	u_γ	a_β	a_γ	b_γ	b_α	c_α	c_β	bcp	cap	abp	$\beta\gamma p$	$\gamma\alpha p$	$\alpha\beta p$	Ab	Ac	Bc	Ba	Ca	Cb	$A\beta$	$A\gamma$	$B\gamma$	$B\alpha$	$C\alpha$	$C\beta$	BC'	CA'	AB'	BC''	CA''	AB''			
g*		0			0			0			0																												
bcp	x	0	0																																				
cap	0	.	0		0				0					0																									
abp	0	0	.																																				
$\beta\gamma p$				x	0	0																																	
$\gamma\alpha p$		0		0	.	0			0					0			0																						
$\alpha\beta p$				0	0	.																																	
Abu							0	0	0	0	.	0	0	.	0	.	.	.																					
Acu							0	0	0	.	0	0	0	0	.	.	.	.			0																		
Bcu							.	0	0	0	0	0	0	0	.	.	.	.																					
Bau		0			0		0	0	0	0	0	.	.	0	0	.	.	.																					
Cau							0	0	.	0	0	0	.	0	0	.	.	.			0																		
Cbu							0	.	0	0	0	0	0	.	0	.	.	.																					
$A\beta x$							0	.	0	0	0	0	.	.	.	0	.	0	0	0	.	.	.	.															
$A\gamma x$							.	0	0	0	0	0	.	.	.	0	0	.	0	0	.	.	.	.															
$B\gamma x$		0			0		0	0	0	.	0	0	.	.	.	0	0	.	.	.	0	0	.	.			0												
$B\alpha x$							0	0	.	0	0	0	.	.	.	.	0	0	.	.	0	0	.	.															
$C\alpha x$							0	0	0	0	0	.	.	.	.	.	0	0	.	.	.	.	0	0															
$C\beta x$							0	0	0	0	.	0	.	.	.	0	.	0	.	.	.	.	0	0													Ap	Bp	Cp
$BCux$							.	.	x	.	.	x							.	.	0	0	0	0	.	.	0	0	0	0							x	0	0
$CAux$		0			0		.	x	.	.	x	.		x			x		0	0	.	.	0	0	0	0	.	.	0	0		0					0	.	0
$ABux$							x	.	.	x	.	.							0	0	0	0	.	.	0	0	0	0	.	.							0	0	.
$BCaap$	0	.	.	.	.	.	0	0	.	.	.	.	x	0	0	.	.	.	.	.	x	0	0	x	x	x	.	0	0	.	x	.	.	0					
$CAbbp$	.	0	.	.	.	.	.	.	0	0	.	.	0	x	0	.	.	.	0	x	.	.	x	0	0	.	x	x	.	0	.	x	.	.	0				
$ABccp$	.	.	0	.	.	.	.	.	.	.	0	0	0	0	x	.	.	.	x	0	0	x	.	.	.	0	0	.	x	x	.	.	x	.	.	0	BC'''	CA'''	AB'''
$BC\alpha\alpha p$	.	.	.	0	.	.	.	.	.	0	0	.	.	.	.	x	0	0	x	x	.	0	0	.	.	.	x	0	0	x	x	.	.	.	x	x	0		
$CA\beta\beta p$	.	.	.	.	0	.	0	.	.	.	.	0	.	.	.	0	x	0	0	.	x	x	.	0	0	x	.	.	x	0	.	x	.	x	.	x	.	0	
$AB\gamma\gamma p$	.	.	.	.	.	0	.	0	0	.	.	.	.	.	.	0	0	x	.	0	0	.	x	x	x	0	0	x	.	.	.	.	x	x	x	.	.	.	0

* The factors (BC), (CA), (AB) do not reduce with any of the above F_1 and F'_2 factors. Also g in the above denotes any of the twelve factors a_x, b_x, …, c_α, c_β.

II.

	$a_x\ b_x\ c_x$	$u_\alpha\ u_\beta\ u_\gamma$	$Ap\ Bp\ Cp$	$a_\beta\, a_\gamma\, b_\gamma\, b_\alpha\, c_\alpha\, c_\beta$	$bcp\ cap\ abp$	$\beta\gamma p\ \gamma\alpha p\ \alpha\beta p$	$Ab\ Ac\ Bc\ Ba\ Ca\ Cb$	$A\beta\ A\gamma\ B\gamma\ B\alpha\ C\alpha\ C\beta$	$BC\ CA\ AB$	$BC'\ CA'\ AB'$	$BC''\ CA''\ AB''$	$BC'''\ CA'''\ AB'''$
$abcu$	0	0	x	0	0	x	0	x	x	x	x	x
$\alpha\beta\gamma x$					x	0	x	0				
Abc	0	x 0 0	0	0	0	. . .	0 0 0 . . 0	0 0	0	. . .	. 0 0	x . .
Bca		0 x 0				. . .	. 0 0 0 0 .	. . 0 0 . .		. . .	0 . 0	. x .
Cab		0 0 x				. . .	0 . . 0 0 0	. . . . 0 0		. . .	0 0 .	. . x
$A\beta\gamma$	x 0 0	0	0	0	. . .	0	0 0	0 0 0 . . 0	0	. . .	x . .	. 0 0
$B\gamma\alpha$	0 x 0				. . .		. . 0 0 . .	. 0 0 0 0 .		. . .	. x .	0 . 0
$C\alpha\beta$	0 0 x				. . .		. . . . 0 0	0 . . 0 0 0		. . .	. . x	0 0 .
F_{12}	x 0 .	x 0 .	0	0 . 0 0 . 0	0 . 0	0 . 0	0 . x x . .	0 . x x . .	. 0 0	0	. 0 .	. 0 .
F_{13}	x . 0	x . 0		. 0 0 . 0 0	0 0 .	0 0 .	. 0 . . x x	. 0 . . x x	. 0 0		. . 0	. . 0
F_{23}	. x 0	. x 0		. 0 0 . 0 0	0 0 .	0 0 .	. . 0 . x x	. . 0 . x x	0 . 0		. . 0	. . 0
F_{21}	0 x .	0 x .		0 0 . 0 0 .	. 0 0	. 0 0	x x . 0 . .	x x . 0 . .	0 . 0		0 . .	0 . .
F_{31}	0 . x	0 . x		0 0 . 0 0 .	. 0 0	. 0 0	x x . . 0 .	x x . . 0 .	0 0 .		0 . .	0 . .
F_{32}	. 0 x	. 0 x		0 . 0 0 . 0	0 . 0	0 . 0	. . x x . 0	. . x x . 0	0 0 .		. 0 .	. 0 .
G_{12}	x 0 x	x x 0	0	x 0 0 0 x x	0 x 0	0 0 x	0 x x x x x	x 0 x x . x	0	. x x	x 0 .	x . 0
G_{13}	x x 0	x 0 x		0 x . x 0 0	0 0 x	0 x 0	x 0 x x . x	0 x x . x x		. x x	x . 0	x 0 .
G_{23}	x x 0	0 x x		x x x 0 0 0	0 0 x	x 0 0	x x 0 x x x	. x x 0 x x		x . x	. x 0	0 x .
G_{21}	0 x x	x x 0		0 0 0 x . x	x 0 0	0 0 x	. x x 0 x x	x x 0 x x .		x . x	0 x .	. x 0
G_{31}	0 x x	x 0 x		0 0 x x x 0	x 0 0	0 x 0	x x x x 0 x	x x . x x 0		x x .	0 . x	. 0 x
G_{32}	x 0 x	0 x x		. x 0 0 0 x	0 x 0	x 0 0	x x . x x 0	x . x x 0 x		x x .	. 0 x	0 . x
H_1	. x x	0	0	. . x 0 0 x	x	x	. . 0 0 0 0	x x x 0 0 x	0 . .	0 . .	0 x x	0 . .
H_2	x . x			0 x . . x 0			0 0 . . 0 0	0 x x x x 0	. 0 .	. 0 .	x 0 x	. 0 .
H_3	x x .			x 0 0 x . .			0 0 0 0 . .	x 0 0 x x x	. . 0	. . 0	x x 0	. . 0
h_1	0	x x x	0	0 0 x . . x	x	x	x x x 0 0 x	. . 0 0 0 0	0 . .	0 . .	0 . .	0 x x
h_2		x x x		. x 0 0 x .			0 x x x x 0	0 0 . . 0 0	. 0 .	. 0 .	. 0 .	x 0 x
h_3		x x x		x . . x 0 0			x 0 0 x x x	0 0 0 0 . .	. . 0	. . 0	. . 0	x x 0
$ABCuu$	x x x	0 0 0	0	x	x	x	0 0 0 0 0 0	x x x x x x	0	0	x	x
$ABCxx$	0 0 0	x x x					x x x x x x	0 0 0 0 0 0				
F_4	0 x x	0 x x	x 0 0	0 0 x 0 0 x	x	x	x x x 0 0 x	x x x 0 0 x	0	x x x	0 x x	0 x x
F_4'	x 0 x	x 0 x	0 x 0	0 x 0 0 x 0			0 x x x x 0	0 x x x x 0		x x x	x 0 x	x 0 x
F_4''	x x 0	x x 0	0 0 x	x 0 0 x 0 0			x 0 0 x x x	x 0 0 x x x		x x x	x x 0	x x 0

III.

	abc	$\alpha\beta\gamma$	Abc	Bca	Cab	$A\beta\gamma$	$B\gamma\alpha$	$C\alpha\beta$	F_{12}	F_{13}	F_{23}	F_{21}	F_{31}	F_{32}	G_{12}	G_{13}	G_{23}	G_{21}	G_{31}	G_{32}	H_1	H_2	H_3	h_1	h_2	h_3	K	k	F_4	F_4'	F_4''
$abcu$	0																														
$\alpha\beta\gamma x$	x	0																													
Abc																															
Bca	0	x		0																											
Cab																															
$A\beta\gamma$			0	.	.																										
$B\gamma\alpha$	x	0	.	0	.		0																								
$C\alpha\beta$			.	.	0																										
F_{12}			0	x	0	0	x	0	0																						
F_{13}			0	0	x	0	0	x	.	0																					
F_{23}	x		0	0	x	0	0	x	x	0	0																				
F_{21}			x	0	0	x	0	0	x	x	.	0																			
F_{31}			x	0	0	x	0	0	x	x	x	0	0																		
F_{32}			0	x	0	0	x	0	0	x	x	x	.	0																	
G_{12}			0	.	x	0	x	.	0	0	x	x	x	x	0																
G_{13}			0	x	.	0	.	x	0	0	x	x	x	x	x	0															
G_{23}	x		x	0	.	.	0	x	x	x	0	0	x	x	x	x	0														
G_{21}			.	0	x	x	0	.	x	x	0	0	x	x	x	x	x	0													
G_{31}			.	x	0	x	.	0	x	x	x	x	0	0	x	x	x	x	0												
G_{32}			x	.	0	.	x	0	x	x	x	x	0	0	x	x	x	x	x	0											
H_1			x	.	.	.	0	0	x	x	x	.	.	x							0										
H_2	x		.	x	.	0	x	0	.	x	x	x	x	.			x				.	0									
H_3			.	.	x	0	0	x	x	.	.	x	x	x							.	.	0								
h_1			.	0	0	x	.	.	x	x	x	.	.	x							.	x	x	0							
h_2	x		0	x	0	.	x	.	.	x	x	x	x	.			x				x	.	x	.	0						
h_3			0	0	x	.	.	x	x	.	.	x	x	x							x	x	.	.	.	0					
K	x			x			x				x						x				0	0	0	x	x	x	0				
k																					x	x	x	0	0	0	x	0			
F_4			x	0	0	x	0	0	x	x	x	0	0	x							0	x	x	0	x	x	x	x	0		
F_4'	x		0	x	0	0	x	0	0	x	x	x	x	0			x				x	0	x	x	0	x	x	x	x	0	
F_4''			0	0	x	0	0	x	x	0	0	x	x	x							x	x	0	x	x	0	x	x	x	x	0

IV. *The Complete System.*

19. From the prepared system of § 14 we may in theory proceed to the complete system for three quadrics. This may be sub-divided into four groups K_1, K_2, K_3, K_4 say, corresponding to the four kinds of factors F_1, F_2, F_3, F_4 of the Prepared System. Each K group is defined as a group containing no factor F_i if i is greater than the suffix of K, while at least one factor with the suffix of K is present in the form.

It appears that the groups K_1, K_4 are small, whereas K_2 and K_3 are unwieldy. No effort will be made to count the members of K_2 and K_3, but it will be shown that they are strictly finite.

As for special types of members, all the *invariants* will be found.

The K_1 Group.

This consists of 12 forms made by squaring the 12 factors of the prepared system F_1 (§ 14).

The K_2 Group.

This consists of the forms made by squaring the 36 F_2 brackets (§ 14), together with all possible chains (i, i) where $i = a, b, c, \alpha, \beta, \gamma, A, B, C$; and also chains whose end elements are either x, p or u. A chain* has much the same significance as in the case of ternary forms, being a convenient abbreviation of a lengthy product. An example should make this clear :—

$\begin{pmatrix} & a & & b & & c & & \alpha & & \gamma & \\ x & & C & & A & & \beta & & B & & u \end{pmatrix}$ is a chain of grade 9, representing

$$a_x(aCu)(Cbu)(bAu)(Acu)(c_\beta)(\alpha\beta p)(\alpha Bx)(B\gamma x)u_\gamma.$$

The grade is the number of different elements not reckoning x, p, u. Each element a, etc. may stand in the upper or lower line. Manifestly all the elements of a chain must differ except possibly the end elements. The grade of a chain may be anything between two and nine inclusive. Theoretically then the K_2 system can be written out: it is finite but

* Cf. Turnbull, "Ternary Quadratic Types," *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 83, and Vol. 18, p. 79.

numerous. It is indeed limited further, since no pair of the three elements a, A, a may be adjacent, the same applying to b, B, β and c, C, γ. On the other hand the juxtaposition of BC would indicate four possible factors (BC), $(BCux)$, $(BCaap)$, $(BCaap)$.

This procedure does not guarantee that all the remaining members of K_2 are irreducible. A detailed application of the fundamental identities would eliminate a considerable number more. One useful step further may be taken by seeking the invariants of the group.

Invariants of the K_2 Group.

The six factors a_β, a_γ, ... together with (BC), (CA), (AB), alone lead to invariants. There are only two invariants properly belonging to *three* quadrics:

$$(BC)(CA)(AB) \text{ denoted by } \Phi_{123},$$

and

$$\begin{pmatrix} a & c & b & a \\ & \beta & a & \gamma \end{pmatrix} \qquad \text{,,} \qquad \Omega.$$

The latter may be written as $\frac{1}{6}\,(\overline{abc}\,.\,a\beta\gamma)^2$.

Before proceeding with the remaining K_3 and K_4 groups, the invariants of the whole system will be calculated.

The Invariants.

20. These forms are composed of the six factors a_β, a_γ, ..., three factors (BC), (CA), (AB), the six F_3 factors (Abc), $(A\beta\gamma)$, ..., and the three F_4 factors $(BCaa)$, etc.

In the reduction system the following relations are relevant:—

$$\left.\begin{aligned} F_4(Abc) &\equiv (Bac)(AC)\,b_a + (Cab)(AB)\,c_a \\ F_4'(Bca) &\equiv (Cab)(AB)\,c_\beta + (Abc)(BC)\,a_\beta \\ F_4''(Cab) &\equiv (Abc)(BC)\,a_\gamma + (Bac)(CA)\,b_\gamma \end{aligned}\right\}. \qquad \text{(I)}$$

Reciprocally

$$F_4(A\beta\gamma) \equiv (Ba\gamma)(AC)\,a_\beta + (Ca\beta)(AB)\,a_\gamma \text{ and two others.} \qquad \text{(II)}$$

Again $\quad F_4 c_\beta \equiv (Bac)(Ca\beta) - (BC)\,a_\beta c_a$ and five others, $\qquad$ (III)

including $\quad F_4 b_\gamma \equiv (Ba\gamma)(Cab) - (BC)\,a_\gamma b_a$. $\qquad$ (IV)

Multiplying (I) by (Abc) and dropping reducible terms,

$$(Bac)(Abc)(AC)b_a+(Cab)(Abc)(AB)c_a \equiv 0 \text{ and reciprocally.} \tag{V}$$

Likewise from (III) there follows

$$(Bac)(Ca\beta)c_\beta \equiv 0; \tag{VI}$$

and from (IV) there follows, since $F_4(Abc)$ is reducible in (I),

$$(Ba\gamma)(Cab)(Abc) \equiv 0. \tag{VII}$$

Finally the product F_4F_4' is reducible thus:—

$F_4F_4' = (BCaa)(CAb\beta) = (B\dot{c}a)\dot{c}_a'(A\dot{c}b)\dot{c}_\beta'$: and now by bracketing A in the first bracket this simplifies.* (VIII)

The invariants are found in the K_2, K_3, K_4 groups. Those in the K_2 group have already been discussed.

As for the other invariants, they may be written as a product MN, where M consists of F_3 and F_4 factors, while N has only F_2 factors. A reference to the possible F_2 factors shows that N may consist of chains of the following types—A, B of course standing for any two of the three quadrics—

$$(A,\ B),\quad (a,\ \beta),\quad (a,\ b),\quad (a,\ \beta),\quad (a,\ a).$$

Moreover these chains can only be each of two sorts,

$$\begin{cases}(AB),\\ (AC)(CB),\end{cases}\quad \begin{cases}a_\beta,\\ \begin{pmatrix}a & b & c\\ & \gamma & a & \beta\end{pmatrix},\end{cases}\quad \begin{cases}\begin{pmatrix}a & b\\ & \gamma\end{pmatrix},\\ \begin{pmatrix}a & c & b\\ & \beta & a\end{pmatrix},\end{cases}\quad \begin{cases}\begin{pmatrix}a & \beta\\ & c\end{pmatrix},\\ \begin{pmatrix}a & \gamma & \beta\\ & b & a\end{pmatrix},\end{cases}\quad \begin{cases}\begin{pmatrix}a & c\\ & \beta & a\end{pmatrix},\\ \begin{pmatrix}a & b\\ & \gamma & a\end{pmatrix};\end{cases}$$

any others being immediately reducible.

21. Again, since N consists of chains, there are in N an even number of unpaired symbols standing as end links of these chains. Hence M also must have an even number of unpaired symbols; whence it follows that M has an even number of F_3 brackets. Also M may have F_4 brackets or not: suppose in the first case that M consists entirely of F_3 brackets.

---

* Analytically this is analogous to the formula (J) in reducing two quadrics. Cf. Turnbull, *ibid.*, p. 81.

Excluding the cases reducible by (VII), M may have two or four F_3 brackets, but cannot have six brackets: when the complementary factors of N are inserted this gives the following forms:—

(i) $(Abc)^2$ and its dual $(A\beta\gamma)^2$.

(ii) $(Abc)(Bca)(A, B)(a, b)$ and its dual $(A\beta\gamma)(B\gamma\alpha)(A, B)(\alpha, \beta)$.

(iii) $(Abc)(A\beta\gamma)(b, \gamma)(c, \beta)$, $(Abc)(A\beta\gamma)(b, \beta)(c, \gamma)$ and $(Abc)(A\beta\gamma)(b, c)(\beta, \gamma)$.

(iv) $(Abc)(B\gamma a)(A, B)(b, \gamma)(c, a)$

" " " $(b, a)(c, \gamma)$

" " " $(b, c)(\gamma, a)$.

(v) $(Abc)(A\beta\gamma)(Bac)(Ba\gamma)\,N$.

Of these, (i) is irreducible; as also is (ii) for the case when (A, B) is (AB). But the other type

$$(Abc)(Bca)(AC)(CB)(a, b)$$

reduces when the final chain is either $\begin{pmatrix} a & & b \\ & \gamma & \end{pmatrix}$ or $\begin{pmatrix} a & & c & & b \\ & \beta & & \alpha & \end{pmatrix}$ by squaring the third of identities (I) or by using (V), respectively.

The next type (iii) gives $(Abc)(A\beta\gamma)\,b_\gamma c_\beta$ and $(Abc)(A\beta\gamma)\begin{pmatrix} b & & c \\ & \alpha & \end{pmatrix}\begin{pmatrix} & a & \\ \beta & & \gamma \end{pmatrix}$ only: any other possible form of chain at once duplicates a link.

The next type (iv) must not contain the link b_a, owing to identity (VI). This leaves only two forms for the chains

$$(AB)\,b_\gamma c_a \quad \text{and} \quad (AC)(CB)\,b_\gamma c_a,$$

of which the former reduces by squaring an identity of type (IV).

Similarly by forming the product of identities (III) and (IV), type (v) reduces.

22. In the second case, suppose M to contain F_4 brackets. By (VIII) it is seen that only one such bracket, say F_4'' may occur. Excluding pro-

ducts reducible by identities (I)–(IV), the invariant is composed of

$$F_4'', \text{ i.e. } (ABc\gamma) \text{ with } (Abc), (A\beta\gamma), (Bac), (Ba\gamma), a_\gamma, b_\gamma, c_\alpha, c_\beta,$$
$$(BC), (CA), (AB).$$

In no case can an invariant be built of F_4'' followed by a product of these other factors, as is seen by trial. So no more invariants exist, except the squares of F_4'', F_4', and F_4.

23. *List of Invariants of Three Quadrics.*

	No. of forms.		Degree.
1	12	Forms Δ, Θ, etc. involving one, or two of the quadrics.	
2	1	$(BC)(CA)(AB) = \Phi_{123}$	(2, 2, 2)
3	1	$\begin{pmatrix} a & c & b & a \\ \beta & \alpha & \gamma \end{pmatrix} = \Omega = a_\beta c_\beta c_\alpha b_\alpha b_\gamma a_\gamma$	(4, 4, 4)
4	6	$(Abc)^2$ and its dual $(A\beta\gamma)^2$	(2, 1, 1) (2, 3, 3)
5	3	$(BCa\alpha)^2 = F_4^2, F_4'^2, F_4''^2$	(4, 2, 2)
6	6	$(Abc)(Bca)(AB)\, a_\gamma b_\gamma$ and its dual $(A\beta\gamma)(B\gamma\alpha)(AB)\, c_\alpha c_\beta$	(3, 3, 4) (5, 5, 4)
7	6	$(Abc)(Bca)(AB)\begin{pmatrix} a & c & b \\ \beta & \alpha \end{pmatrix}$ and its dual $(A\beta\gamma)(B\gamma\alpha)(AB)\begin{pmatrix} \alpha & \gamma & \beta \\ b & a \end{pmatrix}$	(6, 6, 2) (6, 6, 6)
8	3	$(Abc)(A\beta\gamma)\, b_\gamma c_\beta$	(2, 4, 4)
9	6	$(Abc)(B\gamma\alpha)(AC)(CB)\, b_\gamma c_\alpha$	(5, 3, 6)
10	3	$(Abc)(A\beta\gamma)\begin{pmatrix} b & c \\ \alpha \end{pmatrix}\begin{pmatrix} a \\ \beta & \gamma \end{pmatrix}$	(6, 4, 4)

This gives 47 invariants in all.

The K_3 Group.

24. F_3 brackets are of these types

(i) $(abcu)$, $(\alpha\beta\gamma x)$.

(ii) (Abc), $(A\beta\gamma)$.

(iii) F_{ij} $(ij = 1, 2, 3$ and differ).

(iv) G_{ij}.

(v) H_i, h_i.

(vi) K, k.

In accordance with § 16, these six sets may be considered to be of increasing complexity; and to express members of one set in terms of earlier sets is to reduce them. We shall quote results without detailing every proof, as the work is tedious. Investigation shows that no irreducible member can have more than four F_3 brackets, and the cases when 3 or 4 occur are comparatively rare.

(i) $(abcu)$, $(\alpha\beta\gamma x)$.

25. The (r.s.)* shows that only the F_3 factors (Abc), (Bca), (Cab) can exist along with $(abcu)$. The complete system is

$$(abcu)(Abc)(Bca)(A_e B)c_x, \quad (abcu)(Abc)(A, a),$$

$(abcu)\,N$, where N consists of F_2 or F_1 brackets and (A, a) likewise; and where a, b, c may be rearranged. That all three F_3 factors cannot appear simultaneously is proved in § 26.

There is a dual set for $(\alpha\beta\gamma x)$.

(ii) (Abc), $(A\beta\gamma)$.

26. The (r.s.) rules out type $(abcu)$, so this group consists of members involving the six brackets $(Abc)\ldots(A\beta\gamma)$ with F_2 or F_1 brackets. Leaving

* A convenient abbreviation for *reduction system*.

out cases reduced in (VII), § 20, and (v), § 21, there may be the following general types :—

$$
(B)\quad \begin{cases}
(Bac)N, & (Ba\gamma)N, \\
(Bac)(Ca\beta)N, & \\
(Bac)(Bu\gamma)N, & \\
(Bca)^2 \text{ and } (B\gamma a)^2, & \\
(Bac)(Cab)N & \text{and its dual,} \\
(Bac)(Cab)(Abc)N & ,, \\
(Bca)(Abc)(A\beta\gamma)N & ,,
\end{cases}
$$

where N consists of F_2 or F_1 brackets. The two latter forms reduce, leaving in this group the forms containing at most two F_3 brackets. Further reduction is not obvious. Herewith is a proof that $(Bca)(Cab)(Abc)$ reduces, which is typical of subsequent reductions, and shorter than that for $(Bca)(Abc)(A\beta\gamma)$.

$$(Bca)(Cab)(Abc)N \equiv 0.$$

From the (r.s.) we select these identities

(1) $(Abc)(Bau) \equiv -(Bca)(Abu)+(abcu)(BA)$,

(2) $(Abc)(Bax) \equiv h_3 b_a$ and also $h_3(Cab) \equiv 0 \bmod (Bac)$,

(3) $(Bca)(Cab)Ap \equiv (Abc)(BC)''+\text{etc.} \equiv 0$, § 16.

If N contains (Bau), the form reduces by multiplying (1) by (Bac). Hence by symmetry N cannot have any of the six (Bau), (Bcu), ...

If N contains (Bax), identity (2) applies. This rules out six more factors.

Since (3) rules out Ap, Bp, Cp it follows that the only factors in N involving A, B, C are $(BC)^i$, $(CA)^i$, $(AB)^i$, which cannot possibly be paired with the odd A, B, C of the first three brackets. Hence $N \equiv 0$.

(iii) F_{ij}, where $F_{12} = (Apb\beta)$.

27. The six F_{ij} factors reduce in product except for types F_{12}^2, $F_{12}F_{13}$, $F_{12}F_{32}$, which lead to four cases,

$$(\alpha)\ F_{ij}^2, \quad (\beta)\ F_{12}F_{13}M_1, \quad (\gamma)\ F_{12}F_{32}M_2, \quad (\delta)\ F_{12}M_3.$$

Now, by (r.s.),

M_1 may contain	(Abc), $(A\beta\gamma)$,	and F_2, F_1 factors,
M_2 ,,	(Abc), (Cab), $(A\beta\gamma)$, $(C\alpha\beta)$	,,
M_3 ,,	,, ,,	,,

Further investigation admits only the following to be retained :—

(C) $\begin{cases} F_{ij}^2 \text{ and } F_{12}F_{13}(Abc)(A\beta\gamma) \text{ and the like, all quadratic complexes,} \\ {}^*F_{12}F_{13}(Abc)(A\gamma x)[\beta] \text{ where } [\beta] = u_\beta \text{ or } \begin{pmatrix}\beta & x \\ & a\end{pmatrix}, \\ F_{12}F_{13}(bcp)(\beta\gamma p) \text{ and } F_{12}F_{13}\, b_\gamma c_\beta, \\ F_{12}F_{32}(A, C) \text{ where } (A, C) = \begin{pmatrix}A & C \\ & p\end{pmatrix} \text{ or } (AC)^i, \\ {}^*F_{12}(Cab)N, \\ F_{12}(Abc)(A\beta\gamma)N, \\ {}^*F_{12}(Abc)N, \\ F_{12}N, \quad \text{where } N \text{ has } F_2 \text{ or } F_1 \text{ factors.} \end{cases}$

(iv) G_{ij}, where $G_{12} = (Apb\gamma)$.

28. A form containing G_{ij} is reduced when expressed in terms of preceding factors. Taking G_{12} as typical, the (r.s.) admits of

.... F_{12}, F_{13}, (Abc), (Bca), $(A\beta\gamma)$, $(C\alpha\beta)$, N.

But $G_{12}G_{13} \equiv F_{12}F_{13}$; so that $G_{12}F_{12}F_{13} \equiv 0$. Accordingly we need

* These have dual forms.

only consider the types

(1) $G_{12}F_{12}M$,

(2) $G_{12}F_{13}M$,

(3) $G_{12}M$, where M contains neither G_{ij} nor F_{ij}.

Since G_{13} is dual of G_{12} and F_{13} is its own dual, then types (1) and (2) are dual. So (1) and (3) need only be considered. Ultimately we are left with

(D)
- $^*G_{12}F_{12}(A\beta\gamma)(A)$ where $(A) = (Ap)$, $\begin{pmatrix} A & x \\ & b \end{pmatrix}$ or $\begin{pmatrix} A & p \\ & B \end{pmatrix}$,
- $G_{12}(A\beta\gamma)(Abc)\, c_\beta\, (A)$,
- $G_{12}(Abc)\begin{pmatrix} c & \gamma \\ & \beta \end{pmatrix}$, where no independent dual exists, and there are only three of this type for all G_{ij}.
- $G_{12}\begin{pmatrix} b & \beta & \\ & c & \gamma \end{pmatrix}(A)$,
- $G_{12}N$, where N consists of F_2, F_1 factors but contains neither c nor β.
- G_{12}^2.

The brevity of the above list is largely due to identities of the type

$$G_{12}\, i_\beta j_c \equiv G_{13}\, i_\gamma j_b,$$

where i, j are any two different symbols u, a, A, a.

(v) H_i, h_i: where $H_1 = (BCua)$.

29. The factor H_1 reduces with any F_3 bracket except

$$F_{21},\ F_{31},\ H_2,\ H_3,\ h_1,\ (A\beta\gamma),\ (B\gamma a),\ (Ca\beta),\ (Bca),\ (Cab).$$

But owing to relations such as

$$H_1F_{21} \equiv F_4\,(Bp)u_a, \qquad H_1(Bca) \equiv F_4(Bcu),$$

$$h_1F_{21} \equiv F_4\,(Bp)\,a_x, \qquad H_1h_1 \equiv F_4(BC)',$$

$$H_1H_2 \equiv K(Ca\beta),$$

$$H_1(A\beta\gamma) \equiv H_2(B\gamma a) \equiv H_3(Ca\beta),$$

the system may be reduced to the following types:—

$$(\mathrm{E})\quad \begin{cases} {}^*H_1F_{21}(C,a) \text{ where } (C,a) \text{ is } (Cp)a_x,\ \begin{pmatrix} C & & B & \\ & A & & x \end{pmatrix} \text{ or } \begin{pmatrix} C & & a \\ & B & \end{pmatrix}, \\ {}^*H_1H_2H_3u_\alpha u_\beta u_\gamma, \\ {}^*H_1H_2u_\alpha u_\beta(A,B), \\ \quad H_1h_1\begin{pmatrix} a & & a \\ & B & \end{pmatrix}, \\ {}^*H_1^2, \\ {}^*H_1(Bca)(Cab)\,c_\alpha b_\alpha u_\alpha, \\ {}^*H_1(B\gamma a)(Ca\beta)\,u_\alpha u_\beta u_\gamma, \\ {}^*H_1(Bca)N, \\ {}^*H_1(B\gamma a)N, \\ {}^*H_1(A\beta\gamma)N, \\ {}^*H_1N, \text{ where } N \text{ consists of } F_1, F_2 \text{ factors.} \end{cases}$$

This list includes a sextic covariant of degree 3 in the coefficients of each of f, f_1, and f_2, viz.:—

$$h_1h_2h_3a_xb_xc_x = (BCax)(CAbx)(ABcx)\,a_xb_xc_x.$$

$$\text{(vi)}\quad K = (ABuCu).$$

30. The (r.s.) allows the factors H_1, H_2, H_3 and the types

$$u_\alpha \ldots (Ap) \ldots (Abu) \ldots (BC) \ldots (BC)' \ldots.$$

The system then is

$$(\mathrm{F})\quad \begin{cases} {}^*K^2, \\ {}^*KH_1u_\alpha(A) \text{ where } (A) = (Ap),\ \begin{pmatrix} A & & p \\ & C & \end{pmatrix},\ \begin{pmatrix} A' & & p \\ & C & \end{pmatrix},\ \begin{pmatrix} A & & C & \\ & b & & p \end{pmatrix}, \\ {}^*K(BC)^i(CA)^j(Cp) \text{ where } (BC)^i = (BC) \text{ or } (BC)' \text{ or } \begin{pmatrix} B & & C \\ & a & \end{pmatrix}, \\ {}^*K(BC)^i(Ap), \\ {}^*K(Ap)(Bp)(Cp). \end{cases}$$

The product KH_1H_2 is reducible.

The K_4 Group.

31. F_4 factors are of one type of which $(ABc\gamma)$ is representative. The (r.s.) shows that forms to be retained are, besides the three F_4^2, $F_4'^2$, $F_4''^2$,

$$F_4''MN \quad \text{and} \quad F_4''M,$$

where M is a product of F_3 factors, and N of F_2, F_1 factors. Further the (r.s.) admits the symbol a only twice, viz. in the factors (Bca) and a_γ. Similarly for b, α, β. Introducing four new symbols, let

$$F_{23}' = (Bca)a_\gamma, \quad F_{23}'' = (B\gamma\alpha)c_\alpha, \quad F_{13}' = (Abc)b_\gamma, \quad F_{13}'' = (A\beta\gamma)c_\beta;$$

and regarding these as new F_3 brackets, we may then express a member P, containing F_4'', as a product of factors selected from

$$c_x,\ u_\gamma,\ (Ap),\ (Bp),\ (Acu),\ (Bcu),\ (A\gamma x),\ (B\gamma x),\ (BC),\ (CA),\ (AB),\ (AB)'',$$

$$(AB)''' \text{ and } F_3 \text{ brackets, viz. } F_{13},\ F_{13}',\ F_{13}'',\ F_{23},\ F_{23}',\ F_{23}'',\ H_3,\ h_3.$$

Identities show that H_3h_3, H_3F_{13}, $H_3(Bac)$, H_3c_β can each be expressed in terms involving F_4' or reducible terms. Hence if H_3 occurs in P, no other F_3 factor is present. Similarly for h_3.

There are similar reductions for $F_{13}(Bac)$, $F_{13}(B\alpha\gamma)$; which imply that $F_{13}F_{23}'$, $F_{13}F_{23}''$ are here reducible. Clearly $F_{13}F_{13}'$ is reducible: and further, $F_{13}(BC)$, $F_{13}(B\gamma x)$, $F_{13}(Bcu)$ can all be expressed in terms involving either F_{23} or F_4'. If then *both* F_{13}, F_{23} occur in P, the only other factors involving c, γ are $c_x u_\gamma$, and the form is

$$F_4''F_{13}F_{23}c_x u_\gamma;$$

otherwise a form containing F_{13} has besides only tags and chains.

Again, by (III), (VIII) of § 20, we reduce $F_4''(Bac)(A\beta\gamma)$, so that the only remaining type with *two* F_3 brackets is $F_4''(Bac)(Abc)$ and its dual.

The K_4 group is represented then as follows:—

$$(G)\quad \begin{cases} F_4''F_{13}F_{23}c_x u_\gamma, \quad F_4''(Bac)a_\gamma(Abc)b_\gamma[c,\gamma], \\ F_4''F_{13}[B], \quad F_4''(Bac)a_\gamma[B], \quad F_4''H_3[c], \\ F_4''[A,B,c,\gamma], \quad F_4''^2, \end{cases}$$

where the second, fourth, and fifth have dual forms, and the square brackets indicate chains and tags as discussed in the K_3 system.

This exhausts all cases, and the Complete System is contained in the K_1 and K_2 groups of § 19, together with the sets denoted by (A) to (G) in §§ 25–31.

ON THE TRANSFORMATION OF CERTAIN SOLUTIONS OF THE ELECTROMAGNETIC EQUATIONS

By J. BRILL.

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1. We consider the particular form of the electromagnetic equations suitable for a ponderable body at rest with respect to the system with which the time is associated,* the medium being isotropic. For the convenience of our investigation we will replace the independent variables x, y, z, t by the symbols x_1, x_2, x_3, x_4, and thus the equations will assume the form

$$\frac{\partial(ke_1)}{\partial x_4}+\sigma e_1 = c\left(\frac{\partial m_3}{\partial x_2}-\frac{\partial m_2}{\partial x_3}\right), \qquad \frac{\partial(ke_2)}{\partial x_4}+\sigma e_2 = c\left(\frac{\partial m_1}{\partial x_3}-\frac{\partial m_3}{\partial x_1}\right),$$

$$\frac{\partial(ke_3)}{\partial x_4}+\sigma e_3 = c\left(\frac{\partial m_2}{\partial x_1}-\frac{\partial m_1}{\partial x_2}\right), \qquad \frac{\partial(ke_1)}{\partial x_1}+\frac{\partial(ke_2)}{\partial x_2}+\frac{\partial(ke_3)}{\partial x_3}=\rho,$$

$$\frac{\partial(\mu m_1)}{\partial x_4} = c\left(\frac{\partial e_2}{\partial x_3}-\frac{\partial e_3}{\partial x_2}\right), \qquad \frac{\partial(\mu m_2)}{\partial x_4} = c\left(\frac{\partial e_3}{\partial x_1}-\frac{\partial e_1}{\partial x_3}\right),$$

$$\frac{\partial(\mu m_3)}{\partial x_4} = c\left(\frac{\partial e_1}{\partial x_2}-\frac{\partial e_2}{\partial x_1}\right), \qquad \frac{\partial(\mu m_1)}{\partial x_1}+\frac{\partial(\mu m_2)}{\partial x_2}+\frac{\partial(\mu m_3)}{\partial x_3}=0.$$

From the first four equations we readily deduce

$$\frac{\partial(\sigma e_1)}{\partial x_1}+\frac{\partial(\sigma e_2)}{\partial x_2}+\frac{\partial(\sigma e_3)}{\partial x_3}+\frac{\partial\rho}{\partial x_4}=0,$$

which may be integrated in the form†

$$\rho = (\alpha, \beta, \gamma;\ x_1, x_2, x_3), \qquad \sigma e_1 = -(\alpha, \beta, \gamma;\ x_2, x_3, x_4),$$

$$\sigma e_2 = (\alpha, \beta, \gamma;\ x_1, x_3, x_4), \qquad \sigma e_3 = -(\alpha, \beta, \gamma;\ x_1, x_2, x_4).$$

---

* *Vide* Silberstein's *Theory of Relativity*, p. 261.

† To facilitate printing we adopt the notation $(u, v;\ x, y)$ as expressing the Jacobian of u and v with respect to x and y.

We will now introduce the assumptions

$$ke_1 = a(\beta, \gamma;\ x_2, x_3)+u, \qquad ke_2 = -a(\beta, \gamma;\ x_1, x_3)+v,$$

$$ke_3 = a(\beta, \gamma;\ x_1, x_2)+w;$$

$$cm_1 = -a(\beta, \gamma;\ x_1, x_4)+\xi, \qquad cm_2 = -a(\beta, \gamma;\ x_2, x_4)+\eta,$$

$$cm_3 = -a(\beta, \gamma;\ x_3, x_4)+\zeta.$$

Substituting these values in our first four equations and taking account of the values for ρ, σe_1, σe_2, σe_3 given above, we obtain

$$\frac{\partial u}{\partial x_4}-\frac{\partial \zeta}{\partial x_2}+\frac{\partial \eta}{\partial x_3}=0, \qquad \frac{\partial v}{\partial x_4}-\frac{\partial \xi}{\partial x_3}+\frac{\partial \zeta}{\partial x_1}=0,$$

$$\frac{\partial w}{\partial x_4}-\frac{\partial \eta}{\partial x_1}+\frac{\partial \xi}{\partial x_2}=0, \qquad \frac{\partial u}{\partial x_1}+\frac{\partial v}{\partial x_2}+\frac{\partial w}{\partial x_3}=0.$$

These four equations are satisfied in the most general manner by the assumptions

$$u=\frac{\partial X_3}{\partial x_2}-\frac{\partial X_2}{\partial x_3}, \qquad v=\frac{\partial X_1}{\partial x_3}-\frac{\partial X_3}{\partial x_1}, \qquad w=\frac{\partial X_2}{\partial x_1}-\frac{\partial X_1}{\partial x_2},$$

$$\xi=\frac{\partial X_1}{\partial x_4}-\frac{\partial X_4}{\partial x_1}, \qquad \eta=\frac{\partial X_2}{\partial x_4}-\frac{\partial X_4}{\partial x_2}, \qquad \zeta=\frac{\partial X_3}{\partial x_4}-\frac{\partial X_4}{\partial x_3}.$$

We have thus obtained a set of forms, of a quite general type, that satisfy the first four of our equations. The remaining four can be satisfied in a perfectly general manner by the assumptions

$$\mu m_1=\frac{\partial Y_3}{\partial x_2}-\frac{\partial Y_2}{\partial x_3}, \qquad \mu m_2=\frac{\partial Y_1}{\partial x_3}-\frac{\partial Y_3}{\partial x_1}, \qquad \mu m_3=\frac{\partial Y_2}{\partial x_1}-\frac{\partial Y_1}{\partial x_2},$$

$$ce_1=\frac{\partial Y_4}{\partial x_1}-\frac{\partial Y_1}{\partial x_4}, \qquad ce_2=\frac{\partial Y_4}{\partial x_2}-\frac{\partial Y_2}{\partial x_4}, \qquad ce_3=\frac{\partial Y_4}{\partial x_3}-\frac{\partial Y_3}{\partial x_4}.$$

Thus the simultaneous satisfaction of our whole set of equations necessitates that the following shall be identically satisfied

$$c\left\{\ a(\beta, \gamma;\ x_2, x_3)+\frac{\partial X_3}{\partial x_2}-\frac{\partial X_2}{\partial x_3}\right\}-k\left(\frac{\partial Y_4}{\partial x_1}-\frac{\partial Y_1}{\partial x_4}\right)=0, \tag{1}$$

$$c\left\{-a(\beta, \gamma;\ x_1, x_3)+\frac{\partial X_1}{\partial x_3}-\frac{\partial X_3}{\partial x_1}\right\}-k\left(\frac{\partial Y_4}{\partial x_2}-\frac{\partial Y_2}{\partial x_4}\right)=0, \tag{2}$$

$$c\left\{\ a(\beta, \gamma;\ x_1, x_2)+\frac{\partial X_2}{\partial x_1}-\frac{\partial X_1}{\partial x_2}\right\}-k\left(\frac{\partial Y_4}{\partial x_3}-\frac{\partial Y_3}{\partial x_4}\right)=0, \tag{3}$$

$$\mu\left\{-a(\beta,\gamma;\ x_1,x_4)+\frac{\partial X_1}{\partial x_4}-\frac{\partial X_4}{\partial x_1}\right\}-c\left(\frac{\partial Y_3}{\partial x_2}-\frac{\partial Y_2}{\partial x_3}\right)=0, \quad (4)$$

$$\mu\left\{-a(\beta,\gamma;\ x_2,x_4)+\frac{\partial X_2}{\partial x_4}-\frac{\partial X_4}{\partial x_2}\right\}-c\left(\frac{\partial Y_1}{\partial x_3}-\frac{\partial Y_3}{\partial x_1}\right)=0, \quad (5)$$

$$\mu\left\{-a(\beta,\gamma;\ x_3,x_4)+\frac{\partial X_3}{\partial x_4}-\frac{\partial X_4}{\partial x_3}\right\}-c\left(\frac{\partial Y_2}{\partial x_1}-\frac{\partial Y_1}{\partial x_2}\right)=0. \quad (6)$$

To obtain a solution we need to know a, β, γ and the X's. Equations (1) to (6) really impose restrictions on the form of these functions. If we eliminate the Y's we obtain differential equations defining the first set of functions.

2. Now suppose that by means of a point transformation we change the independent variables from x_1, x_2, x_3, x_4 to $\xi_1, \xi_2, \xi_3, \xi_4$, and that we obtain

$$X_1dx_1+X_2dx_2+X_3dx_3+X_4dx_4 = L_1d\xi_1+L_2d\xi_2+L_3d\xi_3+L_4d\xi_4,$$

and $Y_1dx_1+Y_2dx_2+Y_3dx_3+Y_4dx_4 = M_1d\xi_1+M_2d\xi_2+M_3d\xi_3+M_4d\xi_4.$

We then have four equations of the type

$$X_1 = L_1\frac{\partial\xi_1}{\partial x_1}+L_2\frac{\partial\xi_2}{\partial x_1}+L_3\frac{\partial\xi_3}{\partial x_1}+L_4\frac{\partial\xi_4}{\partial x_1},$$

and four of the type

$$Y_1 = M_1\frac{\partial\xi_1}{\partial x_1}+M_2\frac{\partial\xi_2}{\partial x_1}+M_3\frac{\partial\xi_3}{\partial x_1}+M_4\frac{\partial\xi_4}{\partial x_1}.$$

Our equations (1) to (6) then assume the form

$$c\left\{a(\beta,\gamma;\ x_2,x_3)+\sum_{n=1}^{4}(L_n,\xi_n;\ x_2,x_3)\right\}-k\sum_{n=1}^{4}(M_n,\xi_n;\ x_1,x_4)=0, \quad (7)$$

$$c\left\{a(\beta,\gamma;\ x_1,x_3)+\sum_{n=1}^{4}(L_n,\xi_n;\ x_1,x_3)\right\}+k\sum_{n=1}^{4}(M_n,\xi_n;\ x_2,x_4)=0, \quad (8)$$

$$c\left\{a(\beta,\gamma;\ x_1,x_2)+\sum_{n=1}^{4}(L_n,\xi_n;\ x_1,x_2)\right\}-k\sum_{n=1}^{4}(M_n,\xi_n;\ x_3,x_4)=0, \quad (9)$$

$$\mu\left\{a(\beta,\gamma;\ x_1,x_4)+\sum_{n=1}^{4}(L_n,\xi_n;\ x_1,x_4)\right\}+c\sum_{n=1}^{4}(M_n,\xi_n;\ x_2,x_3)=0, \quad (10)$$

$$\mu\left\{a(\beta,\gamma;\ x_2,x_4)+\sum_{n=1}^{4}(L_n,\xi_n:\ x_2,x_4)\right\}-c\sum_{n=1}^{4}(M_n,\xi_n;\ x_1,x_3)=0, \quad (11)$$

$$\mu\left\{a(\beta,\gamma;\ x_3,x_4)+\sum_{n=1}^{4}(L_n,\xi_n;\ x_3,x_4)\right\}+c\sum_{n=1}^{4}(M_n,\xi_n;\ x_1,x_2)=0. \quad (12)$$

We will now define our point transformation as satisfying the eighteen conditions

$$
\begin{array}{rlrl}
(x_2, x_3;\ \xi_2, \xi_3) = & (x_1, x_4;\ \xi_1, \xi_4), & (x_1, x_3;\ \xi_2, \xi_3) = - & (x_2, x_4;\ \xi_1, \xi_4), \\
(x_1, x_2;\ \xi_2, \xi_3) = & (x_3, x_4;\ \xi_1, \xi_4), & c^2(x_1, x_4;\ \xi_2, \xi_3) = - & \mu k(x_2, x_3;\ \xi_1, \xi_4), \\
c^2(x_2, x_4;\ \xi_2, \xi_3) = & \mu k(x_1, x_3;\ \xi_1, \xi_4), & c^2(x_3, x_4;\ \xi_2, \xi_3) = - & \mu k(x_1, x_2;\ \xi_1, \xi_4), \\
(x_2, x_3;\ \xi_1, \xi_3) = - & (x_1, x_4;\ \xi_2, \xi_4), & (x_1, x_3;\ \xi_1, \xi_3) = & (x_2, x_4;\ \xi_2, \xi_4), \\
(x_1, x_2;\ \xi_1, \xi_3) = - & (x_3, x_4;\ \xi_2, \xi_4), & c^2(x_1, x_4;\ \xi_1, \xi_3) = & \mu k(x_2, x_3;\ \xi_2, \xi_4), \\
c^2(x_2, x_4;\ \xi_1, \xi_3) = - & \mu k(x_1, x_3;\ \xi_2, \xi_4), & c^2(x_3, x_4;\ \xi_1, \xi_3) = & \mu k(x_1, x_2;\ \xi_2, \xi_4), \\
(x_2, x_3;\ \xi_1, \xi_2) = & (x_1, x_4;\ \xi_3, \xi_4), & (x_1, x_3;\ \xi_1, \xi_2) = - & (x_2, x_4;\ \xi_3, \xi_4), \\
(x_1, x_2;\ \xi_1, \xi_2) = & (x_3, x_4;\ \xi_3, \xi_4), & c^2(x_1, x_4;\ \xi_1, \xi_2) = - & \mu k(x_2, x_3;\ \xi_3, \xi_4), \\
c^2(x_2, x_4;\ \xi_1, \xi_2) = & \mu k(x_1, x_3;\ \xi_3, \xi_4), & c^2(x_3, x_4;\ \xi_1, \xi_2) = - & \mu k(x_1, x_2;\ \xi_3, \xi_4).
\end{array}
$$

If we now multiply equations (7), (8), (9), (10), (11), (12) respectively by $(x_2, x_3;\ \xi_2, \xi_3)$, $(x_1, x_3;\ \xi_2, \xi_3)$, $(x_1, x_2;\ \xi_2, \xi_3)$, $(x_1, x_4;\ \xi_2, \xi_3)$, $(x_2, x_4;\ \xi_2, \xi_3)$, $(x_3, x_4;\ \xi_2, \xi_3)$, take account of the above relations, divide out common factors from certain of our equations, and add the results, we obtain

$$c\left\{\alpha(\beta, \gamma;\ \xi_2, \xi_3) + \sum_{n=1}^{4} (L_n, \xi_n;\ \xi_2, \xi_3)\right\} - k \sum_{n=1}^{4} (M_n, \xi_n;\ \xi_1, \xi_4) = 0,$$

which reduces to

$$c\left\{\alpha(\beta, \gamma;\ \xi_2, \xi_3) + \frac{\partial L_3}{\partial \xi_2} - \frac{\partial L_2}{\partial \xi_3}\right\} - k\left(\frac{\partial M_4}{\partial \xi_1} - \frac{\partial M_1}{\partial \xi_4}\right) = 0.$$

This is identical in form with equation (1). Similarly we can obtain five other equations respectively identical in form with equations (2) to (6).

If we have a set of functions α, β, γ, X_1, X_2, X_3, X_4 suitable in form for the derivation of a solution of our electromagnetic equations, we can transform α, β, γ by means of a point transformation satisfying our eighteen conditions, and calculate a set of L's from the four equations of the type

$$L_1 = X_1 \frac{\partial x_1}{\partial \xi_1} + X_2 \frac{\partial x_2}{\partial \xi_1} + X_3 \frac{\partial x_3}{\partial \xi_1} + X_4 \frac{\partial x_4}{\partial \xi_1}.$$

The transformed forms of α, β, γ and the set of L's so obtained are suitable in form for the deduction of a new solution of the electromagnetic equations. The new values of k and μ will be obtained by simple transformation, but the new value of σ will be derived from the solution of a differential equation.

CYCLOTOMIC QUINQUE-SECTION FOR EVERY PRIME OF THE FORM $10n+1$ BETWEEN 100 AND 500

By PANDIT OUDH UPADHYAYA.

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THE formula of quinque-section was first given by L. J. Rogers in Vol. 32 (old series) of the *Proceedings*. It was shown by W. Burnside in 1915, also in the *Proceedings*, that the problem of quinque-section depends on the solution of two Diophantine equations, namely:

$$(1) \quad [4p-16-25(A+B)]^2+1125(A-B)^2+450(C^2+D^2) = 12^2 p,$$

$$(2) \quad [4p-16-25(A+B)][A-B]+3(C^2+4CD-D^2) = 0.$$

Burnside solved these two equations for every prime of the form $10n+1$ under 100, and gave the values of p, A, B, C, and D in a tabular form.

The object of this paper is to construct a similar table for every prime of the form $10n+1$ between 100 and 500. The details of calculation are given for one prime only.

I have used the general formula of Burnside, except that I have corrected the coefficient of η to $\frac{(p-1)^3}{5^3}$. In Burnside's paper this is misprinted as $\frac{(p-1)^2}{5^3}$. Cayley gave the quintic for every prime under 100. In order to verify his results I calculated them by the method given by Burnside, and found that there are two discrepancies. For the prime 31 the coefficient of η^2 ought to be -21 and not -2. For the prime 61, the constant term must be -13 and not 23.

The Details of Calculation for the Prime 281.

In the first equation let us substitute the value of p, supposing that

$A+B = 43$; then, by the first equation, we get

$$[4\times 281-16-25\times 43]^2+1125(A-B)^2+450(C^2+D^2) = 144\times 281,$$

or
$$1125(A-B)^2+450(C^2+D^2) = 39375.$$

Now, if $A-B = 1$,

$$C^2+D^2 = 85 = 9^2+2^2 \text{ or } 7^2+6^2.$$

Thus $C = 9$ and $D = 2$ or $C = 7$ and $D = 6$. And if $A-B = 3$,

$$C^2+D^2 = 65 = 8^2+1^2 \text{ or } 7^2+4^2.$$

Thus $C = 8$ and $D = 1$ or $C = 7$ and $D = 4$. Finally, if $A-B = -5$,

$$C^2+D^2 = 25 = 5^2+0^2 \text{ or } 4^2+3^2.$$

Thus $C = 5$ and $D = 0$ or $C = 4$ and $D = 3$.

Of these solutions only $C = 4$, $D = 3$ is a solution of the second equation, as may be verified at once by substitution.

Since $A+B = 43$ and $A-B = -5$, we have $A = 19$ and $B = 24$.

The solution is therefore

$$A = 19, \quad B = 24, \quad C = 4, \quad D = 3.$$

The Determination of the Coefficients of the Quintic.

The formula for the determination of the coefficients of the quintic, as given by Burnside, is as follows:

$$\eta^5+\eta^4-\frac{2(p-1)}{5}\eta^3+\left[\tfrac{1}{3}p(A+B)-\frac{2(p-1)(2p+3)}{3\times 5^2}\right]\eta^2$$

$$+\left[\tfrac{1}{9}p\times\left(\frac{p-1}{5}+A+B\right)^2-pAB-\frac{(p-1)^3}{5^3}\right]\eta$$

$$+\tfrac{1}{5}p\left[\frac{1}{5\times 6^3}\left\{5(A+B)-\frac{4p-4}{5}\right\}^3+\frac{1}{6^2}\left(\frac{2p-2}{5}-A-B\right)^2+\tfrac{1}{4}(A-B)^2\right.$$

$$\left.+\tfrac{1}{8}(A-B)(D^2-C^2)\right]-\frac{(p-1)^3}{5^5}.$$

Only the coefficients of η^3, η^2, η and the constant term depend upon p. When $p = 281$ their values are found to be -112, -191, 2257, and 967 respectively. The quintic is therefore

$$\eta^5+\eta^4-112\eta^3-191\eta^2+2257\eta+967 = 0.$$

TABLE.

p	A	B	C	D	η^5	η^4	η^3	η^2	η	1
101	10	9	3	2	1	1	− 40	93	− 21	− 17
131	10	9	1	6	1	1	− 52	− 89	109	193
151	14	10	2	2	1	1	− 60	− 12	784	128
181	13	14	2	7	1	1	− 72	−123	223	− 49
191	12	13	3	4	1	1	− 76	−359	− 437	− 155
211	14	19	1	2	1	1	− 84	− 59	1661	269
241	16	20	4	4	1	1	− 96	−212	1232	512
251	22	18	2	6	1	1	−100	− 20	1504	1024
271	20	19	1	8	1	1	−108	−401	− 13	845
281	19	24	4	3	1	1	−112	−191	2257	967
311	28	27	7	0	1	1	−124	535	− 413	539
331	23	22	2	− 5	1	1	−132	−887	−1843	−1027
401	34	33	3	10	1	1	−160	369	879	− 29
421	37	32	8	1	1	1	−168	219	3853	−3517
431	34	30	6	− 6	1	1	−172	−724	−1824	1728
461	39	34	2	9	1	1	−184	−129	4551	5419
491	40	39	3	12	1	1	−196	59	2019	1377

I have received very much help from Pandit Shukdeo Chaube in calculation, and from Prof. Narendra Kumar Majumdar in preparing the plan of this paper.

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